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1 Dimensions and Measurement

INTRODUCTION

Like all other sciences, physics is based on experimental observations and quantitative measurements. The main objectives of physics are to identify a limited number of fundamental laws that govern natural phenomena and use them to develop theories that can predict the results of future experiments. The fundamental laws used in developing theories are expressed in the language of mathematics, the tool that provides a bridge between theory and experiment.

PHYSICAL QUANTITY

A quantity which can be measured and by which various physical happenings can be explained and expressed in the form of laws is called a physical quantity. For example, length, mass, time, force, etc.

Measurement is necessary to determine the magnitude of a physical quantity, to compare two similar physical quantities, and to prove physical laws or equations.

A physical quantity is represented completely by its magnitude and unit. For example, 10 m means a length which is 10 times the unit of length. Here 10 represents the numerical value of the given quantity and meter represents the unit of quantity under consideration. Thus, in expressing a physical quantity, we choose a unit and then find that how many times that unit is contained in the given physical quantity, i.e.,

Physical quantity (Q) = Magnitude × Unit = $n \times u$

where n represents the numerical value and u represents the unit. Thus, while expressing a definite amount of physical quantity, it is clear that as the unit u changes, the magnitude (n) will also change but product nu will remain same.

i.e. $nu = \text{constant}$

or $n_1u_1 = n_2u_2 = \text{constant} \Rightarrow n \propto \frac{1}{u}$

i.e., the magnitude of a physical quantity and units are inversely proportional to each other. Larger the unit, smaller will be the magnitude.

TYPES OF PHYSICAL QUANTITY

Ratio (numerical value only) When a physical quantity is a ratio of two similar quantities, it has no unit. For example, Relative density = Density of object/Density of water at 4°C

Refractive index = Velocity of light in air/Velocity of light in medium

Strain = Change in dimension/Original dimension

Angle is an exceptional physical quantity, which though is a ratio of two similar physical quantities (angle = arc/radius) but still requires a unit (degrees or radians) to specify it along with its numerical value.

Scalar (magnitude only): These quantities do not have any direction, e.g., length, time, work, energy, etc.

The magnitude of a physical quantity can be negative. In that case, *negative sign* indicates that the numerical value of the quantity under consideration is negative. It does not specify direction.

Scalar quantities can be added or subtracted with the help of following ordinary laws of addition or subtraction.

Vector (magnitude and direction) Example of vectors are displacement, velocity, acceleration, force, etc.

Vector physical quantities can be added or subtracted according to the vector laws of addition. These laws are different from the laws of ordinary addition.

FUNDAMENTAL AND DERIVED QUANTITIES

Fundamental quantities Out of a large number of physical quantities which exist in nature, there are only few quantities which are independent of all other quantities and do not require the help of any other physical quantity for their definition. Therefore, these are called absolute quantities. These quantities are also called fundamental or base quantities, as all other quantities are based upon and can be expressed in terms of these quantities.

Derived quantities All other physical quantities can be derived by suitable multiplication or division of different powers of fundamental quantities. These are, therefore, called derived quantities.

For example, if length is defined as a fundamental quantity, then area and volume are derived from length and are expressed in term of length with power 2 and 3 over the term of length.

Table: Fundamental quantities in SI system and their units

S. No.	Physical quantity	Name of unit	Symbol of unit
1.	Mass	Kilogram	kg
2.	Length	Meter	m
3.	Time	Second	s
4.	Temperature	Kelvin	K
5.	Luminous Intensity	Candela	Cd
6.	Electric Current	Ampere	A
7.	Amount of Substance	Mole	mol

Table: Supplementary quantities in SI system and their units

S. No.	Physical quantity	Name of unit	Symbol of unit
1.	Plane angle	Radian	rad
2.	Solid angle	Steradian	sr

FUNDAMENTAL AND DERIVED UNITS

Normally each physical quantity requires a unit or standard for its specification so it appears that there must be as many units as there are physical quantities. However, it is not so. It has been found that if in mechanics we arbitrarily choose units of any three physical quantities we can express the units of all other physical quantities in mechanics in terms of these. Arbitrarily, the physical quantities *mass*, *length*, and *time* are choosen for this purpose. So any unit of mass, length, and time in mechanics is called a **fundamental, absolute, or base unit.** Other units which can be expressed in terms of fundamental units are called derived units. For example, light year or km is a fundamental units as it is a unit of length while s^{-1}, m^2, or $kg\,m^{-1}$, are derived units as these are derived from units of time, mass, and length, respectively.

System of units A complete set of units, both fundamental and derived, for all kinds of physical quantities is called system of units. The common systems are given in the table below:

Table: Fundamental quantities in SI system and their units

	MKS System	CGS System	FPS System	SI units
Length	m (meter)	cm (centimeter)	ft (foot)	It is an extended form of MKS system. It includes four more fundamental units (in addition to three basic units) which represent fundamental quantities in electricity, magnetism, heat, and light.
Mass	kg (kilogram)	g (gram)	lb (pound)	
Time	s (second)	s (second)	s (second)	

SI PREFIXES

The magnitudes of physical quantities vary over a wide range. The standard prefixes for magnitude too large or too small to be expressed more compactly for certain powers of 10 are mentioned in the table below.

Table: Prefixes used for different powers of 10

Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol	Power of 10	Prefix	Symbol
10^{18}	exa	E	10^{-1}	deci	d	10^{9}	giga	G	10^{-6}	micro	μ
10^{15}	peta	P	10^{-2}	centi	c	10^{6}	mega	M	10^{-9}	nano	n
10^{12}	tera	T	10^{-3}	milli	m	10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f	10^{1}	deca	da	10^{-18}	atto	a

ILLUSTRATION 1.1

Given that $F = 5$ N. Convert it into CGS system.

Sol.

$$F = 5\frac{\text{kg} \times \text{m}}{\text{s}^2} = (5)\frac{(10^3\,\text{g})\,(100\,\text{cm})}{\text{s}^2}$$

$$= 5 \times 10^5\ \frac{\text{g cm}}{\text{s}^2}\ (\text{in CGS system})$$

This unit $\left(\frac{\text{g cm}}{\text{s}^3}\right)$ is also called dyne.

ILLUSTRATION 1.2

A particle is moving with velocity $v = 90$ km/h. Convert the velocity of the particle into m/s.

Sol.

$$v = 90\ \text{km/h}$$

$$= (90)\frac{(1000\ \text{m})}{(60 \times 60\ \text{s})} = (90)\left(\frac{1000}{3600}\right)\frac{\text{m}}{\text{s}} = 90 \times \frac{5}{10}\ \frac{\text{m}}{\text{s}}$$

$$\Rightarrow \quad v = 25\ \text{m/s}$$

Important Points:

To convert $\frac{\text{km}}{\text{hour}}$ into $\frac{\text{m}}{\text{sec}}$, multiply by $\frac{5}{18}$.

ILLUSTRATION 1.3

Convert 78 pm into μm.

Sol. Let 7 pm = (x) μm or $7 \times (10^{-12})\ \text{m} = (x) \times 10^{-6}\ \text{m}$

We get $x = 7 \times 10^{-6}$

So, 7 pm = (7×10^{-6}) μm

ILLUSTRATION 1.4

A new unit of length is chosen such that the speed of light in vacuum is unity. What is the distance between the Sun and the Earth in terms of the new unit if light takes 8 min and 20 s to cover this distance?

Sol. Here, speed of light in vacuum, $c = 1$ new unit of length s^{-1}

Time taken by light of sun to reach the earth,

$t = 8$ min and 20 seconds $= (8 \times 60 + 20)\text{s} = 500$ s

Therefore, distance between the Sun and Earth

$S = ct = 1$ new unit of length $s^{-1} \times 500$ s

$= 500$ new units of length

Concept Application Exercise 1.1

1. State True or false:
   (a) Radian is the unit of plane angle and it is a supplementary unit.
   (b) The unit is always written in singular form, e.g., foot not feet.
   (c) If a unit is named after a person, the unit is not written with capital initial letter.
   (d) If $A + B = C - D$, then units of A, B, C and D must be same.
2. Fill in the blanks:
   (a) The unit of work and energy are ....
   (b) The unit of power is ....
   (c) The unit of energy per unit volume is.........
3. Fill in the blanks
   (a) The volume of a cube of side 1 cm is equal to....m^3.
   (b) The surface area of a solid cylinder of radius 2.0 cm and height 10 cm is equal to ....$(mm)^2$.
   (c) A vehicle moving with a speed of 18 km h^{-1} coves ...m in 1s.
   (d) The relative density of lead is 11.3 Its density is... g cm^{-3} or kg m^{-3}
4. Fill in the blanks by suitable conversion of units
   (a) 1 kg $m^2 s^{-2}$ = ... g $cm^2 s^{-2}$
   (b) 3.0 m s^{-2} = ... km h^{-2}
   (c) $G = 6.67 \times 10^{-11}$ $Nm^2 (kg)^{-2}$ = ... $(cm)^3 s^{-2} g^{-1}$.

Answers

1. (a) True (b) True (c) True (d) True
2. (a) kg $m^2 s^{-2}$ (b) W (c) kg $m^{-1} s^{-1}$
3. (a) 10^{-6} (b) 1.5×10^4 (c) 5 m (d) 1.13×10^4 kg m^{-3}
4. (a) 10^7 (b) 3.9×10^4 (c) 6.67×10^{-8}

Dimensions of a Physical Quantity

When a derived quantity is expressed in terms of fundamental quantities, it is written as a product of different powers of the fundamental quantities. The powers to which fundamental quantities must be raised in order to express the given physical quantity are called its dimensions.

To make it more clear, consider the physical quantity force:

Force = Mass × Acceleration

$$= \frac{\text{Mass} \times \text{Velocity}}{\text{Time}} = \frac{\text{Mass} \times \text{Length/time}}{\text{Time}}$$

$$= \text{Mass} \times \text{Length} \times (\text{Time})^{-2} \quad \text{(i)}$$

Thus, the dimensions of force are 1 in mass, 1 in length and –2 in time. Here the physical quantity that is expressed in terms of the base quantities is enclosed in square brackets to indicate that the equation is among dimensions and not among magnitudes.Thus, (i) can be written as [force] = $[MLT^{-2}]$.

Such an expression for a physical quantity in terms of the fundamental quantities is called the dimensional equation. If we consider only the RHS of the equation, the expression is termed as dimensional formula. Thus, the dimensional formula for force is $[MLT^{-2}]$.

Methods to Find Dimensions of Physical Constant or Coefficients

As dimensions of a physical quantity are unique, we write any formula or equation incorporating the given constant and then by substituting the dimensional formulae of all other quantities, we can find the dimensions of the required constant or coefficient.

- Height, width, radius, displacement, etc., are a kind of length. So we can say that their dimension is $[L]$. Here [Height] can be read as "dimension of height."

  Area of square = Length × Width

  [Area] = [Length] × [Width]

  $= [L] \times [L] = [L^2]$

  Area of circle $= \pi r^2$

  $[\text{Area}] = [\pi][r^2] = [1][L^2] = [L^2]$

  Here π is not a kind of length or mass or time, so π should not effect the dimension of area.

- Volume of cube = [Area] × [Height]

  $[\text{Volume}] = [L^2] \times [L] = [L^3]$

  $$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$

  $$[\text{Volume}] = \left[\frac{4}{3}\pi\right]\left[r^3\right] = (1)[L^3] = [L^3]$$

  So dimension of volume will be always $[L^3]$ whether it is volume of a cuboid or volume of sphere.

- $$\text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

  $$[\text{Density}] = \frac{[\text{Mass}]}{[\text{Volume}]} = \frac{[M]}{[L^3]} = [M^1L^{-3}]$$

- $$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}}$$

  $$[V] = \frac{[\text{Displacement}]}{[\text{Time}]} = \frac{[L]}{[T]} = [M^0L^1T^{-1}]$$

- $$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time}}$$

  $$[a] = \frac{\Delta V}{\Delta t} = \frac{[LT^{-1}]}{[T]} = [LT^{-2}]$$

- Linear momentum (P) = Mass × Velocity

  $[P] = [M][V] = [M][LT^{-1}] = [M^1L^1T^{-1}]$

- Force = Mass × Acceleration

  $[F] = Ma = [M][LT^{-2}] = [M^1L^1T^{-2}]$

- Work or energy = Force × Displacement

  [Work] = [Force][Displacement]

  $= [M^1L^1T^{-2}][L] = [M^1L^2T^{-2}]$

- $$\text{Power} = \frac{\text{Work}}{\text{Time}}$$

  $$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]} = \frac{[M^1L^2T^{-2}]}{[T]} = [M^1L^2T^{-3}]$$

- Pressure = $\dfrac{\text{Force}}{\text{Area}}$

$$[\text{Pressure}] = \frac{[\text{Force}]}{[\text{Area}]} = \frac{[M^1L^1T^{-2}]}{[L^2]} = [M^1L^{-1}T^{-2}]$$

Dimensions of Angular Quantities

- Angular displacement = $\dfrac{\text{Arc}}{\text{Radius}}$

$$[\theta] = \frac{[\text{Arc}]}{[\text{Radius}]} = \frac{[L]}{[L]} = [M^0L^0T^0] \text{ (Dimensionless)}$$

- Angular velocity (ω) = $\dfrac{\text{Angular displacement}}{\text{Time}}$

$$[\omega] = \frac{[\theta]}{[t]} = \frac{1}{[T]} = [M^0L^0T^{-1}]$$

Angular acceleration = $\dfrac{\text{Charge in angular velocity}}{\text{Time}}$

$$[\alpha] = \frac{[d\omega]}{[dt]} = \frac{[M^0L^0T^{-1}]}{[T]} = [M^0L^0T^{-2}]$$

Torque = Force × Lever arm

[Torque] = [Force] × [Lever arm]

$= [M^1L^1T^{-2}] \times [L] = [M^1L^2T^{-2}]$

Gravitational constant: According to Newton's law of gravitation,

$$F = G\frac{m_1m_2}{r^2} \quad \text{or} \quad G = \frac{Fr^2}{m_1m_2}$$

Substituting the dimensions of all physical quantities,

$$[G] = \frac{[MLT^{-2}][L^2]}{[M][M]} = [M^{-1}L^3T^{-2}]$$

APPLICATIONS OF DIMENSIONAL ANALYSIS

To find the unit of a physical quantity in a given system of units

By writing the definition or formula for the physical quantity, we find its dimensions. Now in the dimensional formula replacing M, L, and T by the fundamental units of the required system, we get the unit of physical quantity. However, sometimes to this unit, we further assign a specific name, e.g., Work = Force × Displacement

So $[W] = [MLT^{-2}] \times [L] = [ML^2T^{-2}]$

So, its units in CGS system will be g cm^2 s^{-2} which is called erg while in MKS system will be kg m^2 s^{-2} which is called joule.

To convert a physical quantity from one system to the other

The measure of a physical quantity is nu = constant.

If a physical quantity X has dimensional formula $[M^aL^bT^c]$ and if the derived units of that physical quantity in two systems are $[M_1^aL_1^bT_1^c]$ and $[M_2^aL_2^bT_2^c]$ and n_1 and n_2 be the numerical values in the two systems, respectively, then

$$n_1[u_1] = n_2[u_2]$$

$$\Rightarrow \quad n_1[M_1^aL_1^bT_1^c] = n_2[M_2^aL_2^bT_2^c]$$

$$\Rightarrow \quad n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

where M_1, L_1, and T_1 are the fundamental units of mass, length, and time in the first (known) system and M_2, L_2, and T_2 are the fundamental units of mass, length, and time in the second (unknown) system, respectively. Thus, knowing the values of fundamental units in two systems and the numerical value in one system, the numerical value in the other system may be evaluated.

ILLUSTRATION 1.5

Convert newton into dyne.

Sol. Newton is the SI unit of force and has dimensional formula $[M^1L^1T^{-2}]$, i.e., $a = 1, b = 1, c = -2$.

So $\quad$ 1 N = 1 kg m s^{-2}

SI system	CGS system
M_1 = 1 kg	M_2 = 1 g
L_1 = 1 m	L_2 = 1 cm
T_1 = 1 s	T_2 = 1 s

By using $n_2 = n_1\left[\dfrac{M_1}{M_2}\right]^a\left[\dfrac{L_1}{L_2}\right]^b\left[\dfrac{T_1}{T_2}\right]^c$

$$= 1\left[\frac{1\text{ kg}}{1\text{ g}}\right]^1\left[\frac{1\text{ m}}{1\text{ cm}}\right]^1\left[\frac{1\text{s}}{1\text{ s}}\right]^{-2}$$

$$= 1\left[\frac{10^3\text{ g}}{1\text{ g}}\right]^1\left[\frac{10^2\text{ cm}}{1\text{ cm}}\right]^1\left[\frac{1\text{ s}}{1\text{ s}}\right]^{-2} = 10^5$$

$\therefore \quad$ 1 N = 10^5 Dyne

ILLUSTRATION 1.6

Convert 1 joule to ergs.

Sol. Joule: SI system; erg: CGS system

Work = Force × Distance = Mass × Acceleration × Length

$$= \text{Mass} \times \frac{\text{Length}}{(\text{Time})^2} \times \text{Length}$$

Dimensions of work = $[W] = [M^1L^2T^{-2}]$

$\therefore \quad a = 1, b = 2, c = -2.$

Now

SI system	CGS system
M_1 = 1 kg	M_2 = 1 g
L_1 = 1 m	L_2 = 1 cm
T_1 = 1 s	T_2 = 1 s

Here $N_1 = 1, N_2 = ?$

$\therefore$ Using $N_2 = N_1\left[\dfrac{M_1}{M_2}\right]^a\left[\dfrac{L_1}{L_2}\right]^b\left[\dfrac{T_1}{T_2}\right]^c$

$$= 1\left[\frac{1\text{kg}}{1\text{ g}}\right]^1\left[\frac{1\text{ m}}{1\text{ cm}}\right]^2\left[\frac{1\text{ s}}{1\text{ s}}\right]^{-2}$$

$$= 1\left[\frac{1000\text{ g}}{1\text{ g}}\right]^1 \left[\frac{100\text{ cm}}{1\text{cm}}\right]^2 = 10^7$$

$\therefore\quad N_2 = 10^7$

So 1 J = 10^7 erg

ILLUSTRATION 1.7

Convert gravitational constant (G) from CGS to MKS system.

Sol. The dimensional formula of G is $[M^{-1}L^3T^{-2}]$, i.e., $a = -1$, $b = 3$, $c = -2$.

SI system	CGS system
$M_1 = 1$ kg	$M_2 = 1$ g
$L_1 = 1$ m	$L_2 = 1$ cm
$T_1 = 1$ s	$T_2 = 1$ s

By using $n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$

Value of G in CGS unit

$= 6.67 \times 10^{-8}$ cm^3 g s^{-2}

$$= 6.67\times10^{-8}\left[\frac{1\text{g}}{1\text{kg}}\right]^{-1}\left[\frac{1\text{cm}}{1\text{m}}\right]^3\left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$$= 6.67\times10^{-8}\left[\frac{\text{g}}{10^3\text{ g}}\right]^{-1}\left[\frac{\text{cm}}{10^2\text{ cm}}\right]^3\left[\frac{\text{s}}{\text{s}}\right]^{-2} = 6.67\times10^{-11}$$

$\therefore\ G = 6.67 \times 10^{-11}$ MKS units

ILLUSTRATION 1.8

In CGS system, the magnitude of the force is 100 dynes. In another system where the fundamental physical quantities are kilogram, meter, and minute, find the magnitude of the force.

Sol. We have $n_1 = 100, M_1 = 1$ g, $L_1 = 1$ cm, $T_1 = 1$ s and $M_2 = 1$ kg, $L_2 = 1$ m, $T_2 = 1$ min. The dimensional formula of forces is $[M^1L^1T^{-2}]$, where $a = 1$, $b = 1$, $c = -2$.

By substituting these values in the following conversion formula, we have

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$$

$$= 100\left[\frac{1\text{g}}{1\text{kg}}\right]^1\left[\frac{1\text{cm}}{1\text{m}}\right]^1\left[\frac{1\text{s}}{1\text{min}}\right]^{-2}$$

$$= 100\left[\frac{1\text{g}}{10^3\text{ g}}\right]^1\left[\frac{1\text{cm}}{10^2\text{ cm}}\right]^1\left[\frac{1\text{s}}{60\text{ s}}\right]^{-2} = 3.6$$

ILLUSTRATION 1.9

To determine the Young's modulus of a wire, the formula is $Y = \frac{F}{A}\cdot\frac{L}{\Delta l}$, where L = length, A = area of cross-section of the wire, and ΔL = change in the length of the wire when stretched with a force F. Find the conversion factor to change it from CGS to MKS system.

Sol. We know that the dimension of Young's modulus is $[ML^{-1}T^{-2}]$, i.e., $a = 1$, $b = -1$, $c = -2$.

CGS unit: g cm^{-1} s^{-2} and MKS unit: kg m^{-1} s^{-2}

By using the conversion formula:

MKS system	CGS system
$M_1 = 1$ kg	$M_2 = 1$ g
$L_1 = 1$ m	$L_2 = 1$ cm
$T_1 = 1$ s	$T_2 = 1$ s

$$n_2 = n_1\left[\frac{M_1}{M_2}\right]^1\left[\frac{L_1}{L_2}\right]^{-1}\left[\frac{T_1}{T_2}\right]^{-2} = \left[\frac{1\text{g}}{1\text{kg}}\right]^1\left[\frac{1\text{cm}}{1\text{m}}\right]^{-1}\left[\frac{1\text{s}}{1\text{s}}\right]^{-2}$$

$\therefore$ Conversion factor

$$\frac{n_2}{n_1} = \left[\frac{1\text{g}}{10^3\text{ g}}\right]^1\left[\frac{1\text{cm}}{10^2\text{ cm}}\right]^{-1}\left[\frac{1\text{s}}{1\text{s}}\right]^{-2} = \frac{1}{10} = 0.1$$

ILLUSTRATION 1.10

A calorie is a unit of heat or energy and it equals about 4.2 J, where 1 J = 1 kg m^2 s^{-2}. Suppose we employ a system of units in which the unit of mass equals α kg, the unit of length equals β m, the unit of time is γ s. Show that a calorie has a magnitude $4.2\ \alpha^{-1}\beta^{-2}\gamma^2$ in terms of the new units.

Sol.

SI system	NEW system
$M_1 = 1$ kg	$M_2 = \alpha$ kg
$L_1 = 1$ m	$L_2 = \beta$ m
$T_1 = 1$ s	$T_2 = \gamma$ s
$n_1 = 4.2$	$n_2 = ?$

dimensional formula of energy is $[ML^2T^{-2}]$

Comparing with $[M^aL^bT^c]$, we find that $a = 1$, $b = 2$, $c = -2$.

Now, $n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$

$$= 4.2\left[\frac{1\text{ kg}}{\alpha\text{ kg}}\right]^1\left[\frac{1\text{ m}}{\beta\text{ m}}\right]^2\left[\frac{1\text{ s}}{\gamma\text{ s}}\right]^{-2} = 4.2\alpha^{-1}\beta^{-2}\gamma^2$$

ILLUSTRATION 1.11

Convert 54 km h^{-1} into m s^{-1}.

Sol. Let $v = 54\text{ km h}^{-1} = n_2\text{ m s}^{-1}$.

$[v] = LT^{-1}$, $a = 0$, $b = 1$, $c = -1$

$$n_2 = 54\left[\frac{1\text{kg}}{1\text{g}}\right]^0\left[\frac{1\text{km}}{1\text{m}}\right]^1\left[\frac{1\text{h}}{1\text{s}}\right]^{-1}$$

$$= 54 \times 1 \times 1000 \times [3600]^{-1} = \frac{54\times1000}{3600} = 15$$

Hence, 54 km h^{-1} = 15 m s^{-1}.

To check the dimensional correctness of a given physical relation

This is based on the *principle of homogeneity*. According to this principle, the dimensions of each term on both sides of an equation must be the same.

If $X = A \pm (BC)^2 \pm \sqrt{DEF}$, then according to the principle of homogeneity, we have

$$[X] = [A] = [(BC)^2] = [\sqrt{DEF}],$$

If the dimensions of each term on both sides are same, the equation is dimensionally correct, otherwise not. A dimensionally correct equation may or may not be physically correct.

ILLUSTRATION 1.12

Let us check the dimensional correctness of the relation $v = u + at$.

Sol. Here u represents the initial velocity, v the final velocity, a the uniform acceleration, and t the time.
The dimensional formula of u is $[M^0LT^{-1}]$.
The dimensional formula of v is $[M^0LT^{-1}]$.
The dimensional formula of at is $[M^0LT^{-2}][T] = [M^0LT^{-1}]$.
Here the dimensions of every term in the given physical relation are the same, hence the given physical relation is dimensionally correct.

ILLUSTRATION 1.13

Check the accuracy of relation $v^2 - u^2 = 2as$, where v and u are final and initial velocities, a is the acceleration, and s is the distance.

Sol. We have $v^2 - u^2 = 2as$. Checking the dimensions on both sides, we get

$$\text{LHS} = [LT^{-1}]^2 - [LT^{-1}]^2 = [L^2T^{-2}] - [L^2T^{-2}] = [L^2\,T^{-2}]$$
$$\text{RHS} = [L^1\,T^{-2}]\,[L] = [L^2\,T^2]$$

Comparing LHS and RHS, we find LHS = RHS.
Hence, the formula is dimensionally correct.

ILLUSTRATION 1.14

Check whether the relation $S = ut + \frac{1}{2}at^2$ is dimensionally correct or not, where symbols have their usual meaning.

Sol. We have $S = ut + \frac{1}{2}at^2$

Checking the dimensions on both sides, LHS = $[M^0L^1T^0]$

$$\text{RHS} = [LT^{-1}]\,[T] + [LT^{-2}]\,[T^2]$$
$$= [M^0\,L^1\,T^0] + [M^0\,L^1\,T^0] = [M^0L^1T^0]$$

Comparing the LHS and RHS, we get

LHS = RHS

Hence the formula is dimensionally correct.

ILLUSTRATION 1.15

Find out the unit and dimensions of the constants a and b in the van der Waal's equation $\left(p + \frac{a}{V^2}\right)(V - b) = RT$, where p is pressure, v is volume, R is gas constant, and T is temperature.

Sol. We can add and subtract only like quantities.

$\Rightarrow$ Dimensions of P = Dimensions of $\frac{a}{V^2}$ (i)

and dimensions of v = Dimensions of b (ii)

From (i),

Dimensions of a = Dimensions of P × Dimensions of V^2

$$[a] = [M^1L^{-1}\,T^{-2}] \times [L^3]^2 = [M^1\,L^5\,T^{-2}]$$

Unit of a = Unit of p × Unit of $V^2 = \frac{N}{m^2} \times m^6 = N\,m^4$

From (ii), $[b] = [V] = [M^0L^3T^0]$

So unit of b = Unit of V = m^3

ILLUSTRATION 1.16

A famous relation in physics relates the moving mass m to the rest mass m_0 of a particle in terms of its speed v and the speed of light c. (This relation first arose as a consequence of the special theory of relativity due to Albert Einstein). A boy recalls the relation almost correctly but forgets where to put the constant c. He writes $m = \frac{m_0}{(1 - v^2)^{1/2}}$. Guess where to put the missing c.

Sol. According to the principle of homogeneity of dimensions, powers of M, L, T on either side of the formula must be equal. For this, on RHS, the denominator $(1 - v^2)^{1/2}$ should be dimensionless. Therefore, instead of $(1 - v^2)^{1/2}$, we should write $(1 - v^2/c^2)^{1/2}$. Hence, the correct formula would be

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}}$$

As a research tool to derive new relations

If one knows the dependency of a physical quantity on other quantities and if the dependency is of the product type, then using the method of dimensional analysis, a relation between the quantities can be derived. Let us understand this point through following examples.

Time period of a simple pendulum

Let the time period of a simple pendulum be a function of the mass of the bob (m), effective length (l), and acceleration due to gravity (g), then assuming the function to be the product of power function of m, l, and g, i.e., $T = Km^x l^y g^z$, where K = dimensionless constant.

If the above relation is dimensionally correct, then by substituting the dimensions of quantities,

$$[T] = [M]^x\,[L]^y\,[LT^{-2}]^z$$

or $$[M^0L^0T^1] = [M^xL^{y+z}T^{-2z}]$$

Equating the exponents of similar quantities, we get $x = 0$, $y = 1/2$, and $z = -1/2$.

So, the required physical relation becomes $T = K\sqrt{l/g}$.

The value of dimensionless constant is found (2π) through experiments, so $T = 2\pi\sqrt{l/g}$.

Stoke's law: When a small sphere moves at low speed through a fluid, the viscous force F, opposing the motion, is found experimentally to depend on the radius r, the velocity of the sphere v, and the viscosity η of the fluid.

So $F = f(\eta, r, v)$

If the function is the product of power functions of η, r, and v, $F = K\eta^x r^y v^z$, where K is the dimensionless constant.

If the above relation is dimensionally correct, then

$[MLT^{-2}] = [ML^{-1}T^{-1}]^x [L]^y [LT^{-1}]^z$

or $[MLT^{-2}] = [M^x L^{-x+y+z} T^{-x-z}]$

Equating the exponents of similar quantities $x = 1$; $-x + y + z = 1$; and $-x - z = -2$. Solving these for x, y, and z, we get $x = y = z = 1$.

So it becomes $F = K\eta rv$

On experimental grounds, $K = 6\pi$, so $F = 6\pi\eta rv$

This is the famous Stoke's law.

ILLUSTRATION 1.17

For a particle to move in a circular orbit uniformly, centripetal force is required which infact depends upon mass (m), velocity (v), and radius (r) of the circle. Express centripetal force in terms of these quantities.

Sol. According to the provided information, $F \propto m^a v^b r^c$

$\Rightarrow \quad F = km^a v^b r^c \quad$ ...(i)

where k is the dimensionless constant of proportionality and a, b, c are the constant powers of m, v, r, respectively.

Now using the principle of homogeneity, comparing the power of like quantities on both the sides, we have

$a = 1$ ...(ii) $\quad b + c = 1$ ...(iii) and $\quad b = 2$ ...(iv)

Using (ii), (iii), and (iv), we have $a = 1$, $b = 2$, and $c = -1$.

Using these values in (i), $F = k\, m^1 v^2 r^{-1}$

$\Rightarrow \quad F = K\dfrac{mv^2}{r}$ which is the desired relation

ILLUSTRATION 1.18

Experiments reveal that the velocity v of water waves may depend on their wavelength λ, density of water ρ, and acceleration due to gravity g. Establish a possible relation between v and λ, g, ρ.

Sol. According to the provided information,

$v \propto \lambda^a \rho^b g^c$

$\Rightarrow \quad v = k\,\lambda^a \rho^b g^c \quad$ ...(i)

where k is constant of proportionality.

Using principle of homogeneity

$$[M^0L^1T^{-1}] = [L]^a [M^1L^{-3}T^0]^b [M^0L^1T^{-2}]^c = [M^b L^{a-3b+c} T^{-2c}]$$

Comparing powers of like quantities on both the sides, we have,

$b = 0$ ...(ii) $\quad a - 3b + c = 1$ ...(iii) $\quad -2c = -1$ ...(iv)

Using (ii), (iii), and (iv), we have, $a = \dfrac{1}{2}$, $b = 0$, $c = 1/2$

Using these values in (i), we have $v = k \cdot \lambda^{1/2} \cdot \rho^0 \cdot g^{1/2}$

$\Rightarrow \quad v = k\sqrt{\lambda g}$ which is the required relation.

ILLUSTRATION 1.19

If the velocity of light (c), gravitational constant (G), and Planck's constant (h) are chosen as fundamental units, then find the dimensions of mass in new system.

Sol. Let $m \propto c^x G^y h^z$ or $m = Kc^x G^y h^z$

By substituting the dimension of each quantity in both the sides,

$$[M^1L^0T^0] = K[LT^{-1}]^x [M^{-1} L^3 T^{-2}]^y [ML^2T^{-1}]^z$$
$$= [M^{-y+z} L^{x+3y+2z} T^{-x-2y-z}]$$

By equating the power of M, L, and T in both the sides: $-y + z = 1$, $x + 3y + 2z = 0$, $-x - 2y - z = 0$.

By solving above three equations, $x = 1/2$, $y = -1/2$, and $z = 1/2$.

$\therefore \quad m \propto c^{1/2} G^{-1/2} h^{1/2}$

ILLUSTRATION 1.20

If velocity (V), force (F), and time (T) are chosen as fundamental quantities, express (a) mass and (b) energy in terms of V, F, and T.

Sol. Let $M = $ (Some number)$(V)^a(F)^b(T)^c$

Equating dimensions of both the sides, we get

$$M^1L^0T^0 = (1)\,[L^1T^{-1}]^a [M^1L^1T^{-2}]^b [T^1]^c$$
$$= M^b L^{a+b} T^{-a-2b+c}$$

Get $a = -1$, $b = 1$, $c = 1$.

$M = $ (Some number) $(V^{-1} F^1 T^1)$ $\qquad [M] = [V^{-1} F^1 T^1]$

Similarly, we can also express energy in terms of V, F, and T.

Let $[E] = $ [Some number] $[V]^a [F]^b [T]^c$

$\Rightarrow \quad [ML^2T^{-2}] = [M^0L^0T^0]\,[LT^{-1}]^a [MLT^{-2}]^b [T]^c$

$\Rightarrow \quad [M^1L^2T^{-2}] = [M^b L^{a+b+c} T^{-a-2b+c}]$

$\Rightarrow \quad 1 = b$; $2 = a + b + c$; $-2 = -a - 2b + c$

Get $a = 1$; $b = 1$; $c = 1$.

$\Rightarrow \quad E = $ (Some number) $V^1F^1T^1$ or $[E] = [V^1][F^1][T^1]$.

Important Points:

- The physical relation involving logarithm, exponential, trigonometric ratios, numerical factors, etc., cannot be derived by the method of dimensional analysis.
- Physical relations involving addition or subtraction sign cannot be derived by the method of dimensional analysis.
- The physical quantities such as angles, trigonometric ratios, logarithm, exponential, numerical factors, etc., are dimensionless.

  Consider a term sin(θ) Here θ is dimensionless and $\sin\theta\left(\dfrac{\text{Perpendicular}}{\text{Hypotenuse}}\right)$ is called dimensionless.

Whatever comes in sin (....) is dimensionless and the entire [sin (.....)] is also dimensionless.

sin (....)

dimensionless dimensionless

$e^{(......)}$

dimensionless dimensionless

$\log_e(....)$

dimensionless dimensionless

ILLUSTRATION 1.21

If $\alpha = \dfrac{F}{V^2}\sin(\beta t)$ (here V = velocity, F = force, t = time)

Find the dimension of α and β.

Sol. The physical quantities such as angles, trigonometric ratios are dimensionless. We are given the relation:

$$\alpha = \frac{F}{V^2}\sin(\beta t)$$

dimensionless dimensionless

As $\sin(\beta t)$ is dimensionless, hence, the dimensional formula for α should be same as dimensional formula for F/V^2.

So, $$[\alpha] = \frac{[F]}{[V^2]} = \frac{[M^1L^1T^{-2}]}{[L^1T^{-1}]^2} = M^1L^{-1}T^0$$

ILLUSTRATION 1.22

$\alpha = \dfrac{FV^2}{\beta^2}\log_e\left(\dfrac{2\pi\beta}{V^2}\right)$ where F = force and V = velocity.

Find the dimension of α and β.

Sol. The physical quantities such as logarithm, exponential, numerical factors, etc., are dimensionless. It means $\log_e\left(\dfrac{2\pi\beta}{V^2}\right)$ should be dimensionless. We are given the relation:

$$\alpha = \frac{FV^2}{\beta}\log_e\left(\frac{2\pi\beta}{V^2}\right)$$

dimensionless dimensionless

$$\Rightarrow \quad [\alpha] = \frac{[F][V^2]}{[\beta^2]} \qquad \text{...(i)}$$

Also $$\frac{[2\pi][\beta]}{[V^2]} = 1$$

Hence $$\frac{[1][\beta]}{L^2T^{-2}} = 1 \Rightarrow [\beta] = L^2T^{-2} \qquad \text{...(ii)}$$

From (i) and (ii) $$[\alpha] = \frac{[M^1L^1T^{-2}][L^2T^{-2}]}{[L^2T^{-2}]^2} \Rightarrow [\alpha] = M^1L^{-1}T^0$$

LIMITATIONS OF DIMENSIONAL ANALYSIS

Although dimensional analysis is very useful, it cannot lead us too far due to the following reasons:

1. If dimensions are given, physical quantity may not be unique as many physical quantities have same dimensions. For example, if the dimensional formula of a physical quantity is $[ML^2T^{-2}]$, it may be work or energy or torque.
2. Numerical constant having no dimensions $[K]$ such as (1/2), 1, 2π, etc., cannot be deduced by the methods of dimensions.
3. The method of dimensions cannot be used to derive relations other than the product of power functions. For example,

   $s = ut + (1/2)\,at^2$ or $y = a\sin\omega t$

   cannot be derived by using this theory (try if you can). However, the dimensional correctness of these can be checked.
4. The method of dimensions cannot be applied to derive formula if in mechanics a physical quantity depends on more than three physical quantities as then there will be less number (= 3) of equations than the unknowns (>3). However, still, we can check the correctness of the given equation dimensionally. For example, $T = 2\pi\sqrt{1/mgl}$ cannot be derived by theory of dimensions but its dimensional correctness can be checked.
5. Even if a physical quantity depends on three physical quantities, out of which two have same dimensions, the formula cannot be derived by the theory of dimensions, e.g., formula for the frequency of a tuning fork $f = (d/L^2)\,v$ cannot be derived by the theory of dimensions but can be checked.

CONCEPT APPLICATION EXERCISE 1.2

1. If $x = at + bt^2$, where x is the distance travelled by the body in kilometer while t is the time in seconds, then find the unit of b.
2. A force F is given by $F = at + bt^2$, where t is time. What are the dimensions of a and b?
3. The position of a particle at time t is given by the relation $x(t) = \left(\dfrac{v_0}{\alpha}\right)(1 - c^{-\alpha t})$, where v_0 is a constant and $\alpha > 0$. Find the dimensions of v_0 and α.
4. Find the dimensions of physical quantity X in the equation Force $= \dfrac{X}{\text{Density}}$.
5. The number of particles is given by $n = -D\dfrac{n_2 - n_1}{x_2 - x_1}$ crossing a unit area perpendicular to X-axis in unit time, where n_1 and n_2 are the number of particles per unit volume for the value of x meant to x_2 and x_1. Find the dimensions of D called diffusion constant.
6. The equation of a wave is given by $Y = A\sin\omega\left(\dfrac{x}{v} - k\right)$, where ω is the angular velocity and v is the linear velocity. Find the dimension of k.

7. The potential energy of a particle varies with distance x from a fixed origin as $U = \frac{A\sqrt{x}}{x^2 + B}$, where A and B are dimensional constants, then find the dimensional formula for AB.
8. You may not know integration, but using dimensional analysis you can check on some results. In the integral $\int \frac{dx}{(2ax - x^2)^{1/2}} = a^n \sin^{-1}\left(\frac{x}{a} - 1\right)$, find the value of n.
9. Convert 1 MW power on a new system having basic units of mass, length, and time as 10 kg, 1 dm, and 1 min, respectively.
10. Suppose we employ a system in which the unit of mass equals 100 kg, the unit of length equals 1 km and the unit of time 100 s and call the unit of energy *eluoj* (*joule* written in reverse order), then what is the relation between *eluoj* and *joule*?
11. If 1 g cm s^{-1} = x Ns, then what is the value of x?
12. With the usual notations, check if the following equation $S_t = u + \frac{1}{2}a(2t - 1)$ is dimensionally correct or not.
13. If the time period (T) of vibration of a liquid drop depends on surface tension (S), radius (r) of the drop, and density (ρ) of the liquid, then find the expression of T.
14. If P represents radiation pressure, C represents the speed of light, and Q represents radiation energy striking a unit area per second, then non-zero integers x, y, and z such that $P^x Q^y C^z$ is dimensionless, find the values of x, y, and z.
15. If velocity (V), force (F), and energy (E) are taken as fundamental units, then find the dimensional formula for mass.

ANSWERS

1. km s^{-2} **2.** $[a] = [MLT^{-3}]$, $[b] = [MLT^{-4}]$
3. $[\alpha] = [T^{-1}]$, $[v_0] = [LT^{-1}]$ **4.** $[M^2L^{-2}T^{-2}]$
5. $[L^2T^{-1}]$ **6.** $[T]$ **7.** $[ML^{11/2}T^{-2}]$ **8.** $n = 0$
9. 2.16×10^{12} unit **10.** $eluoj = 10^4\, joule$
11. 10^{-5}Ns **12.** Dimensionally correct
13. $K\sqrt{\frac{\rho r^3}{S}}$ **14.** $x = 1, y = -1$, and $z = 1$
15. $[V^{-2}F^0E]$

SIGNIFICANT FIGURES

Significant figures in the measured value of a physical quantity tell the number of digits in which we have confidence. Larger the number of significant figures obtained in a measurement, greater is the accuracy of the measurement. The reverse is also true.

The following rules are observed in counting the number of significant figures in a given measured quantity:

All non-zero digits are significant.

Example: 42.3 has three significant figures.
243.4 has four significant figures.
24.123 has five significant figures.

A zero becomes a significant figure if it appears between two non-zero digits.

Example: 5.03 has three significant figures.
5.604 has four significant figures.
4.004 has four significant figures.

Leading zeros or the zeros placed to the left of the number are never significant.

Example: 0.543 has three significant figures.
0.045 has two significant figures.
0.006 has one significant figure.

Trailing zeros or the zeros placed to the right of the number are significant.

Example: 4.330 has four significant figures.
433.00 has five significant figures.
343.000 has six significant figures.

In exponential notation, the numerical portion gives the number of significant figures.

Example: 1.32×10^{-2} has three significant figures.
1.32×10^{4} has three significant figures.

In order to avoid confusion in counting the number of significant figures, we usually express a measured quantity in significant figures, we usually express a measured quantity in scientific notation. By this, the number of significant figures are clearly mentioned and do not change on changing the units. For illustrations, see the table below.

If the original measured quantity is 1500 mm	If the original measured quantity is 1.5 m	If the original measured quantity is 150 cm
1500 mm = 1.500×10^3 mm = 1.500 m = 1.500×10^2 cm = 1.500×10^{-3} km	1.5 m = 1.5×10^3 mm = 1.5×10^2 cm = 1.5×10^{-3} km	150 cm = 1.50×10^3 mm = 1.50 m = 1.50×10^2 cm = 1.50×10^{-3} km

ROUNDING OFF

While rounding off measurements, we use the following rules by convention:

If the digit to be dropped is less than 5, then the preceding digit is left unchanged.

Example: x = 7.82 is rounded off to 7.8, again x = 3.94 is rounded off to 3.9.

If the digit to be dropped is more than 5, then the preceding digit is raised by 1.

Example: x = 6.87 is rounded off to 6.9, again x = 12.78 is rounded off to 12.8.

If the digit to be dropped is 5 followed by digits other than zero, then the preceding digit is raised by 1.

Example: x = 16.351 is rounded off to 16.4, again x = 6.758 is rounded off to 6.8.

If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is left unchanged, if it is even.

Example: $x = 3.250$ becomes 3.2 on rounding off, again $x = 12.650$ becomes 12.6 on rounding off.

If the digit to be dropped is 5 or 5 followed by zeros, then the preceding digit is raised by 1, if it is odd.

Example: $x = 3.750$ is rounded off to 3.8, again $x = 16.150$ is rounded off to 16.2.

SIGNIFICANT FIGURES IN CALCULATION

In most of the experiments, the observations of various measurements are to be combined mathematically, i.e., added, subtracted, multiplied, or divided as to achieve the final result. Since all the observations in measurements do not have the same precision, it is natural that the final result cannot be more precise than the least precise measurement. The following two rules should be followed to obtain the proper number of significant figures in any calculation.

1. The result of an addition or subtraction in the number having different precisions should be rounded off the same number of decimal places as are present in the number having the least number of decimal places. The rule is illustrated by the following examples:

a.
$$\begin{array}{rl} 33.3 & \leftarrow \text{(has only one decimal place)} \\ 3.11 & \\ +\ 0.313 & \\ \hline 36.723 & \leftarrow \text{(answer should be rounded off one decimal place)} \end{array}$$

Answer = 36.7

b.
$$\begin{array}{rl} 3.1421 & \\ 0.241 & \\ +\ 0.09 & \leftarrow \text{(has 2 decimal places)} \\ \hline 3.4731 & \leftarrow \text{(answer should be rounded off 2 decimal places)} \end{array}$$

Answer = 3.47

c.
$$\begin{array}{rl} 62.831 & \leftarrow \text{(has 3 decimal places)} \\ -24.5492 & \\ \hline 38.2818 & \leftarrow \text{(answer should be rounded off 3 decimal places)} \end{array}$$

Answer = 38.282

2. The answer to a multiplication or division is rounded off to the same number of significant figures as is possessed by the least precise term used in the calculation. The rule is illustrated by the following examples:

a.
$$\begin{array}{rl} 142.06 & \\ \times\ 0.23 & \leftarrow \text{(two significant figures)} \\ \hline 32.6738 & \leftarrow \text{(answer should have two significant figures)} \end{array}$$

Answer = 33

b.
$$\begin{array}{rl} 51.028 & \\ \times\ 1.31 & \leftarrow \text{(three significant figures)} \\ \hline 66.84668 & \end{array}$$

Answer = 66.8

c. $\dfrac{0.90}{4.26} = 0.2112676$

Answer = 0.21

ORDER OF MAGNITUDE

In scientific notation, the numbers are expressed as: Number = $M \times 10^x$, Where M is a number that lies between 1 and 10 and x is an integer. The order of magnitude of quantity is the power of 10 required to represent the quantity. For determining this power, the value of the quantity has to be rounded off. While rounding off, we ignore the last digit which is less than 5. If the last digit is 5 or more than five, the preceding digit is increased by 1. For example,

Speed of light in vacuum $= 3 \times 10^8\ \text{m s}^{-1} \approx 10^8\ \text{m s}^{-1}$ (ignoring $3 < 5$)

Mass of electron $= 9.1 \times 10^{-31}$ kg $\approx 10^{-30}$ kg (as $9.1 > 5$).

ILLUSTRATION 1.23

Each side of a cube is measured to be 7.203 m. Find the volume of the cube up to appropriate significant figures.

Sol. Volume $= a^3 = (7.203)^3 = 373.715\ \text{m}^3$

The side of the cube is measured upto 4 significant figures. Thus the final answer must be reported in 4 significant figures.

Hence, volume of the cube, $V = 373.7\ \text{m}^3$.

ILLUSTRATION 1.24

The mass of a box is 2.3 kg. Two marbles of masses 2.15 g and 12.39 g are added to it. Find the total mass of the box to the correct number of significant figures.

Sol. Total mass $= 2.3 + 0.00215 + 0.01239 = 2.31$ kg

The total mass in appropriate significant figures will be 2.3 kg.

ILLUSTRATION 1.25

The mass and volume of a body are 4.237 g and 2.5 cm³, respectively. Find the density of the material of the body in correct significant figures.

Sol. The answer to a multiplication or division is rounded off to the same number of significant figures as possessed by the least precise term used in the calculation. The final result should retain as many significant figures as are there in the original number with the least significant figures. In given question, density should be reported to two significant figures.

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{4.237\ \text{g}}{2.5\ \text{cm}^3} = 1.6948\ \text{g/cm}^3$$

After rounding off the number, we get density = 1.7

ILLUSTRATION 1.26

The length, breadth and thickness of a metal sheet are 4.234 m, 1.005 m and 2.01 cm respectively. Calculate volume of the sheet to correct number of significant figures.

Sol. We are given,

- Length (l) = 4.234 m (4 significant figures)
- Breadth (b) = 1.005 m (4 significant figures)
- Thickness (t) = 2.01 cm = 2.01×10^{-2} m (3 significant figures)

Therefore, volume of the sheet = $l \times b \times t$

$= 4.234 \times 1.005 \times 0.0201\ m^2$

$= 0.0855289\ m^9$

In multiplication or division, the number of significant figures in the product or quotient is same as the smallest number of significant figures in any of the factors

Hence, the volume can contain 3 significant figures, therefore, rounding off, we get : Volume $= 0.0855\ m^3$

CONCEPT APPLICATION EXERCISE 1.3

1. The length, breadth, and thickness of a block are measured as 125.5 cm, 5.0 cm, and 0.32 cm, respectively. Which one of the measurement is most accurate?
2. The length of a rectangular sheet is 1.5 cm and the breadth is 1.203 cm. Find the area of the face of a rectangular sheet to the correct number of significant figures
3. Each side of a cube is measured to be 5.402 cm. Find the total surface area and the volume of the cube in appropriate significant figures.
4. Taking into account the significant figures, what is the value of 9.99 m + 0.0099 m?
5. Find the value of the multiplication 3.124×4.576 correct to three significant figures.
6. If the value of resistance is 10.845 Ω and the value of current is 3.23 A, the potential difference is 35.02935 V. Find its value in significant number.
7. With due regard to significant figures, add the following:
   (a) 953 and 0.324 (b) 953 and 0.625
   (c) 953.0 and 0.324 (d) 953.0 and 0.374
8. With due regard to significant figures, subtract
   (a) 0.35 from 7 (b) 0.65 from 7
   (c) 0.35 from 7.0 (d) 0.65 from 7.0
9. A diamond weighs 3.71 g. It is put into a box weighing 1.4 kg. Find the total weight of the box and diamond to the correct number of significant figures.
10. Calculate the area enclosed by a circle of diameter 1.12 m to the correct number of significant figures.
11. (a) Add 3.8×10^{-6} to 4.2×10^{-5} with due regard to significant figures.
    (b) Subtract 3.2×10^{-6} from 4.7×10^{-4} with due regard to significant figures.
    (c) Subtract 1.5×10^{3} from 4.8×10^{4} with due regard to significant figures.
12. The length, breadth, and thickness of a metal sheet are 4.234 m, 1.005 m, and 2.01 cm, respectively. Give the area and volume of the sheet to the correct number of significant figures.

ANSWERS

1. Thickness 2. $1.8\ cm^2$
3. Total surface area = $175.1\ cm^2$, volume = $157.6\ cm^3$
4. 10.00 m 5. 14.3 6. 35.0 V
7. (a) 953 (b) 954 (c) 953.3 (d) 953.4
8. (a) 7 (b) 6 (c) 6.6 (d) 6.4
9. 1.4 kg 10. $0.99\ m^2$
11. (a) 4.6×10^{-5} (b) 4.7×10^{-4} (c) 4.6×10^{4}
12. Area = $8.72\ m^2$, Volume = $0.0855\ m^3$

ERRORS OF MEASUREMENT

The measuring process is essentially a process of comparison. Inspite of our best efforts, the measured value of a quantity is always somewhat different from its actual value, or true value. This difference in the true value of a quantity is called error of measurement.

Absolute error: Absolute error in the measurement of a physical quantity is the magnitude of the difference between the true value and the measured value of the quantity.

Let a physical quantity be measured n times. Let the measured value be $a_1, a_2, a_3, \ldots, a_n$. The arithmetic mean of these values is

$$a_m = \frac{a_1 + a_2 + \ldots. + a_n}{n},$$

Usually, a_m is taken as the true value of the quantity, if the same is unknown otherwise.

By definition, absolute errors in the measured values of the quantity are

$$\Delta a_1 = a_m - a_1$$
$$\Delta a_2 = a_m - a_2$$
$$\ldots$$
$$\Delta a_n = a_m - a_n$$

The absolute errors may be positive in certain cases and negative in certain other cases.

Mean absolute error: It is the arithmetic mean of the magnitudes of absolute errors in all the measurements of the quantity. It is represented by $\overline{\Delta a}$. Thus,

$$\overline{\Delta a} = \frac{|\Delta a_1| + |\Delta a_2| + \cdots |\Delta a_n|}{n}$$

Hence, the final result of measurement may be written as

$$a = a_m \pm \overline{\Delta a}$$

This implies that any measurement of the quantity is likely to lie between $(a_m + \overline{\Delta a})$ and $(a_m - \overline{\Delta a})$.

Relative error or fractional error: The relative error or fractional error of measurement is defined as the ratio of mean absolute error to the mean value of the quantity measured. Thus, relative error or fractional error

$$= \frac{\text{Mean absolute error}}{\text{Mean value}} = \frac{\overline{\Delta a}}{a_m}$$

Percentage error: When the relative/fractional error is expressed in percentage, we call it percentage error.

Thus, percentage error $= \dfrac{\overline{\Delta a}}{a_m} \times 100\%$

PROPAGATION OF ERRORS

Error in sum of the quantities: Suppose $x = a + b$

Let Δa = absolute error in measurement of a

Δb = absolute error in measurement of b

Δx = absolute error in calculation of x, i.e., sum of a and b.

The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$.

Percentage error in the value of $x = \dfrac{(\Delta a + \Delta b)}{a + b} \times 100\%$

Error in difference of the quantities: Suppose $x = a - b$.

Let Δa = absolute error in measurement of a
Δb = absolute error in measurement of b
Δx = absolute error in calculation of x, i.e., difference of a and b.

The maximum absolute error in x is $\Delta x = \pm(\Delta a + \Delta b)$

Percentage error in the value of $x = \dfrac{(\Delta a + \Delta b)}{a - b} \times 100\%$

Error in product of quantities: Suppose $x = a \times b$.

Let Δa = absolute error in measurement of a
Δb = absolute error in measurement of b
Δx = absolute error in calculation of x, i.e., product of a and b.

The maximum fractional error in x is $\dfrac{\Delta x}{x} = \pm\left(\dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}\right)$

Percentage error in the value of x = (Percentage error in value of a) + (Percentage error in value of b)

Error in division of quantities: Suppose $x = \dfrac{a}{b}$.

Let Δa = absolute error in measurement of a
Δb = absolute error in measurement of b
Δx = absolute error in calculation of x, i.e., division of a and b.

The maximum fractional error in x is $\dfrac{\Delta x}{x} = \pm\left(\dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}\right)$

Percentage error in the value of x = (Percentage error in value of a) + (Percentage error in value of b)

Error in quantity raised to some power: Suppose $x = \dfrac{a^n}{b^m}$.

Let Δa = absolute error in measurement of a
Δb = absolute error in measurement of b
Δx = absolute error in calculation of x

The maximum fractional error in x is $\dfrac{\Delta x}{x} = \pm\left(n\dfrac{\Delta a}{a} + m\dfrac{\Delta b}{b}\right)$

Percentage error in the value of $x = n$(Percentage error in value of a) + m(Percentage error in value of b)

The quantity which have maximum power must be measured carefully because it's contribution to error is maximum.

ILLUSTRATION 1.27

Repeated observations in an experiment gave the values 1.29, 1.33, 1.34, 1.35, 1.32, 1.36, 1.30, and 1.33. Calculate the mean value, absolute error, relative error, and percentage error.

Sol. Here, mean value of quantity measured,

$$v = \frac{1.29 + 1.33 + 1.34 + 1.35 + 1.32 + 1.36 + 1.30 + 1.33}{8}$$

x = 1.3275 = 1.33 (rounded off to two places of decimal).

Absolute errors in measurement are:

$\Delta x_1 = 1.33 - 1.29 = 0.04$; $\Delta x_2 = 1.33 - 1.33 = 0.00$
$\Delta x_3 = 1.33 - 1.34 = -0.01$; $\Delta x_4 = 1.33 - 1.35 = -0.02$
$\Delta x_5 = 1.33 - 1.32 = +0.01$; $\Delta x_6 = 1.33 - 1.36 = -0.03$
$\Delta x_7 = 1.33 - 1.30 = +0.03$; $\Delta x_8 = 1.33 - 1.33 = 0.00$

Mean absolute error, $\overline{\Delta x} = \dfrac{\sum_{i=1}^{i=n} |(\Delta x)_i|}{n}$

$$= \frac{0.04 + 0.00 + 0.01 + 0.02 + 0.01 + 0.03 + 0.03 + 0.00}{8}$$

$$= \frac{0.14}{8} = 0.0175 = 0.02$$

Relative error $= \pm\dfrac{\overline{\Delta x}}{x} = \pm\dfrac{0.02}{1.33} = \pm 0.015 = \pm 0.02$

Percentage error $= \pm 0.015 \times 100 = 1.5\%$

ILLUSTRATION 1.28

A physical parameter a can be determined by measuring the parameters b, c, d, and e using the relation $a = b^\alpha c^\beta / d^\gamma e^\delta$. If the maximum errors in the measurement of b, c, d, and e are $b_1\%$, $c_1\%$, $d_1\%$, and $e_1\%$, then find the maximum error in the value of a determined by the experiment.

Sol. $a = b^\alpha c^\beta / d^\gamma e^\delta$

So maximum error in a is given by

$$\left(\frac{\Delta a}{a} \times 100\right)_{max}$$

$$= \alpha \cdot \frac{\Delta b}{b} \times 100 + \beta \cdot \frac{\Delta c}{c} \times 100 + \gamma \cdot \frac{\Delta d}{d} \times 100 + \delta . \frac{\Delta e}{e} \times 100$$

$$= (\alpha b_1 + \beta c_1 + \gamma d_1 + \delta e_1)\%$$

ILLUSTRATION 1.29

The relative density of material of a body is found by weighing it first in air and then in water. If the weight in air is (5.00 ± 0.05) N and the weight in water is (4.00 ± 0.05) N. Find the relative density along with the maximum permissible percentage error.

Sol. Weight in air = (5.00 ± 0.05) N

Weight in water = (4.00 ± 0.05) N

Loss of weight in water = (1.00 ± 0.1) N

Now relative density $= \dfrac{\text{Weight in air}}{\text{Weight loss in water}}$

i.e., $\text{RD} = \dfrac{5.00 \pm 0.05}{1.00 \pm 0.1}$

Now relative density with maximum permissible error

$$= \frac{5.00}{1.00} \pm \left(\frac{0.05}{5.00} + \frac{0.1}{1.00}\right) \times 100$$

$$= 5.0 \pm (1 + 10)\% = 5.0 \pm 11\%$$

ILLUSTRATION 1.30

The period of oscillation of a simple pendulum in the experiment is recorded as 2.63 s, 2.56 s, 2.42 s, 2.71 s, and 2.80 s. Find the average absolute error.

Sol. Average value $= \dfrac{2.63 + 2.56 + 2.42 + 2.71 + 2.80}{5} = 2.62$ s

Now $|\Delta T_1| = 2.63 - 2.62 = 0.01$
$|\Delta T_2| = 2.62 - 2.56 = 0.06$
$|\Delta T_3| = 2.62 - 2.42 = 0.20$

$$|\Delta T_4| = 2.71 - 2.62 = 0.09$$
$$|\Delta T_5| = 2.80 - 2.62 = 0.18$$

Mean absolute error

$$\Delta T = \frac{|\Delta T_1| + |\Delta T_2| + |\Delta T_3| + |\Delta T_4| + |\Delta T_5|}{5}$$
$$= \frac{0.54}{5} = 0.108 = 0.11 \text{ s}$$

ILLUSTRATION 1.31

If there is a positive error of 50% in the measurement of velocity of a body, find the error in the measurement of kinetic energy.

Sol. Kinetic energy, $E = \frac{1}{2}mv^2$

$$\therefore \quad \frac{\Delta E}{E} \times 100 = \left(\frac{\Delta m}{m} + \frac{2\Delta v}{v}\right) \times 100$$

Here $\Delta m = 0$ and $\frac{\Delta v}{v} \times 100 = 50\%$

$$\therefore \quad \frac{\Delta E}{E} \times 100 = 2 \times 50 = 100\%$$

ILLUSTRATION 1.32

The initial and final temperatures of water as recorded by an observer are (40.6 ± 0.2)°C and (78.3 ± 0.3)°C. Calculate the rise in temperature with proper error limits.

Sol. $\theta_1 = (40.6 \pm 0.2)°C, \quad \theta_2 = (78.3 \pm 0.3)°C.$

Rise in temperature, $\theta = \theta_2 - \theta_1 = 78.3 - 40.6 = 37.7°C$

$$\Delta\theta = \pm(\Delta\theta + \Delta\theta_2) = \pm(0.2 + 0.3) = 0.5°C$$

Hence, rise in temperature = (37.7 ± 0.5)°C

ILLUSTRATION 1.33

The length and breadth of a rectangle are (5.7 ± 0.1) cm and (3.4 ± 0.2) cm, respectively Calculate the area of rectangle with error limits.

Sol. Here, $l = (5.7 \pm 0.1)$ cm, $b = (3.4 \pm 0.2)$ cm.

Area, $A = l \times b = 5.7 \times 3.4 = 19.38 \text{ cm}^2 = 19.0 \text{ cm}^2$ (rounding off to two significant figures)

$$\therefore \quad \frac{\Delta A}{A} = \pm\left(\frac{\Delta l}{l} + \frac{\Delta b}{b}\right)$$
$$= \pm\left(\frac{0.1}{5.7} + \frac{0.2}{3.4}\right) = \pm\left(\frac{0.34 + 1.14}{5.7 \times 3.4}\right) = \pm\frac{1.48}{19.38}$$
$$\Rightarrow \quad \Delta A = \pm\frac{1.48}{19.38} \times A = \pm\frac{1.48}{19.38} \times 19.38 = \pm 1.48$$

$\Delta A = \pm 1.5$ (rounding off to two significant figures);

Area = (19.0 ± 1.5) cm^2.

ILLUSTRATION 1.34

A physical quantity x is calculated from the relation $x = \frac{a^2b^3}{c\sqrt{d}}$.

If the percentage error in a, b, c, and d are 2%, 1%, 3%, and 4%, respectively, what is the percentage error in x?

Sol. As $x = \frac{a^2 b^3}{c\sqrt{d}}$,

$$\therefore \quad \frac{\Delta x}{x} = \pm\left[2\frac{\Delta a}{a} + 3\frac{\Delta b}{b} + \frac{\Delta c}{c} + \frac{1}{2}\frac{\Delta d}{d}\right]\frac{\Delta x}{x} \times 100$$
$$= \pm\left[2 \times 2\% + 3 \times 1\% + 3\% + \frac{1}{2} \times 4\%\right]$$
$$= \pm[4\% + 3\% + 3\% + 2\%] = \pm 12\%.$$

ILLUSTRATION 1.35

The length and breadth of a field are measured as: $l = (120 \pm 2)$ m and $b = (100 \pm 5)$ m, respectively. What is the area of the field?

Sol. Now $\frac{\Delta A}{A} = \frac{\Delta l}{l} + \frac{\Delta b}{b} = \left(\frac{2}{120} + \frac{5}{100}\right) = 0.0667;$

$$\Delta A = 0.0667 \times A$$

Now $A = l \cdot b = 120 \times 100 = 12000 \text{ m}^2$

$\Rightarrow \quad \Delta A = 0.0667 \times 12000 = 800.4 \text{ m}^2$

Area of the field $= A \pm \Delta A$

$$= 12000 \pm 800.4 = (1.2 \pm 0.08) \times 10^4 \text{ m}^2$$

ILLUSTRATION 1.36

In an experiment of simple pendulum, the time period measured was 50 s for 25 vibrations when the length of the simple pendulum was taken 100 cm. If the least count of stop watch is 0.1 s and that of meter scale is 0.1 cm. Calculate the maximum possible error in the measurement of value of g. If the actual value of g at the place of experiment is 9.7720 m s^{-2}, calculate the percentage error.

Sol. The time period of a simple pendulum is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{or} \quad T^2 = \frac{4\pi^2 l}{g} \quad \text{or} \quad g = \frac{4\pi^2 l}{T^2}$$

As 4 and π are constants, the maximum permissible error in g is given by $\frac{\Delta g}{g} = \frac{\Delta l}{l} + \frac{2\Delta T}{T}$

Here $\Delta L = 0.1$ cm, $L = 1$ m $= 100$ cm, $\Delta T = 0.1$ s, $T = 50$ s.

$$\therefore \quad \frac{\Delta g}{g} = \frac{0.1}{100} + 2\left(\frac{0.1}{50}\right) = \frac{0.1}{100} + \left(\frac{0.1}{25}\right);$$
$$\frac{\Delta g}{g} \times 100 = \left[\frac{0.1}{100} + \frac{0.1}{25}\right] \times 100 = 0.1 + 0.4 = 0.5\%$$

Now $g = \frac{4\pi^2 l^2}{T^2}$. Here $T = \frac{50}{25} = 2$. Therefore,

$$g' = \frac{4 \times (3.14)^2 \times (1)^2}{(2)^2} = 9.8596 \text{ m s}^{-2};$$

Actual value $g = 9.7720$ m s^{-1}.

$$\therefore \quad \text{Percentage error} = \frac{g' - g}{g} \times 100$$
$$= \frac{9.8596 - 9.7720}{9.7720} \times 100 = 0.8964\%$$

ILLUSTRATION 1.37

The distance covered by a body in time (5.0 ± 0.6) s is (40.0 ± 0.4) m. Calculate the speed of the body. Also determine the percentage error in the speed.

Sol. Here $s = 40.0 \pm 0.4$ m and $t = 5.0 \pm 0.6$ s. Therefore, speed,

$$v = \frac{s}{t} = \frac{40.0}{5.0} = 8.0 \text{ m s}^{-1}$$

As $v = \dfrac{s}{t}$, therefore, $\dfrac{\Delta v}{v} = \dfrac{\Delta s}{s} + \dfrac{\Delta t}{t}$

Here $\Delta s = 0.4$ m, $s = 40.0$ m, $\Delta t = 0.6$ s, $t = 5.0$ s.

$$\therefore \quad \frac{\Delta v}{v} = \frac{0.4}{40.0} + \frac{0.6}{5.0} \quad \Rightarrow \Delta v = [0.01 + 0.12] \times 8.0 = 1.04$$

Hence, $v = (8.0 \pm 1.04)$ m s^{-1}; percentage error

$$\left(\frac{\Delta v}{v} \times 100\right) = 0.13 \times 100 = 13\%$$

ILLUSTRATION 1.38

In resonance tube experiment, the velocity of sound is given by $v = 2f_0(l_2 - l_1)$. We found $l_1 = 25.0$ cm and $l_2 = 75.0$ cm. If there is no error in frequency, what will be the maximum permissible error in the speed of sound? (Take $f_0 = 325$ Hz)

Sol. $V = 2f_0\,(l_2 - l_1)$

$(dV) = 2f_0\,(dl_2 - dl_1)$

$(dV)_{\text{max}} = \text{max of } [2f_0(\pm \Delta l_2 \pm \Delta l_2] = 2f_0\,(\Delta l_2 + \Delta l_1)$

$l_1 = 25.0 \text{ cm} \Rightarrow l_1 = 0.1$ cm (place value of last number)

$l_1 = 75.0 \text{ cm} \Rightarrow l_2 = 0.1$ cm (place value of last number)

So the maximum permissible error in the speed of sound $(dV)_{\text{max}}$

$= 2(325\text{Hz})\,(0.1 \text{ cm} + 0.1 \text{ cm}) = 1.3 \text{ m s}^{-1}$

Value of $V = 2f_0\,(l_2 - l_1) = 2(325\text{Hz})\,(75.0 \text{ cm} - 25.0 \text{ cm}) = 325 \text{ m s}^{-1}$

So $\quad V = (325 \pm 1.3)$ m s^{-1}.

ILLUSTRATION 1.39

If the measured value of resistance $R = 1.05\ \Omega$, wire diameter $d = 0.60$ mm, and length $l = 75.3$ cm, then find the maximum permissible error in resistivity, $\rho = \dfrac{R(\pi d^2/4)}{l}$.

Sol.
$$\left(\frac{d\rho}{\rho}\right)_{\text{max}} = \frac{\Delta R}{R} + 2\frac{\Delta d}{d} + \frac{\Delta l}{l}$$

$R = 1.05\ \Omega \rightarrow \Delta R = 0.01\ \Omega$ (least count)

$\Delta d = 0.60 \text{ mm} \rightarrow \Delta d = 0.01$ mm (least count)

$l = 75.3 \rightarrow \Delta l = 0.1$ cm (least count)

$$\left(\frac{d\rho}{\rho}\right)_{\text{max}} = \frac{0.01\ \Omega}{1.05\ \Omega} + 2\left(\frac{0.01 \text{ mm}}{0.60 \text{ mm}}\right) + \frac{0.1 \text{ cm}}{75.3 \text{ cm}}$$

ILLUSTRATION 1.40

In Ohm's law experiment, the potential drop across a resistance was measured as $V = 5.0$ V and the current was measured as $i = 2.00$ A. Find the maximum permissible error in resistance.

Sol. $R = \dfrac{V}{i} = V \times i^{-1}$

$$\left(\frac{dR}{R}\right)_{\text{max}} = \frac{\Delta V}{V} + \frac{\Delta i}{i}$$

$V = 5.0 \text{ V} \quad \rightarrow \quad \Delta V = 0.1$ V

$i = 2.00 \text{ A} \quad \rightarrow \quad \Delta i = 0.01$ A

$$\%\left(\frac{dR}{R}\right)_{\text{max}} = \left(\frac{0.1}{5.0} + \frac{0.01}{2.00}\right) \times 100\% = 2.5\% = \text{Value of } R$$

From the observation, $R = \dfrac{V}{i} = \dfrac{5.0}{2.00} = 2.5\ \Omega$.

So, we can write $R = (2.5 \pm 2.5\%)\ \Omega$

ILLUSTRATION 1.41

In Searle's experiment to find Young's modulus, the diameter of wire is measured as $D = 0.05$ cm, the length of wire is $L = 125$ cm, and when a weight, $m = 20$ kg is the put, extension in wire was found to be 0.100 cm. Find the maximum permissible error in Young's modulus (Y).

Sol.
$$\frac{mg}{\pi d^2/4} = Y\left(\frac{x}{l}\right) \quad \Rightarrow \quad Y = \frac{mgl}{(\pi/4)d^2 x}$$

$$\left(\frac{dY}{Y}\right)_{\text{max}} = \frac{\Delta m}{m} + \frac{\Delta l}{l} + 2\frac{\Delta d}{d} + \frac{\Delta x}{x}$$

$m = 20.0 \text{ kg} \quad \Rightarrow \quad \Delta m = 0.1$ kg

$l = 125 \text{ m} \quad \Rightarrow \quad \Delta l = 1$ cm

$d = 0.050 \text{ cm} \quad \Rightarrow \quad \Delta d = 0.001$ cm

$x = 0.100 \text{ cm} \quad \Rightarrow \quad \Delta x = 0.001$ cm

$$\left(\frac{dY}{Y}\right)_{\text{max}} = \left(\frac{0.1 \text{ kg}}{20.0 \text{ kg}} + \frac{1 \text{ cm}}{125 \text{ cm}} + \frac{0.001 \text{ cm}}{0.05 \text{ cm}} + \frac{0.001 \text{ cm}}{0.100 \text{ cm}}\right) \times 100\%$$

$= 4.3\%$

ILLUSTRATION 1.42

To find the value of g using simple pendulum, $T = 2.00$ s and $l = 1.00$ m were measured. Estimate maximum permissible error in g. Also find the value of g.

Sol. $T = 2\pi\sqrt{\dfrac{l}{g}} \quad \Rightarrow \quad g = \dfrac{4\pi^2 l}{T^2}$

$$\left(\frac{dg}{g}\right)_{\text{max}} = \frac{\Delta l}{l} + 2\frac{\Delta T}{T} = \left(\frac{0.01}{1.00} + 2\frac{0.01}{2.00}\right) \times 100\% = 2\%$$

$$g = \frac{4\pi^2 l}{T^2} = \frac{4 \times 10 \times 1.00}{(2.00)^2} = 10.0 \text{ m s}^{-2}$$

$$\left(\frac{dg}{g}\right)_{\text{max}} = \frac{2}{100} \quad \text{or} \quad \frac{dg_{\text{max}}}{10.0} = \frac{2}{100}$$

$(dg)_{\text{max}} = 0.2 = \text{max error in } g$

So $\quad g = (10.0 \pm 0.2)$ m s^{-2}

Concept Application Exercise 1.1

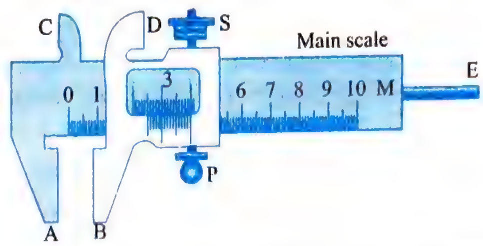
1. Which of the following length measurements is most precise and why?
(a) 2.0 cm (b) 2.00 cm (c) 2.000 cm
2. In a number without decimal, what is the significance of zeros on the right of non-zero digits?
3. A research worker takes 100 observations in an experiment. If he repeats the same experiment by taking 500 observations, how is the probable error affected?
4. Which quantity in a given formula should be measured most accurately? Why?
5. A body travels uniformly a distance of (13.8 ± 0.2) m in a time (4.0 ± 0.3) s. Find the velocity of the body within error limits and the percentage error.
6. The error in the measurement of the radius of a sphere is 1%. Find the error in the measurement of volume.
7. Given $R_1 = 5.0 \pm 0.2\ \Omega$, and $R_2 = 10.0 \pm 0.1\ \Omega$. What is the total resistance in parallel with possible % error?
8. The value of resistance is 10.845 Ω and the current is 3.23 A. On multiplying them, we get the potential difference = 35.02935 V. What is the value of potential difference in terms of significant figures?
9. The length of one rod is 2.53 cm and that of the other is 1.27 cm. The least count of the measuring instrument is 0.01 cm. If the two rods are put together end to end, find the combined length.
10. The pressure on a square plate is measured by measuring the force on the plate and the length of the sides of the plate by using the formula $P = F/l^2$. If the maximum errors in the measurement of force and length are 4% and 2%, respectively, then what is the maximum error in the measurement of pressure?
11. The density of a cube is measured by measuring its mass and the length of its sides. If the maximum errors in the measurement of mass and length are 3% and 2%, respectively, then find the maximum error in the measurement of the density of cube.
12. The resistance $R = V/i$, where $V = 100 \pm 5$ V and $i = 10 \pm 0.2$ A. What is the total error in R?
13. The length of a cylinder is measured with a meter rod having least count 0.1 cm. Its diameter is measured with Vernier calipers having least count 0.01 cm. Given that length is 5.0 cm. and radius is 2 cm. Find the percentage error in the calculated value of the volume.
14. According to Joule's law of heating, heat produced $H = I^2 Rt$, where I is current, R is resistance, and t is time. If the errors in the measurement of I, R, and t are 3%, 4%, and 6%, respectively, find error in the measurement of H.
15. A physical quantity P is given by $P = \dfrac{A^3 B^{1/2}}{C^{-4} D^{3/2}}$. Which quantity among A, B, C, and D brings in the maximum percentage error in P?

Answers

1. (c) **2.** Not significant
3. Probable error reduces to 1/5
4. Quantity having higher powers
5. $3.45\ \text{ms}^{-1}$, 8.95% **6.** 3%
7. $3.3\ \Omega \pm 7\%$ **8.** 35.0 V
9. (3.80 ± 0.02) cm **10.** 8%
11. 9% **12.** 7%
13. 3% **14.** 16% **15.** Quantity C

VERNIER CALLIPERS

It consists of a main scale graduated in cm/mm over which an auxiliary scale (or Vernier scale) can slide along the length (figure below). The division of the Vernier scale being either slightly longer and shorter than the divisions of the main scale.



In the common form, the divisions on the Vernier scale V are smaller in size than the smallest division on the main scale M, but in some special cases, the size of the Vernier division may be larger than the main scale division.

Let n Vernier scale divisions (VSD) coincide with $(n - 1)$ main scale divisions (MSD) then

$$n\ \text{VSD} = (n-1)\ \text{MSD}$$

or
$$1\ \text{VSD} = \left(\frac{n-1}{n}\right) \text{MSD}$$

$$1\ \text{MSD} - 1\ \text{VSD} = 1\ \text{MSD} - \left(\frac{n-1}{n}\right)\text{MSD} = \frac{1}{n}\text{MSD}$$

The difference between the values of one main scale division and one Vernier scale division is known as Vernier constant (VC) or the Least count (LC). This is the smallest distance that can be accurately measured with the Vernier scale.

Thus,
$$\text{VC} = \text{LC} = 1\ \text{MSD} - 1\ \text{VSD} = \left(\frac{1}{n}\right)\text{MSD}$$

$$= \frac{\text{Smallest division on main scale}}{\text{Number of divisions on Vernier scale}}$$

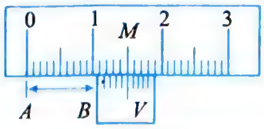
In the ordinary Vernier callipers, one MSD is 1 mm and 10 VSDs coincide with nine MSD.

$$1\ \text{VSD} = \frac{9}{10}\text{MSD} = 0.9\ \text{mm}$$

$$\text{VC} = 1\ \text{MSD} - 1\ \text{VSD} = 1\ \text{mm} - 0.9\ \text{mm} = 0.1\ \text{mm} = 0.01\text{cm}$$

Reading a Vernier Callipers

If we have to measure a length AB, the end A coincides with the zero of main scale. Suppose the end B lies between 1.0 cm and 1.1 cm on the main scale. Then, $1.0\ \text{cm} < AB < 1.1$ cm



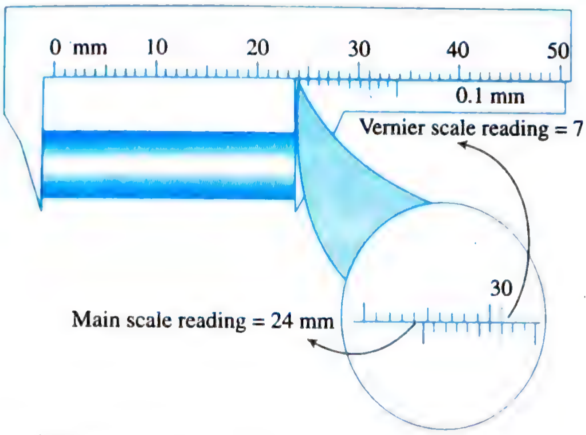
Let the fifth division of Vernier scale coincides with 1.5 cm of main scale. Then, $AB = 1.0 + 5 \times \text{VC} = (1.0 + 5 \times 0.01)$ cm = 1.05 cm

Thus, we can make the following formula:

Total reading $= N + n \times \text{VC}$

Here, N = main scale reading on the left of the zero of the Vernier scale.

n = Number of Vernier division which just coincides with any of the main scale division.



Thickness of the object = 24. - - - - - - -

= 24 mm + 7 (0.1 mm)

According to the formula,
Thickness of the object = Main scale reading + Vernier scale reading Least count

Mark on Vernier scale which exactly coincides with some mark of main scale.

1 Main scale devision − 1 Vernier scale division

= 1 mm − 0.9 mm
= 0.1 mm

ILLUSTRATION 1.43

1 cm on the main scale of vernier callipers is divided into 10 equal parts. If 20 divisions of vernier scale coincide with 10 small divisions of the main scale, what will be the least count of callipers?

Sol. 20 divisions of Vernier scale = 10 divisions of main scale

$$1\ \text{V.S.D.} = \left(\frac{10}{20}\right)\text{M.S.D.} = \left(\frac{1}{2}\right)\text{M.S.D.}$$

$$\text{L.C.} = 1\ \text{M.S.D.} - 1\ \text{V.S.D.}$$

$$= 1\text{M.S.D.} - \left(\frac{1}{2}\right)\text{M.S.D.}$$

$$= \left(1 - \frac{1}{2}\right)\text{M.S.D.}$$

$$= \frac{1}{2}\text{M.S.D.} = \frac{1}{2} \times 0.1\,\text{cm} = 0.05\,\text{cm}\ (\because 1\ \text{M.S.D.} = \frac{1}{10}\,\text{cm} = 0.1\,\text{cm})$$

ILLUSTRATION 1.44

The Vernier scale of a travelling microscope has 50 divisions which coincide with 49 main scale divisions. If each main scale division is 0.5 mm, calculate the minimum inaccuracy in the measurement of distance.

Sol. According to the problem, 50 divisions of Vernier scale coincide with 49 main scale divisions.

$$50\ \text{VSD} = 49\ \text{MSD} \Rightarrow 1\text{MSD} = \frac{50}{49}\text{VSD} \text{ or } 1\,\text{VSD} = \frac{49}{50}\text{MSD}$$

where MSD = Main scale division and VSD = Vernier scale division. We know that

Minimum inaccuracy = Vernier constant

$$= 1\text{MSD} - 1\text{VSD}$$

$$= 1\text{MSD} - \frac{49}{50}\text{ MSD} = \frac{1}{50}\text{ MSD}$$

It is given in the problem that = 1 MSD = 0.5 mm

Hence, minimum inaccuracy $= \frac{1}{50} \times 0.5\,\text{mm} = \frac{1}{100} = 0.01$ mm

Zero Error and Zero Correction

If the zero of the vernier scale does not coincide with the zero of main scale when jaw B touches A and the straight edge of D touches the straight edge of C, then the instrument has an error called zero error. Zero error is always algebraically subtracted from measured length.

Zero correction has a magnitude equal to zero error but its sign is opposite to that of the zero error. Zero correction is always algebraically added to measured length.

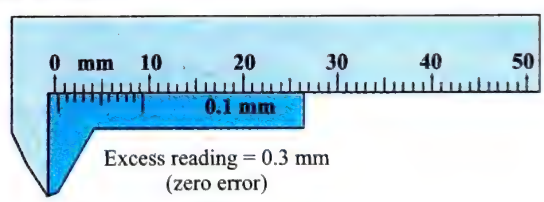
Zero error → Algebraically subtracted
Zero correction → Algebraically added

Positive and Negative Zero Error

If the zero of Vernier scale lies to the right of the zero of main scale, then the zero error is positive, and if it lies to the left of zero of the main scale, then the zero error is negative (when jaws A and B are in contact).

Positive zero error $= (N + x \times \text{VC})$

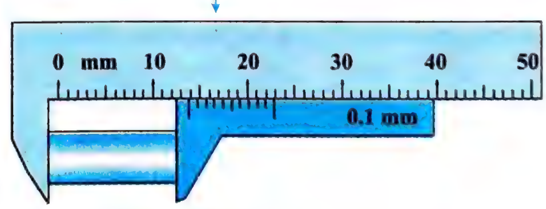
Here N is the main scale reading on the left of zero of Vernier scale and x is the Vernier scale division which coincides with any main scale division.



↓

If we put an object between the jaws

↓



It gives 13.8 mm reading.

↓

In which there is 0.3 mm excess reading, which has to be removed (subtracted).

So, actual thickness = 13.8 − 0.3 = 13.5 mm

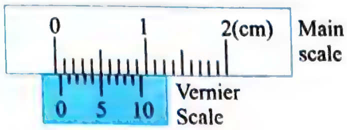
Observed reading — Excess reading (zero error)

So, we can formulate it as

Actual reading = Observed reading − Excess reading (zero error)

Calculating Positive Zero Error

In the given figure, one division of the main scale is of 1 mm and the jaws of vernier are touching each other. The fifth division of vernier scale coincides with a main scale division.



In this case, L.C. $= \frac{1}{10} = 0.1\,\text{mm}$

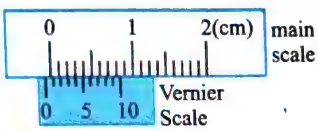
Hence, the zero error $= +5 \times 0.1 = 0.5$ mm and,

Zero correction $= -0.5$ mm.

Here this error is to be subtracted from the reading taken for measurement.

Calculating Negative Zero Error

In this situation, the jaws of vernier are touching each other. The zero of the vernier scale lies to the left of the zero of the main scale. And the fourth division of vernier scale coincides with a main scale division as shown in the figure. Also



$$1\text{ L.C.} = \frac{1}{10}\text{mm} = 0.1\,\text{mm}$$

Hence, the zero error $= -4 \times 0.1 = -0.4$ mm

and zero correction $=$ 0.4 mm

In this case, the error is to be added in the reading taken for measurement.

ILLUSTRATION 1.45

Consider the following data: 10 MSDs = 1 cm, 10 VSDs = 9 MSDs, zero of Vernier scale is to the right of the zero marking of the main scale with 6 VSDs coinciding with MSDs and the actual reading for length measurement is 4.3 cm with 2 VSDs coinciding with main scale graduations. Estimate the length.

Sol. In this case, Vernier constant $= \frac{1\text{ mm}}{10} = 0.1$ mm

Zero error $= 6 \times 0.1 = +0.6$ mm

Correction $= -0.6$ mm

Actual length $= (4.3 + 2 \times 0.01) +$ Correction $= 4.32 - 0.06 = 4.26$ cm

ILLUSTRATION 1.46

The side of a cube is measured by Vernier callipers (10 divisions of the vernier scale coincide with 9 divisions of the main scale, where 1 division of main scale is 1 mm). The main scale reads 10 mm and first division of vernier scale coincides with the main scale. The mass of the cube is 2.736 g. Find the density of the cube in appropriate significant figures.

Sol. Least count of Vernier callipers

$$= \frac{\text{1 division of main scale}}{\text{Number of divisions in Vernier scale}} = \frac{1}{10} = 0.1\text{ mm}$$

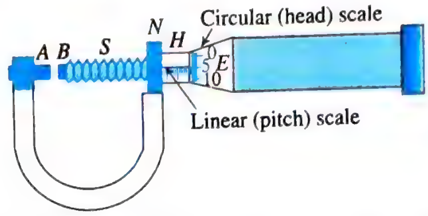
Side of cube $= 10\text{ mm} + 1 \times 0.1\text{ mm} = 1.01$ cm

Now, density $= \frac{\text{Mass}}{\text{Volume}} = \frac{2.736\text{ g}}{(1.01)^3\text{ cm}^3} = 2.66\text{ g cm}^{-3}$

(to correct number of significant figures)

Screw Gauge (or Micrometer Screw)

In general, Vernier callipers can measure accurately up to 0.01 cm and for greater accuracy micrometer screw devices, e.g., screw gauge spherometer, are used. These consist of accurately cut screw which can be moved in a closely fitting fixed nut by turning it axially (see figure below). The instrument is provided with two scales:



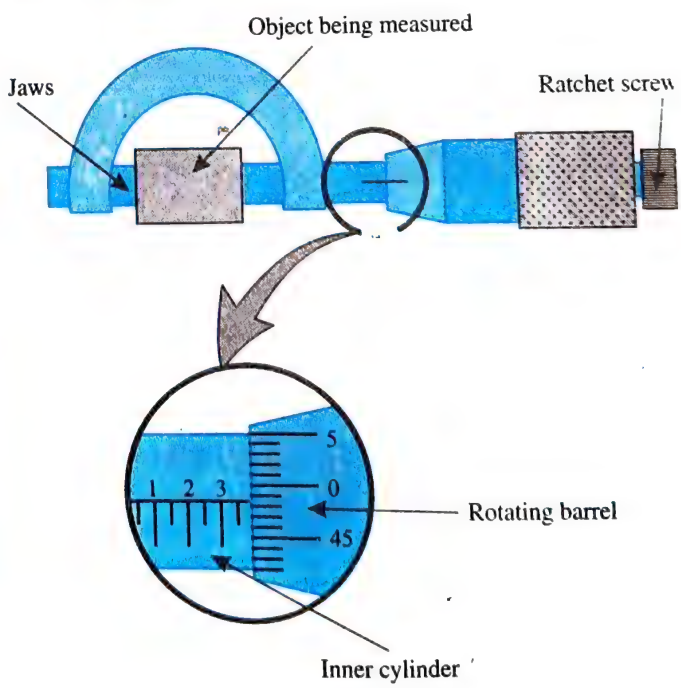
(a) The main scale or pitch scale graduated along the axis of the screw

(b) The cap-scale or head scale H round the edge of the screw head

Constants of the Screw Gauge

Pitch: The translational motion of the screw is directly proportional to the total rotation of the head. The pitch of the instrument is the distance between two consecutive threads of the screw which is equal to the distance moved by the screw due to one complete rotation of the cap. If the screw advances by 5 mm for 10 rotations of the cap, then the pitch = 5/10 = 0.5 mm.

Least count: In this case also, the minimum (or least) measurement (or count) of length is equal to one division on the head scale which is equal to the pitch divided by the total cap divisions. Thus, in the aforesaid illustration, if the total cap division is 100, then least count = 0.5 mm/100 = 0.005 mm.



In case of the figure given:

- Main scale has $\frac{1}{2}$ mm marks.
- Circular scale has 50 divisions.
- In complete rotation, the screw advances by $\frac{1}{2}$ mm.

Main scale reading = 3.5 mm

Circular scale reading = 48

$$\text{Least count, L.C.} = \frac{\text{Pitch}}{\text{Total no. of divisions on the circular scale}} = \frac{\frac{1}{2}\text{ mm}}{50} = 0.01\text{ mm}$$

Object thickness = Main scale reading + Circular scale reading × L.C.

$= 3.5\text{ mm} + 48 \times 0.01 = 3.98\text{ mm}$

Zero Error and Zero Correction

In a perfect instrument, the zero of the main scale coincides with the line of graduation along the screw axis with no zero error, otherwise the instrument is said to have zero error which is equal to the cap reading with the gap closed. This error is positive when zero line or the reference line of the cap lies above the line of graduation and vice versa. The corresponding corrections will be just opposite.

Zero correction is the invert of zero error:

Zero correction = – (zero error)

Actual reading = Observed reading – Zero error

= Observed reading + Zero correction

Let us take an instrument in which in a complete rotation, the spindle of the instrument advances by 1 mm. There are 100 divisions on circular scale.

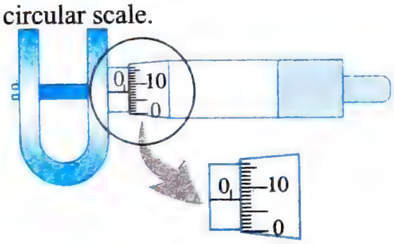


Fig. (a)

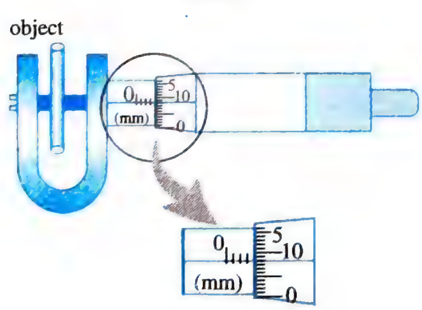


Fig. (b)

$$\text{Least count, L.C.} = \frac{\text{Pitch}}{\text{Total no. of divisions on the circular scale}} = \frac{1\text{ mm}}{100} = 0.01\text{ mm}$$

In figure (a), the instrument has positive zero error.

Excess reading (Zero error) = 7 × L.C. = 7 × 0.01 = 0.07 mm

This error has to removed (subtracted).

In figure (b), the reading of instrument

= Main scale reading + Circular scale reading × L.C

= 3 mm + 7 × 0.01 = 3.07 mm

In this reading, there is 0.07 mm excess reading which has to be subtracted.

Hence, actual reading = Observed reading – Excess reading

= 3.07 – 0.07 = 3.00 mm

Illustration 1.47

A screw gauge having 100 equal dimensions and a pitch of length 1 mm is used to measure the diameter of a wire of length 5.6 cm. The main scale reading is 1 mm and the 47th circular division coincides with the main scale. Find the diameter of the wire.

Sol. $\text{Least count} = \frac{\text{Pitch}}{\text{No. of divisions on circular scale}}$

$= \frac{1}{100}\text{ mm} = 0.01\text{ mm}$

Wire diameter

= Main scale reading + Circular scale reading × L.C.

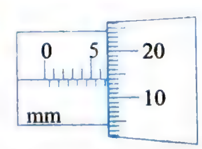
= 1 mm + 47 × LC

= 1 mm + 47 × 0.01 = 1.47 mm

Illustration 1.48

In a complete rotation, the spindle of a screw gauge advances by 0.5 mm. There are 50 divisions on circular scale. The main scale has 0.5 mm marks (is graduated to 0.5 mm or has least count. = 0.5 mm) Read the screw gauge shown below in fig (a) and (b):

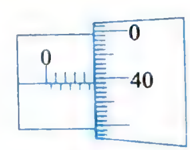
Sol.



Object thickness

$= 6.5\text{ mm} + 14\left(\frac{0.5\text{ mm}}{50}\right)$

$= 6.64$ mm



Object thickness

$= 4.5\text{ mm} + 39\left(\frac{0.5\text{ mm}}{50}\right)$

$= 4.89$ mm

Important Points:

- A measurement of a physical quantity is said to be accurate if the systematic error in its measurement is relatively very low. On the other hand, the measurement of a physical quantity is said to be precise if the random error is small.
- Errors are always additive in nature.
- For greater accuracy, the quantity with higher power should have least error.
- Absolute error is not dimensionless quantity.
- Relative error is a dimensionless quantity.
- $\text{Least Count} = \frac{\text{Value of 1 part on main scale }(s)}{\text{Number of parts on vernier scale }(n)}$
- Least count of Vernier callipers

$$= \left\{\begin{array}{c}\text{Value of 1 part of}\\ \text{main scale }(s)\end{array}\right\} - \left\{\begin{array}{c}\text{Value of 1 part of}\\ \text{Vernier scale }(v)\end{array}\right\}$$

Least count of vernier calliper = 1 MSD – 1 VSD

where MSD = Main scale division and VSD = Vernier scale division

- Least count of screw guaze $= \dfrac{\text{Pitch } (p)}{\text{Number of parts on circular scale } (n)}$
- Smaller the least count, higher is the accuracy of measurement.
- Larger the number of significant figures after the decimal in a measurement, higher is the accuracy of measurement.
- Significant figures do not change if we measure a physical quantity in different units.
- When we add or subtract two measured quantities, the absolute error in the final result is equal to the sum of the absolute errors in the measured quantities.
- When we multiply or divide two measured quantities, the relative error in the final result is equal to the sum of the relative errors in the measured quantities.

Solved Examples

EXAMPLE 1.1

$\frac{\alpha}{t^2} = Fv + \frac{\beta}{x^2}$

Find the dimension formula for $[\alpha]$ and $[\beta]$ (here t = time, F = force, v = velocity, x = distance).

Sol. Since $[Fv] = M^1L^2T^{-3}$,

So $\left[\frac{\beta}{x^2}\right]$ should also be $M^1L^2T^{-3}$; $\frac{[\beta]}{[x^2]} = M^1L^2T^{-3}$

$[\beta] = M^1L^4T^{-3}$ and $\left[Fv + \frac{\beta}{x^2}\right]$ will also have dimension $M^1L^2T^{-3}$.

So, $\frac{[\alpha]}{[t^2]} = M^1L^2T^{-3}$; $[\alpha] = M^1L^2T^{-1}$

EXAMPLE 1.2

A body of mass m hung at one end of the spring executes simple harmonic motion. The force constant of a spring is k while its period of vibration is T. Prove by dimensional method that the equation $T = 2\pi m/k$ is incorrect. Derive the correct equation, assuming that they are related by a power law.

Sol. The given equation is $T = \frac{2\pi m}{k}$

Taking the dimensions of both sides, we have

$$[T] = \frac{[M]}{[ML^0T^{-2}]} = T^2$$

As the dimensions of two sides are not equal, hence the equation is incorrect.

Let the correct relation be $T = Cm^a k^b$, where C is constant.

Equating the dimensions of both sides, we get

$$[T] = [M]^a [MT^{-2}]^b$$

or $$[M^0L^0T] = [M^{a+b} L^0T^{-2b}]$$

Comparing the powers of M, L, and T on both sides, we get $a + b = 0$ and $-2b = 1$.

Therefore, $b = -\frac{1}{2}$ and $a = \frac{1}{2}$

$$\therefore \quad T = Cm^{1/2}k^{-1/2} = C\sqrt{\left(\frac{m}{k}\right)}$$

This is the correct equation.

EXAMPLE 1.3

The radius of the earth is 6.37×10^6 m and its mass is 5.975×10^{24} kg. Find the earth's average density to appropriate significant figures.

Sol. Given mass of the earth (M) and radius of the earth (R)

Hence, volume of the earth (V)

$$= \frac{4}{3} \times \pi R^3 = \frac{4}{3} \times (3.142) \times (6.37 \times 10^6)^3 \text{ m}^3$$

$$\text{Average density (D)} = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{V} = \frac{5.975 \times 10^{24}}{\frac{4}{3} \times (3.142) \times (6.37 \times 10^6)^3}$$

$$= 0.005517 \times 10^6 \text{ kg m}^{-3}$$

$$= 5.52 \times 10^3 \text{ kg m}^{-3} \quad \text{(to three significant figures)}$$

The density is accurate only up to three significant figures which is the accuracy of the least accurate factor, namely, the radius of the earth.

EXAMPLE 1.4

It has been observed that velocity of ripple waves produced in water depends upon their wavelength (λ), density of water (ρ), and surface tension (T). Prove that $v^2 \propto T/\lambda\rho$.

Sol. According to the problem,

$$v \propto \lambda^a \rho^b T^c$$

$$v = k\lambda^a \rho^b T^c$$

where k is a dimensionless constant.

$$LT^{-1} = L^a(ML^{-3})^b (MT^{-2})^c$$

$$\Rightarrow \quad M^0L^1T^{-1} = M^{b+c} L^{a-3b} T^{-2c}$$

Using the principle of homogeneity, we get

$$b + c = 0, a - 3b = 1, -2c = -1$$

Solving these equations, we get

$$a = -\frac{1}{2}, b = -\frac{1}{2}, c = \frac{1}{2}$$

So, the relation becomes $v = k\lambda^{-1/2}\rho^{-1/2}T^{1/2} \Rightarrow v^2 \propto \frac{T}{\lambda\rho}$

EXAMPLE 1.5

In an experiment for determining the value of acceleration due to gravity (g) using a simple pendulum, the following observations were recorded:

Length of the string (l) = 98.0 cm

Diameter of the bob (d) = 2.56 cm

Time for 10 oscillations (T) = 20.0 s

Calculate the value of g with maximum permissible absolute error and the percentage relative error.

Sol. Time period for a simple pendulum is $T = 2\pi\sqrt{\dfrac{l_{eff}}{g}}$ ...(i)

where l_{eff} is the effective length of the pendulum equal to $\left(l+\dfrac{d}{2}\right)$

and time period equals $T = \dfrac{20.0}{10} = 2.00$ s

From (i), we get $g = \dfrac{4\pi^2(l_{eff})}{T^2}$

To calculate actual value of g.

$$\text{Since } g = \frac{4\pi^2(l_{eff})}{T^2} = \frac{4\pi^2\left(l+\frac{d}{2}\right)}{T^2} = \frac{4\pi^2(l+r)}{T^2}$$

$$\Rightarrow \quad g = \frac{4\pi^2(98+1.28)}{(2.00)^2} = 980 \text{ cm s}^{-2} = 9.80 \text{ m s}^{-2}$$

Error in the value of g:

$$\frac{\Delta g}{g} = \frac{\Delta l_{eff}}{l_{eff}} + 2\left(\frac{\Delta T}{T}\right) = \frac{\Delta l + \Delta r}{l+r} + 2\left(\frac{\Delta T}{T}\right)$$

Further, since errors can never exceed the least count of the measuring instrument. So, $\Delta l = 0.1$ cm and $\Delta r = 0.01$ cm.

$$\Rightarrow \quad \frac{\Delta g}{g} = \left(\frac{0.1+0.01}{98.0+1.28}\right) + 2\left(\frac{0.1}{20.0}\right)$$

$$\Rightarrow \quad = 0.0011 + 0.01 = 0.0111$$

Thus, percentage error $\dfrac{\Delta g}{g} \times 100\% = 1.1\%$

and absolute error $= \Delta g = g(0.011) = 0.11 \text{ m s}^{-2}$

So, $g = (9.80 \text{ m s}^{-2} \pm 1.1\%) = (9.80 \pm 0.11) \text{ m s}^{-2}$

EXAMPLE 1.6

The length and breadth of a rectangular sheet are 16.2 cm and 10.1 cm, respectively. Find the area of the sheet in appropriate significant figures and error.

Sol. According to the problem, length, $l = (16.2 \pm 0.1)$ cm

Breadth, $b = (10.1 \pm 0.1)$cm

Area, $A = l \times b = (16.2 \text{ cm}) \times (10.1 \text{ cm}) = 163.62 \text{ cm}^2$

As per the rule, area will have only three significant figures and error will have only one significant figure. Rounding off, we get, area $A = 164 \text{ cm}^2$.

If ΔA is error in an area, then relative error is calculated as $\Delta A/A$. Then

$$\frac{\Delta A}{A} = \frac{\Delta l}{l} + \frac{\Delta b}{b}$$

$$= \frac{0.1 \text{ cm}}{16.2 \text{cm}} + \frac{0.1 \text{ cm}}{10.1 \text{ cm}} = \frac{1.01+1.62}{16.2\times 10.1} = \frac{2.63}{163.62}$$

$$\Rightarrow \quad \Delta A = A \times \frac{2.63}{163.62} \text{ cm}^2 = 63.62 \times \frac{2.63}{163.62} = 2.63 \text{ cm}^2$$

$\Delta A = 3 \text{ cm}^2$ (By rounding off to one significant figure)

Area, $A = A \pm \Delta A = (164 \pm 3) \text{ cm}^2$

EXAMPLE 1.7

You measure two quantities as $A = 1.0$ m $\pm$ 0.2 m, and B = 2.0 m $\pm$ 0.2 m. Find the correct value for $\sqrt{AB}$ in appropriate significant figures and error.

Sol. According to the problem,

A = 1.0 m $\pm$ 0.2 m, B = 2.0 m $\pm$ 0.2 m

Let $Z = \sqrt{AB} = \sqrt{(1.0)(2.0)} = 1.414$ m

Rounding off to two significant digits, Z = 1.4 m

$$\text{As} \quad \frac{\Delta Z}{Z} = \frac{1}{2}\frac{\Delta A}{A} + \frac{1}{2}\frac{\Delta B}{B} = \frac{1}{2}\left(\frac{0.2 \text{ m}}{1 \text{m}}\right) + \frac{1}{2}\left(\frac{0.2 \text{ m}}{2 \text{m}}\right) = 0.15$$

$$\Rightarrow \quad \Delta Z = Z(0.15) = 1.4 \text{ m}(0.15) = 0.212 \text{ m}$$

Rounding off to one significant digit, $\Delta Z = 0.2$ m

The correct value for $\sqrt{AB} = 1.4 \pm 0.2$m.

EXAMPLE 1.8

The time for 20 oscillations of a pendulum is measured as $t_1 = 39.6$ s; $t_2 = 39.9$ s and $t_3 = 39.5$ s. What is the precision in the measurements? What is the accuracy of the measurement?

Sol. According to the problem, the time for 20 oscillations of pendulum, $t_1 = 39.6$s, $t_2 = 39.9$s and $t_3 = 39.5$s.

(a) It is quite obvious from these observations that the least count of the watch is 0.1 s. As measurements have only one decimal place, precision in the measurement = Least count of the measuring instrument = 0.1 s

Precision in 20 oscillations = 0.1 s

Therefore, precision in 1 oscillation $= \dfrac{0.1}{20} = 0.005$ s

(b) Mean value of time for 20 oscillations is given by

$$t = \frac{t_1+t_2+t_3}{3} = \frac{39.6+39.9+39.5}{3} = 39.66 \text{ s}$$

Mean time period of the second pendulum $= \dfrac{39.66}{20} \approx 1.98$ s

Rounding off the time period of second pendulum = 2 s

Measured time period of the second pendulum

= 2 − 0.005 = 1.995 s

Accuracy of measurement is the máximum observed error and is given by = 1.995 − 1.980 = 0.015 s

EXAMPLE 1.9

A physical quantity X is related to four measurable quantities a, b, c and d as follows $X = a^2 b^3 c^{5/2} d^{-2}$. The percentage error in the measurement of a, b, c and d are 1%, 2%, 3% and 4%, respectively. What is the percentage error in quantity X? If the value of X calculated on the basis of the above relation is 2.763, to what value should you round off the result?

Sol. Percentage error in quantity X is given by $\dfrac{\Delta x}{x} \times 100$.

According to the problem, physical quantity is $X = a^2b^3c^{5/2}d^{-2}$

Percentage error in $a = \left(\frac{\Delta a}{a} \times 100\right) = 1\%$

Percentage error in $b = \left(\frac{\Delta b}{b} \times 100\right) = 2\%$

Percentage error in $c = \left(\frac{\Delta c}{c} \times 100\right) = 3\%$

Percentage error in $d = \left(\frac{\Delta d}{d} \times 100\right) = 4\%$

Maximum percentage error in X is

$$\frac{\Delta X}{X}\times 100 = \pm\left[2\left(\frac{\Delta a}{a}\times 100\right)+3\left(\frac{\Delta b}{b}\times 100\right) + \frac{5}{2}\left(\frac{\Delta c}{c}\times 100\right)+2\left(\frac{\Delta d}{d}\times 100\right)\right]$$

$$= \pm\left[2(1) + 3(2) + \frac{5}{2}(3) + 2(4)\right]\%$$

$$= \pm\left[2 + 6 + \frac{15}{2} + 8\right] = \pm 23.5\%$$

$\therefore$ Percentage error in quantity $X = \pm 23.5\%$

Mean absolute error in $X = \pm 0.235 = \pm 0.24$ (rounding-off upto two significant digits)

On the basis of these values, value of X should have two significant digits only. Therefore, $X = 2.8$

EXAMPLE 1.10

If the velocity of light c, Planck's constant h and gravitational constant G are taken as fundamental quantities, then express mass, length and time in terms of dimensions of these quantities.

Sol. We have to apply principle of homogeneity to solve this problem. Principle of homogeneity states that in a correct equation, the dimensions of each term added or subtracted must be same, i.e., dimensions of LHS and RHS should be equal.

We know that dimensions of

$$[h] = [ML^2T^{-1}];\ [c] = [LT^{-1}];\ [G] = [M^{-1}L^3T^{-2}]$$

(i) Let $m \propto c^a h^b G^c \Rightarrow \quad m = kc^a h^b G^c$ ...(i)

where, k is a dimensionless constant of proportionality.

Substituting dimensions of each term in Eq. (i), we get

$$[ML^0T^0] = [LT^{-1}]^a \times [ML^2T^{-1}]^b[M^{-1}L^3T^{-2}]^c$$
$$= [M^{b-c}L^{a+2b+3c}T^{-a-b-2c}]$$

Comparing powers of same terms on both sides, we get

$b - c = 1$ ...(ii)

$a + 2b + 3c = 0$ ...(iii)

$-a - b - 2c$ ...(iv)

Adding Eqs. (ii), (iii) and (iv), we get $2b = 1 \Rightarrow b = \frac{1}{2}$

Substituting value of b in Eq. (ii), we get $c = -\frac{1}{2}$

From Eq. (iv), $a = -b - 2c$

Substituting values of b and c, we get $a = \frac{1}{2} - 2\left(-\frac{1}{2}\right) = \frac{1}{2}$

Putting values of a, b and c in Eq. (i), we get

$$m = kc^{1/2}h^{1/2}G^{-1/2} \Rightarrow m = k\sqrt{\frac{ch}{G}}$$

(ii) Let $L \propto c^a h^b G^c \quad \Rightarrow L = kc^a h^b G^c$ ...(v)

where k is a dimensionless constant.

Substituting dimensions of each term in Eq. (v), we get

$$[M^0LT^0] = [LT^{-1}]^a \times [ML^2T^{-1}]^b \times [M^{-1}L^3T^{-2}]^c$$
$$= [M^{b-c}L^{a+2b+3c}T^{-a-b-2c}]$$

On comparing powers of same terms, we get

$b - c = 0$ ...(vi)

$a + 2b + 3c = 1$ ...(vii)

$-a - b - 2c = 0$ ...(viii)

Adding Eqs. (vi), (vii) and (viii), we get $2b = 1 \Rightarrow b = \frac{1}{2}$

Substituting value of b in Eq. (vi), we get $c = \frac{1}{2}$

From Eq. (viii), $a = -b - 2c$

Substituting values of b and c, we get

$$a = -\frac{1}{2} - 2\left(\frac{1}{2}\right) = -\frac{3}{2}$$

Putting values of a, b and c in Eq. (v), we get

$$L = kc^{-3/2}h^{1/2}G^{1/2} \Rightarrow L = k\sqrt{\frac{hG}{c^3}}$$

(iii) Let $T \propto c^a h^b G^c \Rightarrow T = kc^a k^b G^c$ ...(ix)

where k is a dimensionless constant.

Substituting dimensions of each term in Eq. (ix), we get

$$[M^0L^0T^1]$$
$$= [LT^{-1}]^a \times [ML^2T^{-1}]^b[M^{-1}L^3T^{-2}]^c$$
$$= [M^{b-c}L^{a+2b+3c}T^{-a-b-2c}]$$

On comparing powers of same terms, we get

$b - c = 0$ ...(x)

$a + 2b + 3c = 0$ ...(xii)

$-a - b - 2c = 1$ ...(xiii)

Adding Eqs. (x), (xi) and (xii), we get $2b = 1 \Rightarrow b = \frac{1}{2}$

Substituting value of b in Eq. (x), we get $c = b = \frac{1}{2}$

From Eq. (xii), $a = -b - 2c - 1$

Substituting values of b and c, we get

$$a = -\frac{1}{2} - 2\left(\frac{1}{2}\right) - 1 = -\frac{5}{2}$$

Putting values of a, b and c in Eq. (ix), we get

$$T = kc^{-5/2}h^{1/2}G^{1/2} \Rightarrow T = k\sqrt{\frac{hG}{c^5}}$$

EXAMPLE 1.11

An artificial satellite is revolving around a planet of mass M and radius R in a circular orbit of radius r. From Kepler's third law about the period of a satellite around a common central body, square of the period of revolution T is proportional to the cube of the radius of the orbit r.

Using dimensional analysis, show that $T = \frac{k}{R}\sqrt{\frac{r^3}{g}}$, where k is a dimensionless constant and g is acceleration due to gravity.

According to Kepler's third law, $T^2 \propto a^3$, i.e., square of time period (T^2) of a satellite revolving around a planet is proportional to cube of the radius of the orbit (a^3).

We have to apply Kepler's third law,

$$T^2 \propto r^3 \Rightarrow T \propto r^{3/2}$$

Also, T depends on R and g.

Let $\quad T \propto r^{3/2} g^a R^b \Rightarrow T = kr^{3/2} R^a g^b \quad$ ...(i)

where k is a dimensionless constant of proportionality.

Writing the dimensions of various quantities on both the sides, we get

$$[M^0 L^0 T] = [L]^{3/2} [LT^{-2}]^a [L]^b = [M^0 L^{a+b+3/2} T^{-2a}]$$

On comparing the dimensions of both sides, we get

$$a + b + \frac{3}{2} = 0 \quad \text{...(}$$

$$-2a = 1 \Rightarrow a = -\frac{1}{2} \quad \text{...(}$$

From Eq. (ii), we get

$$b - \frac{1}{2} + \frac{3}{2} = 0 \Rightarrow b = -1$$

Substituting the values of a and b in Eq. (i), we get $T = kr^{3/2}R^{-1}$

$$\Rightarrow \quad T = \frac{k}{R}\sqrt{\frac{r^3}{g}}$$

Single Correct Answer Type

1. A physical quantity depends upon five factors, all of which have dimensions; then method of dimensional analysis
   (1) Can be applied
   (2) Cannot be applied
   (3) Depends upon factors involved
   (4) Both (1) and (3)

2. A student when discussing the properties of a medium (except vacuum) writes
   Velocity of light in vacuum = Velocity of light in medium
   This formula is
   (1) Dimensionally correct (2) Dimensionally incorrect
   (3) Numerically incorrect (4) Both (1) and (3)

3. Given that T stands for time period and l stands for the length of simple pendulum. If g is the acceleration due to gravity, then which of the following statements about the relation $T^2 = (l/g)$ is correct?
   (1) It is correct both dimensionally as well as numerically.
   (2) It is neither dimensionally correct nor numerically.
   (3) It is dimensionally correct but not numerically.
   (4) It is numerically correct but not dimensionally.

4. Suppose refractive index μ is given as $\mu = A + \dfrac{B}{\lambda^2}$
   where A and B are constants and λ is wavelength, then dimensions of B are same as that of
   (1) Wavelength (2) Volume
   (3) Pressure (4) Area

5. The best method to reduce random error is
   (1) To change the instrument used for measurement
   (2) To take help of experienced observer
   (3) To repeat the experiment many times and to take the average results
   (4) None of the above

6. Of the following quantities, which one has the dimensions different from the remaining three?
   (1) Energy density
   (2) Force per unit area
   (3) Product of charge per unit volume and voltage
   (4) Angular momentum per unit mass

7. The quantities A and B are related by the relation $A/B = m$, where m is the linear mass density and A is the force, the dimensions of B will be
   (1) Same as that of pressure (2) Same as that of work
   (3) That of momentum (4) Same as that of latent heat

8. Which of the following quantities has its unit as newton-second?
   (1) Energy (2) Torque
   (3) Momentum (4) Angular momentum

9. Which of the following is the most precise instrument for measuring length?
   (1) Meter rod of least count 0.1 cm
   (2) Vernier callipers of least count 0.01 cm
   (3) Screw gauge of least count 0.001 cm
   (4) Data is not sufficient to decide

10. The equation of the stationary wave is
    $$y = 2A\sin\left(\frac{2\pi ct}{\lambda}\right)\cos\left(\frac{2\pi x}{\lambda}\right)$$
    Which of the following statements is wrong?
    (1) The unit of ct is same as that of λ.
    (2) The unit of x is same as that of λ.
    (3) The unit of $2\pi c/\lambda$ is same as that of $2\pi x/\lambda t$.
    (4) The unit of c/λ is same as that of x/λ.

11. Given that: $y = A\sin\left[\left(\frac{2\pi}{\lambda}\right)(ct - x)\right]$, where y and x are measured in the unit of length. Which of the following statements is true?
    (1) The unit of λ is same as that of x and A.
    (2) The unit of λ is same as that of x but may not be same as that of A.
    (3) The unit of c is same as that of $2\pi/\lambda$.
    (4) The unit of $(ct - x)$ is same as that of $2\pi/\lambda$.

12. In the relation $\dfrac{dy}{dt} = 2\omega\sin(\omega t + \phi_0)$, the dimensional formula for $\omega t + \phi_0$ is
    (1) MLT (2) MLT^0
    (3) ML^0T^0 (4) $M^0L^0T^0$

13. The length l, breadth b, and thickness t of a block of wood were measured with the help of a measuring scale. The results with permissible errors (in cm) are
    $l = 15.12 \pm 0.01$, $b = 10.15 \pm 0.01$, and $t = 5.28 \pm 0.01$.
    The percentage error in volume up to proper significant figures is
    (1) 0.28% (2) 0.35%
    (3) 0.48% (4) 0.64%

14. A physical quantity x depends on quantities y and z as follows: $x = Ay + B\tan(Cz)$, where A, B, and C are constants. Which of the followings do not have the same dimensions?
    (1) x and B (2) C and z^{-1}
    (3) y and B/A (4) x and A

15. The relative density of a material of a body is found by weighing it first in air and then in water. If the weight of the body in air is $W_1 = 8.00 \pm 0.05$ N and the weight in water is $W_2 = 6.00 \pm 0.05$ N, then the relative density $\rho_r = W_1/(W_1 - W_2)$ with the maximum permissible error is
    (1) $4.00 \pm 0.62\%$ (2) $4.00 \pm 0.82\%$
    (3) $4.00 \pm 3.2\%$ (4) $4.00 \pm 5.62\%$

16. Force F is given in terms of time t and distance x by $F = A \sin Ct + B \cos Dx$. Then the dimensions of A/B and C/D are

(1) $[M^0L^0T^0]$, $[M^0L^0T^{-1}]$
(2) $[MLT^{-2}]$, $[M^0L^{-1}T^0]$
(3) $[M^0L^0T^0]$, $[M^0LT^{-1}]$
(4) $[M^0L^1T^{-1}]$, $[M^0L^0T^0]$

17. In the relation $y = r \sin(\omega t - kx)$, the dimensions of ω/k are

(1) $[M^0L^0T^0]$
(2) $[M^0L^1T^{-1}]$
(3) $[M^0L^0T^1]$
(4) $[M^0L^1T^0]$

18. Given that $Y = a \sin \omega t + bt + ct^2 \cos \omega t$. The unit of abc is same as that of

(1) y
(2) y/t
(3) $(y/t)^2$
(4) $(y/t)^3$

19. The frequency (n) of vibration of a string is given as $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$, where T is tension and l is the length of vibrating string, then the dimensional formula is

(1) $[M^0L^1T^1]$
(2) $[M^0L^0T^0]$
(3) $[M^1L^{-1}T^0]$
(4) $[ML^0T^0]$

20. If frequency F, velocity V, and density D are considered fundamental units, the dimensional formula for momentum will be

(1) DVF^2
(2) DV^2F^{-1}
(3) $D^2V^2F^2$
(4) DV^4F^{-3}

21. The potential energy of a particle varies with distance x as $U = \frac{Ax^{1/2}}{x^2 + B}$, where A and B are constants. The dimensional formula for $A \times B$ is

(1) $M^1L^{7/2}T^{-2}$
(2) $M^1L^{11/2}T^{-2}$
(3) $M^1L^{5/2}T^{-2}$
(4) $M^1L^{9/2}T^{-2}$

22. If force F, acceleration a, and time T are taken as the fundamental physical quantities, the dimensions of length on this system of units are

(1) FAT^2
(2) FAT
(3) FT
(4) AT^2

23. The position x of a particle at time t is given by $x = \frac{V_0}{a}(1 - e^{-at})$, where V_0 is constant and $a > 0$. The dimensions of V_0 and a are

(1) M^0LT^{-1} and T^{-1}
(2) M^0LT^0 and T^{-1}
(3) M^0LT^{-1} and LT^{-2}
(4) M^0LT^{-1} and T

24. If x and a stand for distance, then for what value of n is the given equation dimensionally correct? The equation is

$$\int \frac{dx}{\sqrt{a^2 - x^n}} = \sin^{-1}\frac{x}{a}$$

(1) 0
(2) 2
(3) −2
(4) 1

25. The time dependence of a physical quantity P is given by $P = P_0 e^{-\alpha t^2}$, where α is a constant and t is time. Then constant α is/has

(1) Dimensionless
(2) Dimensions of T^{-2}
(3) Dimensions of P
(4) Dimensions of T^2

26. The frequency f of vibrations of a mass m suspended from a spring of spring constant k is given by $f = Cm^xk^y$, where C is a dimensionless constant. The values of x and y are, respectively,

(1) $\frac{1}{2}, \frac{1}{2}$
(2) $-\frac{1}{2}, -\frac{1}{2}$
(3) $\frac{1}{2}, -\frac{1}{2}$
(4) $-\frac{1}{2}, \frac{1}{2}$

27. A student writes four different expressions for the displacement y in a periodic motion as a function of time t, a as amplitude, T as time period. Which of the following can be correct?

(1) $y = aT \sin \frac{2\pi t}{T}$

(2) $y = a \sin Vt$

(3) $y = \frac{a}{T} \sin \frac{t}{a}$

(4) $y = \frac{a}{\sqrt{2}}\left[\sin\frac{2\pi t}{T} + \cos\frac{2\pi t}{T}\right]$

28. If C (the velocity of light) g, (the acceleration due to gravity), and P (the atmospheric pressure) are the fundamental quantities in MKS system, then the dimensions of length will be same as that of

(1) C/g
(2) C/P
(3) PCg
(4) C^2/g

29. The relation $\tan\theta = v^2/rg$ gives the angle of banking of the cyclist going round the curve. Here v is the speed of the cyclist, r is the radius of the curve, and g is the acceleration due to gravity. Which of the following statements about the relation is true?

(1) It is both dimensionally as well as numerically correct.
(2) It is neither dimensionally correct nor numerically correct.
(3) It is dimensionally correct but not numerically.
(4) It is numerically correct but not dimensionally.

30. A physical quantity X is represented by $X = (M^xL^{-y}T^{-z})$. The maximum percentage errors in the measurement of M, L, and T, respectively, are $a\%$, $b\%$ and $c\%$. The maximum percentage error in the measurement of X will be

(1) $(ax + by - cz)\%$
(2) $(ax - by - cz)\%$
(3) $(ax + by + cz)\%$
(4) $(ax - by + cz)\%$

31. The velocity of transverse wave in a string is $v = \sqrt{T/m}$ where T is the tension in the string and m is the mass per unit length. If $T = 3.0$ kgf, the mass of string is 2.5 g and the length of string is $v = 1.000$ m, then the percentage error in the measurement of velocity is

(1) 0.5
(2) 0.7
(3) 2.3
(4) 3.6

32. Write the dimensions of a/b in the relation $P = \frac{a - t^2}{bx}$, where P is the pressure, x is the distance, and t is the time.

(1) $M^{-1}L^0T^{-2}$
(2) ML^0T^{-2}
(3) ML^0T^2
(4) MLT^{-2}

33. Write the dimensions of $a \times b$ in the relation $E = \dfrac{b - x^2}{at}$, where E is the energy, x is the displacement, and t is the time.

(1) ML^2T (2) $M^{-1}L^2T^1$
(3) ML^2T^{-2} (4) MLT^{-2}

34. If the velocity of light C, the universal gravitational constant G, and Planck's constant h are chosen as fundamental units, the dimensions of mass in this system are

(1) $h^{1/2}C^{1/2}G^{-1/2}$ (2) $h^{-1}C^{-1}G$
(3) hCG^{-1} (4) hCG

35. The effective length of a simple pendulum is the sum of the following three: length of string, radius of bob, and length of hook.
In a simple pendulum experiment, the length of the string, as measured by a meter scale, is 92.0 cm. The radius of the bob combined with the length of the hook, as measured by a vernier callipers, is 2.15 cm. The effective length of the pendulum is

(1) 94.1 cm (2) 94.2 cm
(3) 94.15 cm (4) 94 cm

36. The moment of inertia of a body rotating about a given axis is 12.0 kg m^2 in the SI system. What is the value of the moment of inertia in a system of units in which the unit of length is 5 cm and the unit of mass is 10 g?

(1) 2.4×10^3 (2) 6.0×10^3
(3) 5.4×10^5 (4) 4.8×10^5

37. If the velocity (V), acceleration (A), and force (F) are taken as fundamental quantities instead of mass (M), length (L), and time (T), the dimensions of Young's modulus (Y) would be

(1) FA^2V^{-4} (2) FA^2V^{-5}
(3) FA^2V^{-3} (4) FA^2V^{-2}

38. The percentage errors in the measurement of mass and speed are 2% and 3%, respectively. How much will be the maximum error in the estimation of KE obtained by measuring mass and speed?

(1) 5% (2) 1%
(3) 8% (4) 11%

39. An experiment measures quantities a, b, and c, and then X is calculated from $X = \dfrac{a^{1/2}b^2}{c^3}$. If the percentage errors in a, b, and c are ±1%, ±3%, and ±2%, respectively, then the percentage error in X can be

(1) ±12.5% (2) ±7%
(3) ±1% (4) ±4%

40. The resistance of a metal is given by $R = V/I$, where V is potential difference and I is the current. In a circuit, the potential difference across resistance is $V = (8 \pm 0.5)$ V and current in resistance, $I = (4 \pm 0.2)$ A. What is the value of resistance with its percentage error?

(1) $(2 \pm 5.6\%)\ \Omega$ (2) $(2 \pm 0.7\%)\ \Omega$
(3) $(2 \pm 35\%)\ \Omega$ (4) $(2 \pm 11.25\%)\ \Omega$

41. The mass of the liquid flowing per second per unit area of cross section of the tube is proportional to P^x and v^y, where P is the pressure difference and v is the velocity, then the relation between x and y is

(1) $x = y$ (2) $x = -y$
(3) $y^2 = x$ (4) $y = -x^2$

42. A physical quantity x is calculated from $x = ab^2/\sqrt{c}$. Calculate the percentage error in measuring x when the percentage errors in measuring a, b, and c are 4, 2, and 3%, respectively.

(1) 7% (2) 9%
(3) 11% (4) 9.5%

43. The specific resistance ρ of a circular wire of radius r, resistance R, and length l is given by $\rho = \pi r^2 R/l$. Given: $r = 0.24 \pm 0.02$ cm, $R = 30 \pm 1\ \Omega$, and $l = 4.80 \pm 0.01$ cm. The percentage error in ρ is nearly

(1) 7% (2) 9%
(3) 13% (4) 20%

44. Using mass (M), length (L), time (T), and electric current (A) as fundamental quantities, the dimensions of permittivity will be

(1) $[MLT^{-1}A^{-1}]$ (2) $[MLT^{-2}A^{-2}]$
(3) $[M^{-1}L^{-3}T^4A^2]$ (4) $[M^2L^{-2}T^{-2}A]$

45. Assuming that the mass m of the largest stone that can be moved by a flowing river depends upon the velocity v of the water, its density ρ, and the acceleration due to gravity g. Then m is directly proportional to

(1) v^3 (2) v^4
(3) v^5 (4) v^6

46. A spherical body of mass m and radius r is allowed to fall in a medium of viscosity η. The time in which the velocity of the body increases from zero to 0.63 times the terminal velocity (v) is called time constant (τ). Dimensionally, τ can be represented by

(1) $\dfrac{mr^2}{6\pi\eta}$ (2) $\sqrt{\dfrac{6\pi mr\eta}{g^2}}$
(3) $\dfrac{m}{6\pi\eta rv}$ (4) None of these

47. A liquid drop of density ρ, radius r, and surface tension σ oscillates with time period T. Which of the following expressions for T^2 is correct?

(1) $\dfrac{\rho r^3}{\sigma}$ (2) $\dfrac{\rho\sigma}{r^3}$
(3) $\dfrac{r^3\sigma}{\rho}$ (4) None of these

48. A highly rigid cubical block A of small mass M and side L is fixed rigidly on the other cubical block of same dimensions and of modulus of rigidity η such that the lower face of A completely covers the upper face of B. The lower face of B is rigidly held on a horizontal surface. A small force F is applied perpendicular to one of the side faces of A. After the force is withdrawn, block A executes small oscillations, the time period of which is given by

(1) $2\pi\sqrt{M\eta L}$ (2) $2\pi\sqrt{M\eta/L}$
(3) $2\pi\sqrt{ML/\eta}$ (4) $2\pi\sqrt{M/\eta L}$

49. The mass of a body is 20.000 g and its volume is 10.00 cm^3. If the measured values are expressed to the correct significant figures, the maximum error in the value of density is
(1) 0.001 g cm^{-3} (2) 0.010 g cm^{-3}
(3) 0.100 g cm^{-3} (4) None of these

50. The length of a strip measured with a meter rod is 10.0 cm. Its width measured with a vernier callipers is 1.00 cm. The least count of the meter rod is 0.1 cm and that of vernier callipers is 0.01 cm. What will be the error in its area?
(1) ± 0.01 cm^2 (2) ± 0.1 cm^2
(3) ± 0.11 cm^2 (4) ± 0.2 cm^2

51. The relative density of a material is found by weighing the body first in air and then in water. If the weight in air is (10.0 ± 0.1) gf and the weight in water is (5.0 ± 0.1) gf, then the maximum permissible percentage error in relative density is
(1) 1 (2) 2
(3) 3 (4) 5

52. The dimensional formula for a physical quantity x is $[M^{-1}L^3T^{-2}]$. The errors in measuring the quantities M, L, and T, respectively, are 2%, 3%, and 4%. The maximum percentage of error that occurs in measuring the quantity x is
(1) 9 (2) 10
(3) 14 (4) 19

53. The heat generated in a circuit is given by $Q = I^2Rt$, where I is current, R is resistance, and t is time. If the percentage errors in measuring I, R, and t are 2%, 1%, and 1%, respectively, then the maximum error in measuring heat will be
(1) 2% (2) 3%
(3) 4% (4) 6%

54. The internal and external diameters of a hollow cylinder are measured with the help of a Vernier callipers. Their values are 4.23 ± 0.01 cm and 3.87 ± 0.01 cm, respectively. The thickness of the wall of the cylinder is
(1) 0.36 ± 0.02 cm (2) 0.18 ± 0.02 cm
(3) 0.36 ± 0.01 cm (4) 0.18 ± 0.01 cm

55. To determine the Young's modulus of a wire, the formula is $Y = \frac{F}{A} \times \frac{L}{\Delta L}$; where L = length, A = area of cross-section of the wire, ΔL = change in length of the wire when stretched with a force F. The conversion factor to change it from CGS to MKS system is
(1) 1 (2) 10
(3) 0.1 (4) 0.01

56. An experiment shows that two perfectly neutral parallel metal plates separated by a small distance d attract each other via a very weak force, known as the Casimir force. The force per unit area of the plates, F, depends only on the Planck constant h, on the speed of light c, and on d. Which of the following has the best chance of being correct for F?
(1) $F = \frac{hc}{d^2}$ (2) $F = \frac{hc}{d^4}$
(3) $F = \frac{hd^2}{c}$ (4) $F = \frac{d^4}{hc}$

57. In the context of accuracy of measurement and significant figures in expressing results of experiment, which of the following is/are correct
(a) Out of the two measurements 50.14 cm and 0.00025 ampere, the first one has greater accuracy.
(b) If one travels 478 km by rail and 397 m by road, the total distance travelled is 478 km.
(1) Only (a) is correct (2) Only (b) is correct
(3) Both are correct (4) None of them is correct

58. The relative density of material of a body is found by weighing it first in air and then in water. If the weight in air is (5.00 ± 0.05) Newton and weight in water is (4.00 ± 0.05) Newton. Then the relative density along with the maximum permissible percentage error is
(1) $5.0 \pm 11\%$ (2) $5.0 \pm 1\%$
(3) $5.0 \pm 6\%$ (4) $1.25 \pm 5\%$

59. The focal length f of a mirror is given by $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ where u and v represent object and image distances respectively.
(1) $\frac{\Delta f}{f} = \frac{\Delta u}{u} + \frac{\Delta v}{v}$
(2) $\frac{\Delta f}{f} = \frac{\Delta u}{v} + \frac{\Delta v}{u}$
(3) $\frac{\Delta f}{f} = \frac{\Delta u}{u} + \frac{\Delta v}{v} - \frac{\Delta(u+v)}{u+v}$
(4) $\frac{\Delta f}{f} = \frac{\Delta u}{u} + \frac{\Delta v}{v} + \frac{\Delta u}{u+v} + \frac{\Delta v}{u+v}$

60. The external and internal diameters of a hollow cylinder are measured to be (4.23 ± 0.01) cm and (3.89 ± 0.01) cm. The thickness of the wall of the cylinder is
(1) (0.34 ± 0.02) cm (2) (0.17 ± 0.02) cm
(3) (0.17 ± 0.01) cm (4) (0.34 ± 0.01) cm

61. A gas bubble from an expression under water, oscillates with a period T proportional to $p^a d^b E^c$, where p is the static pressure, d is the density of water and E is the total energy of explosion. Find the value of a:
(1) $\frac{1}{2}$ (2) $\frac{1}{3}$
(3) $-\frac{1}{4}$ (4) $-\frac{5}{6}$

62. In the relation $P = \frac{\alpha}{\beta} e^{-\alpha Z/K\theta}$, P is the pressure, Z is the distance, K is the Boltzmann constant and α is the temperature. The dimensional formula of β will be:
(1) $[M^0L^2T^0]$ (2) $[M^1L^2T^1]$
(3) $[M^1L^0T^{-1}]$ (4) $[M^0L^2T^{-1}]$

63. A bus travels distance x_1 when accelerates from rest at constant rate a_1 for some time and after that travels a distance x_2 when decelerates at a constant rate a_2 to come to rest. A student established a relation $x_1 + x_2 = \frac{a_1 a_2 t^2}{2(a_1 + a_2)}$. Choose the correct option(s).

(1) The relation is dimensionally correct
(2) The relation is dimensionally incorrect
(3) The relation may be dimensionally correct
(4) None of the above

64. A body travels uniformly a distance of $(S \pm \Delta S)$ in a time $(t \pm \Delta t)$. What may be the condition so that body within the error limits move with a velocity $\left(\frac{S}{t} \pm \frac{\Delta S}{\Delta t}\right)$;

(1) $\frac{\Delta t}{t} + \frac{S(\Delta t)^2}{(\Delta S)t^2} = \pm 1$ (2) $\frac{\Delta t}{t} + \frac{S\,\Delta t}{\Delta S\, t} = \pm 1$

(3) $\frac{\Delta t}{t} + \frac{(\Delta S)t}{S(\Delta t)} = \pm 1$ (4) $\frac{\Delta t}{t} + \frac{S^2 \Delta t}{(\Delta S)^2 t} = \pm 1$

Multiple Correct Answers Type

1. Which of the following pairs has/have the same dimensions?
(1) Torque and work
(2) Angular momentum and Planck's constant
(3) Energy and Young's modulus
(4) Light year and wavelength

2. Which of the following pairs has/have different dimensions?
(1) Frequency and angular velocity.
(2) Tension and surface tension.
(3) Density and energy density.
(4) Linear momentum and angular momentum.

3. Pressure is dimensionally
(1) Force per unit area
(2) Energy per unit volume
(3) Momentum per unit area per second
(4) Momentum per unit volume

4. Choose the correct statement(s).
(1) A dimensionally correct equation must be correct.
(2) A dimensionally correct equation may be correct.
(3) A dimensionally incorrect equation must be incorrect.
(4) A dimensionally incorrect equation may be correct.

5. The values of measurement of a physical quantity in five trials were found to be 1.51, 1.53, 1.53, 1.52, and 1.54. Then
(1) Average absolute error is 0.01.
(2) Relative error is 0.01.
(3) Percentage error is 0.01%.
(4) Percentage error is 1%.

6. If S and V are one main scale and one Vernier scale and $n - 1$ divisions on the main scale are equivalent to n divisions of the Vernier, then
(1) The least count is S/n.
(2) The Vernier constant is S/n.
(3) The same vernier constant can be used for circular Verniers also.
(4) The same vernier constant cannot be used for circular Verniers.

7. Which of following pairs has/have the same dimensions? (L = inductance, C = capacitance, R = resistance)
(1) $\frac{L}{R}$ and CR (2) LR and CR
(3) $\frac{L}{R}$ and $\sqrt{LC}$ (4) RC and $\frac{1}{LC}$

8. Which of the following pairs has/have the same dimensions?
(1) h/e and magnetic flux (2) h/e and electric flux
(3) Electric flux and q/ε_0 (4) Electric flux and $\mu_0 I$

9. Consider three quantities: $x = \frac{E}{b}$, $y = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, and $z = \frac{l}{CR}$.
Here, l is the length of a wire, C is the capacitance, and R is a resistance. All other symbols have usual meanings. Then
(1) x and y have the same dimensions.
(2) x and z have the same dimensions.
(3) y and z have the same dimensions.
(4) None of the above three pairs have the same dimensions.

10. Which of the following ratios express(es) pressure?
(1) Force/Area (2) Energy/Volume
(3) Energy/Area (4) Force/Volume

11. The velocity, acceleration and force in two systems of units are related as under:
(i) $v' = \frac{\alpha^2}{\beta} v$ (ii) $a' = (\alpha\beta)a$ (iii) $F' = \left(\frac{1}{\alpha\beta}\right)F$
All the primed symbols belong to one system and unprinted ones belong to the other system. α and β are dimensionless constants. Which of the following is/are correct?
(1) Length standards of the systems are related by:
$$L' = \left(\frac{\alpha^3}{\beta^3}\right)L$$
(2) Mass standards of the two systems are related by:
$$M' = \left(\frac{1}{\alpha^2\beta^2}\right)M$$
(3) Time standards of the two systems are related by:
$$T' = \left(\frac{\alpha}{\beta^2}\right)T$$
(4) Momentum standards of the systems are related by:
$$P' = \left(\frac{1}{\beta^3}\right)P$$

12. The gas equation for n moles of a real gas is:
$\left(P + \frac{a}{V^2}\right)(V - b) = nRT$ where P is the pressure, V is the volume, T is the absolute temperature, R is the molar gas constant and a, b are arbitrary constants. Which of the following has/have the same dimensions as those of PV?
(1) nRT (2) a/V
(3) Pb (4) ab/V^2

13. If $y = a^3 \sin^{-1}\left(\frac{x}{b} - 1\right)$, where y is in metre. Then,
(1) the dimensions of a is $[M^0 L^{\frac{1}{3}} T^0]$
(2) the unit of $\sin^{-1}\left(\frac{x}{b} - 1\right)$ is radian
(3) the dimensions of x are same as those of b
(4) the unit of a is metre

14. The speed of a particle depends on the time t as
$v = \sqrt{AB} + Bt + \frac{C}{D+t}$. Then,
(1) A represents distance
(2) B represents acceleration
(3) C represents distance
(4) D represents time

15. If pressure p of a gas is given in terms of time t and distance x as $p = A\sin bt + B\sin ct$, then
(1) dimensions of $\frac{A}{B}$ are same as $\frac{b}{c}$
(2) dimensions of A is $[ML^{-1}T^{-2}]$
(3) dimensions of A and B are same
(4) dimensions of b and c are same

16. If $\left(A + \frac{B}{C^2}\right)(D - x) = y$, then
(1) dimensions of A and B must be same
(2) dimensions of A and B may be same
(3) dimensions of D and x must be same
(4) dimensions of AD and y must be same

17. Error in measurement of radius of a sphere is 1%, then
(1) the error in measurement of volume is 3%
(2) the error in measurement in volume is 1%
(3) the error in measurement of surface area is 2%
(4) the error in measurement of surface area is 6%

18. Error in measurement of volume of a sphere is 9%, then
(1) the error in measurement of radius is 2%
(2) the error in measurement of radius is 3%
(3) the error in measurement surface area is 4%
(4) the error in measurement surface area is 6%

19. If p, ρ, g and h denote pressure, density, acceleration due to gravity and height, respectively. In the case of flow of fluid, $\frac{p}{\rho g} + \frac{v^2}{2g} + h = u_0$, where v denotes velocity. Mark the correct option(s).
(1) the equation is dimensionally correct
(2) if dimensions of u_0 are $[M^0LT^0]$, the equation is dimensionally correct
(3) dimensions of $\frac{p}{\rho g}$ are same as that of $\frac{v^2}{2g}$
(4) none of the above

20. The pair(s) of physical quantities that has/have the same dimensions is(are)
(1) volumetric strain and coefficient of friction
(2) disintegration constant of a radioactive substance and frequency of light wave
(3) heat capacity and gravitational potential
(4) Planck's constant and torque

21. If P, Q, R are physical quantities having different dimensions, which of the following combinations can never be a meaningful quantity?
(1) $(P - Q)/R$ (2) $PQ - R$
(3) $(PR - Q^2)/R$ (4) $(R + Q)/P$

22. Photon is quantum of radiation with energy $E = h\nu$, where ν is frequency and h is Planck's constant. The dimensions of h are the same as that of
(1) linear impulse (2) angular impulse
(3) linear momentum (4) angular momentum

23. If in a new hypothetical system of units, the new unit of mass is 5 kg, new unit of length is 10 meter and new unit of time is 20 seconds, choose the correct statement(s).
(1) 1 SI unit of velocity is equal to 2 new units of velocity.
(2) 1 SI unit of power is equal to 16 new units of power.
(3) 1 SI unit of energy is equal to 5/4 new unit of energy.
(4) 1 SI unit of force is equal to 10 new units of force.

24. If the unit of velocity is 4 m s^{-1}, the unit of acceleration is 24 m s^{-2}, and the unit of force is 6 N, then choose the correct statement(s).
(1) Unit of energy is 4 J.
(2) Unit of momentum is 1 Ns.
(3) Unit of power is 24 W.
(4) Unit of momentum is 2/3 Ns.

Linked Comprehension Type

For Problems 1–5

The van der Waals' equation of state for some gases can be expressed as:

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

where P is the pressure, V is the molar volume, and T is the absolute temperature of the given sample of gas and a, b, and R are constants.

1. The dimensions of a are
(1) ML^5T^{-2} (2) $ML^{-1}T^{-2}$
(3) L^3 (4) L^6

2. The dimensions of constant b are
(1) ML^5T^{-2} (2) $ML^{-1}T^{-2}$
(3) L^3 (4) L^6

3. Which of the following does not have the same dimensional formula as that for RT?
(1) PV (2) Pb
(3) $\frac{a}{V^2}$ (4) $\frac{ab}{V^2}$

4. The dimensional representation of ab/RT is
(1) ML^5T^{-2} (2) $M^0L^3T^0$
(3) $ML^{-1}T^{-2}$ (4) None of these

5. In the above problem, the dimensional formula for RT is same as that of
(1) Energy (2) Force
(3) Specific heat (4) Latent heat

For Problems 6–8

Dimensional methods provide three major advantages in verification, derivation, and changing the system of units. Any empirical formula that is derived based on this method has to be verified and proportionality constants found by experimental means. The presence or absence of certain factors—non dimensional constants or variables—cannot be identified by this method. So every dimensionally correct relation cannot be taken as perfectly correct.

6. If α kilogram, β meter, and γ second are the fundamental units, 1 cal can be expressed in new units as [1 cal = 4.2 J]
(1) $\alpha^{-1}\beta^{2}\gamma$ (2) $\alpha^{-1}\beta^{-2}\gamma$
(3) $4.2\alpha^{-1}\beta$ (4) $4.2\alpha^{-1}\beta^{-2}\gamma^{2}$

7. The time period of oscillation of a drop depends on surface tension σ, density of the liquid ρ, and radius r. The relation is
(1) $\sqrt{\frac{\rho r^2}{\sigma}}$ (2) $\sqrt{\frac{r^2}{\rho\sigma}}$
(3) $\sqrt{\frac{r^3\rho}{\sigma}}$ (4) $\sqrt{\frac{\rho\sigma}{r^3}}$

8. The energy of an SHM is dependent on mass m, frequency f, and amplitude A of oscillation. The relation is
(1) Mf/A^2 (2) MfA^{-2}
(3) Mf^2A^{-2} (4) Mf^2A^2

For Problems 9–11

The accuracy of measurement also lies in the way the result is expressed. The number of digits to which a value is to be expressed is one digit more than number of sure numbers. Rules do exist to deal with number of digits after an operation is carried out on the given values. The error can be minimised by many trials and using the correct methods and instruments.

9. If the length and breadth are measured as 4.234 and 1.05 m, the area of the rectangle is
(1) 4.4457 m^2 (2) 4.45 m^2
(3) 4.446 m^2 (4) 0.4446 m^2

10. The order of magnitude of 147 is
(1) 1 (2) 2
(3) 3 (4) 4

11. The number of significant figures can reduce in
(1) Addition (2) Subtraction
(3) Multiplication (4) Division

For Problems 12–14

A student forgot newton's formula for speed of sound but he knows there were speed (v), pressure (p) and density (d) in the formula. He then start using dimensional analysis method to find the actual relation:

$$v = kp^x d^y$$

where k is a dimension-less constant. On the basis of above passage answer the following questions:

12. The value of x is
(1) 1 (2) 1/2
(3) −1/2 (4) 2

13. The value of y is
(1) 1 (2) 1/2
(3) −1/2 (4) 2

14. If the density increases, the speed of sound will
(1) increase (2) decrease
(3) unchanged (4) none of these

For Problems 15–17

A physical quantity X depends on another physical quantities as $X = YFe^{-\beta r^2} + ZW\sin(\alpha r)$ where r, F and W represents distance, force and work respectively and Y and Z are unknown physical quantities and α, β are positive constants.

15. If Y represents displacement, then $\dim\left(\frac{\alpha YZ}{\beta F}\right)$ is equal to
(1) $M^{-1}LT^{2}$ (2) $M^{-1}L^{2}T^{-2}$
(3) $M^{1}L^{1}T^{-2}$ (4) None of these

16. If Y represents velocity, then dim (X) is equal to
(1) $ML^{2}T^{-3}$ (2) $M^{-1}L^{2}T^{-3}$
(3) $ML^{2}T^{-2}$ (4) None of these

17. If Z represents frequency, then choose the correct alternative.
(1) The dimension of X is $[ML^{1}T^{-3}]$
(2) The dimension of Y is $[M^{0}LT^{-1}]$
(3) The dimension of β is $[M^{0}L^{-1}T^{0}]$
(4) The dimension of α is $[M^{0}L^{1}T^{0}]$

For Problems 18 and 19

Suppose a student is trying to make a measurement system named mach system, so that he can use it like a code measurement system. If he takes unit of mass as A kg, the unit of length as B meter, the unit of time as C second and the unit of charge as d coulomb.

18. In mach system, 1 ohm is
(1) $[A^{-1}B^{-2}Cd^{-2}]$ (2) $[ABCd^{-2}]$
(3) $[A^2B^2Cd^2]$ (4) $[A^{-1}B^2C^{-1}d^2]$

19. In mach system, 1 farad is
(1) $[ABC^2d^{-2}]$ (2) $[AB^2C^{-2}d^{-2}]$
(3) $[A^{-2}B^2C^{-2}d^{-2}]$ (4) $[A^{-2}BC^{-2}d^{-3}]$

Matrix Match Type

1. If R is resistance, L is inductance, C is capacitance, H is latent heat, and s is specific heat, then match the quantity given in Column I with the dimensions given in Column II.

Column I	Column II
i. LC	a. L^2T^{-2}
ii. LR	b. $L^2T^{-2}K^{-1}$
iii. H	c. T^2
iv. s	d. $M^2L^4T^{-5}A^{-4}$

2. There are four Vernier scales, whose specifications are given in Column I and the least count is given in Column II. Match the Columns I and II with correct specification and corresponding least count (s = value of main scale division, n = number of marks on Vernier). Assume $(n-1)$ main scale divisions are equal to n Vernier divisions.

Column I	Column II
i. $s = 1$ mm, $n = 10$	a. 0.05 mm
ii. $s = 0.5$ mm, $n = 10$	b. 0.01 mm
iii. $s = 0.5$ mm, $n = 20$	c. 0.1 mm
iv. $s = 1$ mm, $n = 100$	d. 0.025 mm

3. Match the columns.

Column I	Column II
i. Backlash error	a. Always subtracted
ii. Zero error	b. Least count = 1 MSD − 1 VSD
iii. Vernier callipers	c. May be negative or positive
iv. Error in screw gauge	d. Due to loose fittings

4. Using significant figures, match the following

Column I	Column II
i. 0.12345	a. 5
ii. 0.12100 cm	b. 4
iii. 47.23 ÷ 2.3	c. 1
iv. 3×10^8	d. 2

5. Suppose two students are trying to make a new measurement system so that they can use it like a code measurement system and others do not understand it. Instead of taking 1 kg, 1m and 1 sec as basic unit they took unit of mass as a kg, the unit of length as b m and unit of time as γ second. They called power in new system as ACME. Then match the two columns.

Column I	Column II
i. 1N in new system	a. $\alpha^{-1}\beta^{-2}\gamma^2$
ii. 1J in new system	b. $\alpha^{-1}\beta^{-1}\gamma^2$
iii. 1 Pascal (SI unit of pressure) in new system	c. $\alpha^{-1}\beta\gamma^2$
iv. a ACME in watt	d. $\alpha^2\beta^2\gamma^{-3}$

Codes:

	i.	ii.	iii.	iv.
(1)	b	a	c	d
(2)	c	b	d	a
(3)	a	b	d	c
(4)	c	d	a	b

. Match the column I with the column II and mark the correct option from the given codes.
Some physical quantities are given in column I and some SI units in which these quantities may be expressed are given in column II. Match the physical quantities in column II and select the code.

Column I	Column II
i. Gravitational field intensity $I = \frac{GM_e}{R_e^2}$; where G = gravitational constant, M_e = mass of the earth, R_e = radius of the earth	a. $\frac{\text{Joule}}{\text{mol-Kelvin}}$
ii. Electric field intensity, $E = \frac{F}{Q}$ Where F = force due to charge, Q = electric charge	b. $\frac{\text{Newton}}{\text{kg}}$
iii. $R = \frac{V}{I}$; where V = potential difference across a resistance, I = current in the resistance	c. $\frac{\text{Newton}}{\text{Coulomb}}$
iv. Universal gas constant $R = \frac{PV}{nR}$.	d. $\frac{\text{Volt}}{\text{Ampere}}$

Codes:

	i.	ii.	iii.	iv.
(1)	b	a	c	d
(2)	c	b	d	a
(3)	b	c	d	a
(4)	c	d	a	b

7. Match the Column I with Column II and mark the correct from the given codes.

Column I	Column II
i. Modulus of rigidity	a. Voltmeter per ampere
ii. Latent heat	b. Ohm-metre
iii. Electric resistivity	c. Joule per kg
iv. Gravitational potential	d. Newton per metre^2

Codes:

	i	ii	iii	iv
(1)	d	c	a, b	c
(2)	a	b	c	d
(3)	b	d	c	a
(4)	a	d	c	a

8. If β = bulk modulus of electricity, A = amplitude, K = angular wave number, r = radius, Y = Young's modulus, m = mass, ω = angular frequency, F = force, and S = area, match Column I with Column II and mark the correct option from the codes given below.

Column I	Column II
i. βAK	a. Watt/metre^2
ii. $\rho A^2\omega^2 C$	b. Joule
iii. $\frac{F^2 l}{SY}$	c. Newton per metre^2
iv. $\sqrt{\frac{r^3}{GM}}$	d. Second

Codes:

	i	ii	iii	iv
(1)	a	b	c	d
(2)	c	a	b	d
(3)	b	a	d	c
(4)	a, b	c, d	a	b

Numerical Value Type

1. If equation $\int \frac{dt}{\sqrt{3at-2t^2}} = a^x \sin^{-1}\left(\frac{t^2}{a^2}-1\right)$, find the value of x.

2. If unit of mass becomes 2 times, the unit of length becomes 4 times and the unit of time becomes 4 times in the unit of Planck's constant. Due to this unit of Planck's constant becomes n times. Find the value of n.

3. A stone is lying at rest in a river. The minimum mass of stone, $m = k\rho v^x g^{-3}$ is needed for remaining at rest. Here, k = constant having no unit, g = acceleration due to gravity v = river flow velocity, ρ = density of water. Find the value of x.

4. If the unit of velocity is run, the unit of time is second and unit of force is strength in a hypothetical system of unit. In this system of unit, the unit of mass is (strength (second)^y (run)^z. Find the value of y/x.

5. If force F, velocity v and time T are taken as fundamental units. Find the dimension of force in the dimensional formula of pressure.

6. $\int \frac{dx}{\left(\frac{1}{a}\right)^2 + x^2} = a^n + \tan^{-1}(ax)$. Find the value of n.

7. If volume is written as, $V = Kg^x c^y h^z$. Here, K is dimensionless constant and g, c, h are gravitational constant, speed of light and Planck's constant, respectively. Find the value of x/z.

8. Acceleration due to gravity on the surface of the earth is $g = \frac{GM}{R^2}$. The gravitational constant G is exactly known. But percentage error in measurement of the mass of earth M and radius of the earth R are 1% and 2%. Respectively. The maximum percentage error is measurement of acceleration due to gravity on the surface of the earth in n%. Find the value of n.

9. During measurement of kinetic energy T, the percentage error in measurement of mass of particle and momentum of particle are 2% and 3%., respectively. The percentage error in measurement of kinetic energy is n%. Find the value of n.

10. If $y = 2.21 \times 0.3$, then find the number of significant digits in the value of y.

11. If $x = 0.72 + 0.8 + 3.87 - 1.089$, then find number of significant digits in the value of x.

12. The length of a wire is 2.17 cm and radius is 0.46 cm. Find number of significant digits in the value of volume of wire.

13. The density of a material in CGS system of units is 4 g/cc. In a system of units in which unit of length is 2 cm and unit of mass is 16 g, find the numerical value of density of material.

14. In a new system of units mass, acceleration and frequency are taken as fundamental units. If unit of mass is 100 g, unit of acceleration is 2 m/s^2 and unit of frequency is 4 sec^{-1} in the new system of units, then find the value of 0.1 J in this system.

15. In a new system of units, the net force applied on a block of mass 10 kg moving with acceleration 10 m/s^2 is given as 100 unit of force. When the same block is moving at a speed of 20 m/s, its kinetic energy become 20 unit of energy. A liquid has a surface tension of 10 SI units. What is the magnitude of its surface tension in new system? (Dimensional formula of surface tension is ML^0T^{-2}.)

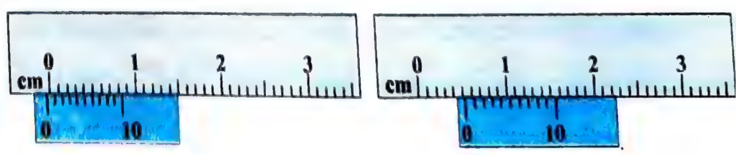
16. In an ancient civilization, the unit of length was half of meter, the unit of mass was double the kilogram and the unit of time was half of second. Find the ratio of their unit of energy and Joule.

17. In a new system of unit, 1 unit of mass is 10 kg, 1 unit of length is 10 m and 1 unit of time is 1 s. If 1 unit of power in this unit is equal to x Watt, find x.

18. An experiment was conducted to measure the velocity of water splash when a car passes over a water layer on the road. It is found that velocity depends on the weight of car. power delivered by its engine and thickness of water layer. The velocity of splash was found to be 3 m/s when a car weighing 1000 N passes over a layer of thickness 3 mm and its engine is supplying a power of 800 kW. Find the velocity (in m/s) of splash when car weighing 500 N passes over a layer of thickness 4mm and its engine is supplying a power of just 400 kW.

19. In a new system of units, the unit of mass is 100 g, unit of length is 4 m and unit of time is 2 s. Find the numerical value of 10 J in this system.

20. The main scale of a Vernier calipers reads in millimeter and its Vernier is divided into 10 divisions which coincides with 9 divisions of the main scale. Find the reading for shown situation in mm.



Archives

JEE MAIN

Single Correct Answer Type

1. In an experiment, the angles are required to be measured using an instrument. 29 divisions of the main scale exactly coincide with 30 divisions of the vernier scale. If the smallest division of the main scale is half-a-degree (=0.5°), then the least count of the instrument is

(1) one minute (2) half minute
(3) one degree (4) half degree

(AIEEE 2009)

2. Resistance of a given wire is obtained by measuring the current flowing in it and the voltage difference applied across it. If the percentage errors in the measurements of the current and the voltage difference are 3% each, then error in the value of resistance of the wire is:

(1) 6% (2) zero
(3) 1% (4) 3%

(AIEEE 2012)

3. Let $[\varepsilon_0]$ denote the dimensional formula of the permittivity of vacuum. If M = mass, L = length, T = time and A = electric current, then:

(1) $[\varepsilon_0]=[M^{-1}L^{-3}T^4A^2]$ (2) $[\varepsilon_0]=[M^{-1}L^2T^1A^{-2}]$
(3) $[\varepsilon_0]=[M^{-1}L^2T^1A]$ (4) $[\varepsilon_0]=[M^{-1}L^{-3}T^2A]$

(JEE Main 2013)

4. A student measures the time period of 100 oscillations of a simple pendulum four times. The data set is 90 s, 91 s, 95 s and 92 s. If the minimum division in the measuring clock is 1 s, then the reported mean time should be:

(1) 92 ± 2 s (2) 92 ± 5.0 s
(3) 92 ± 1.8 s (4) 92 ± 3 s

(JEE Main 2016)

5. A screw gauge with a pitch of 0.5 mm and a circular scale with 50 divisions is used to measure the thickness of a thin sheet of Aluminium. Before starting the measurement, it is found that when the two jaws of the screw gauge are brought in contact, the 45th division coincides with the main scale line and that the zero of the main scale is barely visible. What is the thickness of the sheet if the main scale reading is 0.5 mm and the 25th division coincides the main scale line?

(1) 0.75 mm (2) 0.80 mm
(3) 0.70 mm (4) 0.50 mm

(JEE Main 2016)

6. The density of a material in the shape of a cube is determined by measuring three sides of the cube and its mass. If the relative errors in measuring the mass and length are respectively 1.5 % and 1 %, the maximum error in determining the density is

(1) 6% (2) 2.5%
(3) 3.5% (4) 4.5%

(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. A Vernier callipers has 1 mm marks on the main scale. It has 20 equal divisions on the vernier scale, which match with 16 main scale divisions. For this Vernier callipers, the least count is

(1) 0.02 mm (2) 0.05 mm
(3) 0.1 mm (4) 0.2 mm

(IIT JEE 2010)

2. The density of a solid ball is to be determined in an experiment in an experiment. The diameter of the ball is measured with a screw gauge, whose pitch is 0.5 mm and there are 50 divisions on the circular scale. The reading on the main scale is 2.5 mm and that on the circular scale is 20 divisions. If the measured mass of the ball has a relative error of 2%, the relative percentage error in the density is

(1) 0.9% (2) 2.4%
(3) 3.1% (4) 4.2%

(IIT JEE 2011)

3. In the determination of Young's modulus $\left(Y=\dfrac{4MLg}{\pi l d^2}\right)$ by using Searle's method, a wire of length $L = 2$ m and diameter $d = 0.5$ mm is used. For a load $M = 2.5$ kg, an extension $l = 0.25$ mm in the length of the wire is observed. Quantities d and l are measured using a screw gauge and a micrometer, respectively. They have the same pitch of 0.5 mm. The number of divisions on their circular scale is 100. The contributions to the maximum probable error of the Y measurement

(1) due to the errors in the measurements of d and l are the same.
(2) due to the error in the measurement of d is twice that due to the error in the measurement of l.
(3) due to the error in the measurement of l is twice that due to the error in the measurement of d.
(4) due to the error in the measurement of d is four times that due to the error in the measurement of l.

(IIT JEE 2012)

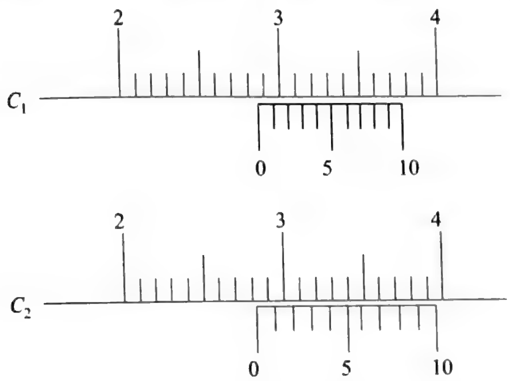
4. The diameter of a cylinder is measured using a Vernier callipers with no zero error. It is found that the zero of the Vernier scale lies between 5.10 cm and 5.15 cm of the main scale. The Vernier scale has 50 divisions equivalent to 2.45 cm. The 24th division of the Vernier scale exactly coincides with one of the main scale divisions. The diameter of the cylinder is

(1) 5.112 cm (2) 5.124 cm
(3) 5.136 cm (4) 5.148 cm

(JEE Advanced 2013)

5. Using the expression $2d \sin \theta = \lambda$, one calculates the values of d by measuring the corresponding angles θ in the range 0° to 90°. The wavelength λ is exactly known and the error in θ is constant for all values of θ. As θ increases from 0°,
(1) the absolute error in d remains constant.
(2) the absolute error in d increases.
(3) the fractional error in d remains constant.
(4) the fractional error in d decreases.
(JEE Advanced 2013)

6. There are two Vernier calipers both of which have 1 cm divided into 10 equal divisions on the main scale. The Vernier scale of one of the calipers (C_1) has 10 equal divisions that correspond to 9 main scale divisions. The Vernier scale of the other caliper (C_2) has 10 equal divisions that correspond to 11 main scale divisions. The readings of the two calipers are shown in the figure. The measured values (in cm) by calipers C_1 and C_2, respectively, are



(1) 2.87 and 2.87 (2) 2.87 and 2.86
(3) 2.87 and 2.83 (4) 2.85 and 2.82
(JEE Advanced 2016)

7. A person measures the depth of a well by measuring the time interval between dropping a stone and receiving the sound of impact with the bottom of the well. The error in his measurement of time is $\delta T = 0.01$ second and he measures the depth of the well to be $L = 20$ meters. Take the acceleration due to gravity $g = 10$ ms^{-2} and the velocity of sound is 300 ms^{-1}. Then the fractional error in the measurement, $\delta L/L$, is closest to
(1) 0.2% (2) 5%
(3) 3% (4) 1%
(JEE Advanced 2017)

Multiple Correct Answers Type

1. A student uses a simple pendulum of exactly 1 m length to determine g, the acceleration due to gravity. He uses a stop watch with the least count of 1 s for this and records 40 s for 20 oscillations. For this observation, which of the following statement(s) is(are) true?
(1) Error ΔT in measuring T, the time period, is 0.05 s
(2) Error ΔT in measuring T, the time period, is 1 s
(3) Percentage error in the determination of g is 5%
(4) Percentage error in the determination of g is 2.5%
(IIT JEE 2010)

2. Plank's constant h, speed of light c and gravitational constant G are used to from a unit of length L and a unit of mass M. Then the correct options(s) is (are)
(1) $M \propto \sqrt{c}$ (2) $M \propto \sqrt{G}$
(3) $L \propto \sqrt{h}$ (4) $L \propto \sqrt{G}$
(JEE Advanced 2015)

3. In terms of potential difference V, electric current I, permittivity ε_0, permeability μ_0 and speed of light c, the dimensionally correct equations(s) is (are)
(1) $\mu_0 I^2 = \varepsilon V^2$ (2) $\mu_0 I = \mu_0 V$
(3) $I = \varepsilon_0 V$ (4) $\mu_0 c I = \varepsilon_0 V$
(JEE Advanced 2015)

4. A length-scale (l) depends on the permittivity (ε) of a dielectric material, Boltzmann constant k_B, the absolute temperature T, the number per unit volume (n) of certain charged particles, and the charge (q) carried by each of the particles, Which of the following expressions(s) for l is(are) dimensionally correct?
(1) $l = \sqrt{\left(\dfrac{nq^2}{\varepsilon k_B T}\right)}$ (2) $l = \sqrt{\left(\dfrac{\varepsilon k_B T}{nq^2}\right)}$
(3) $l = \sqrt{\left(\dfrac{q^2}{\varepsilon n^{2/3} k_B T}\right)}$ (4) $l = \sqrt{\left(\dfrac{q^2}{\varepsilon n^{1/3} k_B T}\right)}$
(JEE Advanced 2016)

5. In an experiment to determine the acceleration due to gravity g, the formula used for the time period of a period of a periodic motion is $T = 2\pi\sqrt{\dfrac{7(R-r)}{5g}}$. The values of R and r are measured to be (60 ± 1) mm and (10 ± 1) mm respectively. In five successive measurements, the time period is found to be 0.52 s, 0.56 s, 0.57 s, 0.54 s and 0.59 s. The least count of the watch used for the measurement of time period is 0.01 s. Which of the following statement(s) is (are) true?
(1) The error in the measurement of r is 10%
(2) The error in the measurement of T is 3.57%
(3) The error in the measurement of T is 2%
(4) The error in the determined value of g is 11%
(JEE Advanced 2016)

Linked Comprehension Type

For Questions 1 and 2

In electromagnetic theory, the electric and magnetic phenomena are related to each other. Therefore, the dimensions of electric and magnetic quantities must also be related to each other. In the questions below, $[E]$ and $[B]$ stand for dimensions of electric and magnetic fields respectively, while $[\epsilon_0]$ and $[\mu_0]$ stand for dimensions of the permittivity and permeability of free space, respectively. $[L]$ and $[T]$ are dimensions of length and time, respectively. All the quantities are given in SI units.
(JEE Advanced 2018)

1. The relation between $[E]$ and $[B]$ is
(1) $[E]=[B][L][T]$ (2) $[E]=[B][L]^{-1}[T]$
(3) $[E]=[B][L][T]^{-1}$ (4) $[E]=[B][L]^{-1}[T]^{-1}$

2. The relation between $[\epsilon_0]$ and $[\mu_0]$ is
(1) $[\mu_0]=[\varepsilon_0][L]^2[T]^{-2}$
(2) $[\mu_0]=[\varepsilon_0][L]^{-2}[T]^2$
(3) $[\mu_0]=[\varepsilon_0]^{-1}[L]^2[T]^{-2}$
(4) $[\mu_0]=[\varepsilon_0]^{-1}[L]^{-2}[T]^2$

Matrix Match Type

1. Match List I with List II and select the correct answer using the codes given below the lists:

	List I		List II
p.	Boltzmann constant	1.	$[ML^2T^{-1}]$
q.	Coefficient of viscosity	2.	$[ML^{-1}T^{-1}]$
r.	Planck constant	3.	$[MLT^{-3}K^{-1}]$
s.	Thermal conductivity	4.	$[ML^2T^{-2}K^{-1}]$

Codes:

	p	q	r	s
(1)	3	1	2	4
(2)	3	2	1	4
(3)	4	2	1	3
(4)	4	1	2	3

(JEE Advanced 2013)

Numerical Value Type

1. To find the distance d over which a signal can be seen clearly in foggy conditions, a railways engineer uses dimensional analysis and assumes that the distance depends on the mass density ρ of the fog, intensity (power/area) S of the light from the signal and its frequency f. The engineer finds that d is proportional to $S^{1/n}$. The values of n is

(JEE Advanced 2014)

2. The energy of a system as a function of time t is given as $E(t) = A^2 \exp(-at)$, wher $\alpha = 0.2\ s^{-1}$. The measurement of A has an errror of 1.25%. If the error in the measurement of time is 1.50%, the percentage error in the value of $E(t)$ at $t = 5$ s is

(JEE Advanced 2015)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2)	2. (4)	3. (3)	4. (4)	5. (3)
6. (4)	7. (4)	8. (3)	9. (3)	10. (4)
11. (2)	12. (4)	13. (2)	14. (4)	15. (4)
16. (3)	17. (2)	18. (4)	19. (3)	20. (4)
21. (2)	22. (4)	23. (1)	24. (2)	25. (2)
26. (4)	27. (4)	28. (4)	29. (1)	30. (3)
31. (4)	32. (2)	33. (2)	34. (1)	35. (2)
36. (4)	37. (1)	38. (3)	39. (1)	40. (4)
41. (2)	42. (4)	43. (4)	44. (3)	45. (4)
46. (4)	47. (1)	48. (4)	49. (4)	50. (4)
51. (4)	52. (4)	53. (4)	54. (2)	55. (3)
56. (2)	57. (3)	58. (1)	59. (4)	60. (3)
61. (4)	62. (1)	63. (1)	64. (1)	

Multiple Correct Answers Type

1. (1),(2),(4)	2. (2),(3),(4)
3. (1),(2),(3)	4. (2),(4)
5. (1),(2),(4)	6. (1),(2),(3)
7. (1),(3)	8. (1),(3)
9. (1),(2),(3)	10. (1),(2)
11. (1),(2),(3),(4)	12. (1),(2),(3),(4)
13. (1),(2),(3)	14. (1),(2),(3),(4)
15. (1),(2),(3),(4)	16. (2),(3),(4)
17. (1),(3)	18. (2),(4)
19. (2),(3)	20. (1),(2),(3)
21. (1),(4)	22. (2),(4)
23. (1),(2)	24. (1),(2),(3),(4)

Linked Comprehension Type

1. (1)	2. (3)	3. (3)	4. (4)	5. (1)
6. (4)	7. (3)	8. (4)	9. (2)	10. (2)
11. (2)	12. (2)	13. (3)	14. (2)	15. (1)
16. (1)	17. (2)	18. (1)	19. (2)	

Matrix Match Type

1. i $\rightarrow$ c; ii $\rightarrow$ d; iii $\rightarrow$ a; iv $\rightarrow$ b
2. i $\rightarrow$ c; ii $\rightarrow$ a; iii $\rightarrow$ d; iv $\rightarrow$ b
3. i $\rightarrow$ d; ii $\rightarrow$ a, c; iii $\rightarrow$ b; iv $\rightarrow$ c, d
4. i $\rightarrow$ a; ii $\rightarrow$ a; iii $\rightarrow$ d; iv $\rightarrow$ c

5. (1) 6. (3) 7. (1) 8. (2)

Numerical Value Type

1. (0)	2. (8)	3. (6)	4. (1)	5. (1)
6. (1)	7. (1)	8. (5)	9. (7)	10. (1)
11. (2)	12. (2)	13. (2)	14. (4)	15. (1000)
16. (0.50)	17. (1000)	18. (3)	19. (25)	20. (6.9)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (1) 2. (3) 3. (1) 4. (1) 5. (2)
6. (4)

JEE Advanced

Single Correct Answer Type

1. (4) 2. (3) 3. (1) 4. (2) 5. (4)
6. (3) 7. (4)

Multiple Correct Answers Type

1. (1),(3) 2. (1),(3),(4) 3. (1),(3) 4. (2),(4)
5. (1),(2),(4)

Linked Comprehension Type

1. (3) 2. (4)

Matrix Match Type

1. p $\rightarrow$ 4; q $\rightarrow$ 2; r $\rightarrow$ 1; s $\rightarrow$ 3

Numerical Value Type

1. (3) 2. (4)

2 Basic Mathematics

INTRODUCTION

Mathematics is the supporting tool for physics. The elementary knowledge of basic maths is useful in problem-solving in physics. The basic knowledge of elementary algebra, trigonometry, coordinate geometry, and calculus is must before going into the depth of physics.

ELEMENTARY ALGEBRA

COMMON FORMULAE

1. $(a+b)^2 = a^2 + b^2 + 2ab$
2. $(a-b)^2 = a^2 + b^2 - 2ab$
3. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
4. $(a+b)(a-b) = a^2 - b^2$
5. $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
6. $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
7. $(a+b)^2 - (a-b)^2 = 4ab$
8. $(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$

POLYNOMIAL, LINEAR, AND QUADRATIC EQUATIONS

Real Polynomial

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and x be a real variable, then $f(x, y) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is called a *real polynomial.*

Degree or index of a polynomial: The highest power appearing in a polynomial is called its degree. For example, $f(x) = x^3 + 8x + 3$ is a polynomial of degree 3.

Students must note here that it is not necessary that the highest power must be of a single variable only. For example, $f(x, y) = 3x^2y + y^2 + 2$ is a polynomial of degree 3 because of the variable y in the term x^2y. We add the powers of the variables in a term to find the degree of a polynomial irrespective of the nature of variables. Thus, in the present case, x^2y has power equal to $2 + 1 = 3$. Hence, the degree of the given polynomial is 3.

Linear Equations

Equations having terms of unit degree are called linear equations, e.g., $x + y = 2$ or $2x + 3 = 5$. Such equations always represent a straight line on the graph.

Quadratic Equations

Equations of second degree are called quadratic equations. The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$.

Roots of a quadratic equation: The solutions to a quadratic equation are called its roots. Roots are those values of a variable such as x for which the given quadratic equation reduces to zero. As a rule, a quadratic equation always has two roots, which may or may not be equal.

The roots of a quadratic equation are generally represented by a and b. Let $ax^2 + bx + c = 0$ be a quadratic equation.

1. Its roots are $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$; $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

   Hence, its solution is given by $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.

2. Sum of its roots is given by $\alpha + \beta = \dfrac{-b}{a}$.
3. Product of its roots is given by $\alpha\beta = \dfrac{c}{a}$.
4. Difference of its roots is given by $\alpha - \beta = \dfrac{\sqrt{b^2-4ac}}{a}$.

BINOMIAL EXPRESSION AND THEOREM

An algebraic expression containing two terms is called a **binomial expression**. For example, $(a+b)$, $(2x-3y)$, $\left(x+\dfrac{1}{y}\right)$, $\left(x+\dfrac{3}{x}\right)$, etc., are binomial expressions.

Binomial Theorem for Positive Integral Index

The general form of a binomial expression is $(x+a)^n$, where n is any positive integer (called index) and x and a are real numbers. Binomial theorem states

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}\cdot a^1 + {}^nC_2x^{n-2}\cdot a^2 + \cdots + {}^nC_rx^{n-r}\cdot a^r + \cdots + {}^nC_n \cdot a^n$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and $n! = n(n-1)(n-2)\ldots \times 3 \times 2 \times 1$ is the product of first n natural numbers (e.g., $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$).

Important Points:

- The total number of terms in the expansion is $n + 1$.
- In every successive term in the expansion, the power of x decreases by 1 and that of a increases by 1, so that the sum of the powers of x and a in each term is always equal to n.

- ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients.
- The expression $n!$ is read as "factorial n." So,

$${}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n(n-1)(n-2)\cdots 3\times 2\times 1}{1(n-1)(n-2)\cdots 3\times 2\times 1} = n$$

Similarly, $${}^nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2\times 1}, \ldots, {}^nC_n = 1$$

Binomial Theorem for Any Index

If n is positive, negative, or fraction and x is any real number such that $-1 < x < 1$, i.e., x lies between -1 and $+1$, then according to the binomial theorem,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \infty \text{ terms}$$

Note

- If n is a positive integer, then the expansion will have $(n+1)$ terms.
- If n is a negative integer or a fraction, then the number of terms in the expansion will be infinite, i.e, there will be no last term.
- If $|x| \ll 1$, then only the first two terms of the expansion are significant because the values of higher order terms become very small and can be neglected. Thus, in this case, the binomial expansion reduces to the following simplified forms when $|x| \ll 1$:
  $(1+x)^n = 1 + nx$, $(1+x)^{-n} = 1 - nx$,
  $(1-x)^n = 1 - nx$, $(1-x)^{-n} = 1 + nx$

ILLUSTRATION 2.1

Calculate $(1001)^{1/3}$.

Sol. We can write 1001 as $1001 = 1000\left(1+\frac{1}{1000}\right)$, so that we have

$$(1001)^{1/3} = \left[1000\left(1+\frac{1}{1000}\right)\right]^{1/3} = 10\left[1+\frac{1}{1000}\right]^{1/3}$$

$$= 10(1+0.001)^{1/3} = 10\left(1+\frac{1}{3}\times 0.001\right)$$

$$= 10.003333$$

ILLUSTRATION 2.2

Expand $(1+x)^{-3}$.

Sol. $$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-3-1)x^2}{2!} + \frac{(-3)(-3-1)(-3-2)}{3!}x^3 + \cdots$$

$$= 1 - 3x + \frac{12}{2}x^2 - \frac{60}{3\times 2}x^3 + \cdots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \cdots$$

ILLUSTRATION 2.3

The value of acceleration due to gravity (g) at height h above the surface of earth is given by $g' = \frac{gR^2}{(R+h)^2}$. If $h << R$, then prove that $g' = g\left(1-\frac{2h}{R}\right)$.

Sol. We are given

$$g' = g\left(\frac{R}{R+h}\right)^2 = g\left(\frac{1}{1+\frac{h}{R}}\right)^2 = \frac{g}{\left(1+\frac{h}{R}\right)^2} = g\left(1+\frac{h}{R}\right)^{-2}$$

As we are given $h << R$, hence, $h/R \ll 1$

We can write $g' = g\left(1-\frac{2h}{R}\right)$

CONCEPT APPLICATION EXERCISE 2.1

1. Expand $(1+x)^{-2}$.
2. Using binomial expansion, simplify the expressi[on] $Q\left[\left(1+\frac{\Delta x}{x}\right)^3 - 1\right]$, assuming Δx to be small in compari[son] to x.
3. Write down the first four terms of the expansion $\left(1+x^2\right)^{-1}$
4. If x is small enough, show that $\frac{1+x}{1-x} \approx 1 + 2x$.
5. The mass m of a body moving with a velocity v is given b[y] $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where m_0 = rest mass of body = 10 kg and c = speed of light $= 3\times 10^8$ m/s. Find the value of m at $v = 3\times 10^7$ m/s.

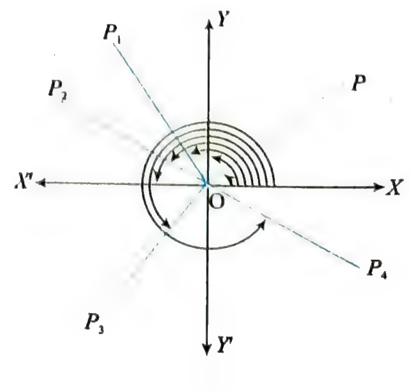
ANSWERS

1. $1-2x+3x^2-4x^3$	**2.** $3Q\Delta x/x$	
3. $1-2x^2+3x^4-4x^6$	**4.** $1+2x$	**5.** 10.05 kg

ELEMENTARY TRIGONOMETRY

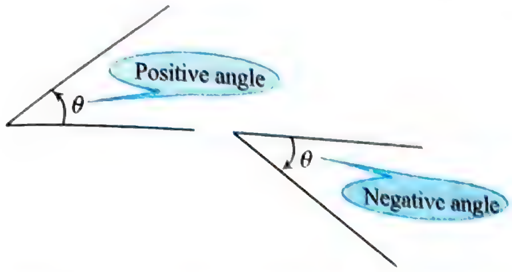
ANGLE AND SYSTEMS OF MEASUREMENT OF AN ANGLE

An angle is measured by the amount of revolution that a revolvin[g] line makes in passing from its initial position to its final positio[n]. For example, in the given figure, when line OP again coincide[s] with OX, after making a complete revolution and having passe[d] through the stages, etc., we say that it has traced out an angl[e] equal to 360° or four right angles.



Positive and Negative Angle

Angle is taken positive when rotation is in anticlockwise direction and angle is taken negative when rotation is in clockwise direction.



System of Measurement of an Angle

There are different types of measurement of an angle.

Sexagesimal system: In this system,

1 right angle = 90° (degree); 1 degree = 60′ (minutes) 1 minute = 60″ (seconds)

Centesimal system: In this system,

1 right angle = 100 grades (100 g)

1 g = 100′

1 = 100″

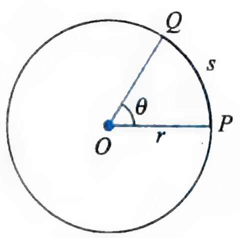
Circular system: In this system, angle is measured in rad.

π rad = 180°

Consider a particle moving from position P to position Q along a circle of radius r with center at O (see figure). Then,

$$\text{Angle } \theta = \frac{\text{Arc length } PQ}{\text{Radius of circle}} = \frac{s}{r} \Rightarrow s = r\theta$$

If the length of the arc PQ is equal to the radius r of the circle, then $\theta = 1$ rad.



Rad:

When a body completes one revolution, $\theta = 2\pi$ rad. So,

$$2\pi \text{ rad} = 360°$$

or $\quad 2 \times 3.14 \text{ rad} = 360°$

or $\quad 1 \text{ rad} = \dfrac{360°}{2 \times 3.14} = 57.3°$

ILLUSTRATION 2.4

Convert 45° to radians.

Sol. As we know 180° is equal to π rad, hence,

$$1° = \frac{\pi}{180°} \text{ rad}$$

$$\Rightarrow \quad 45° = 45° \times \frac{\pi}{180°} = \frac{\pi}{4} \text{ rad}$$

ILLUSTRATION 2.5

Convert $\pi/6$ rad to degrees.

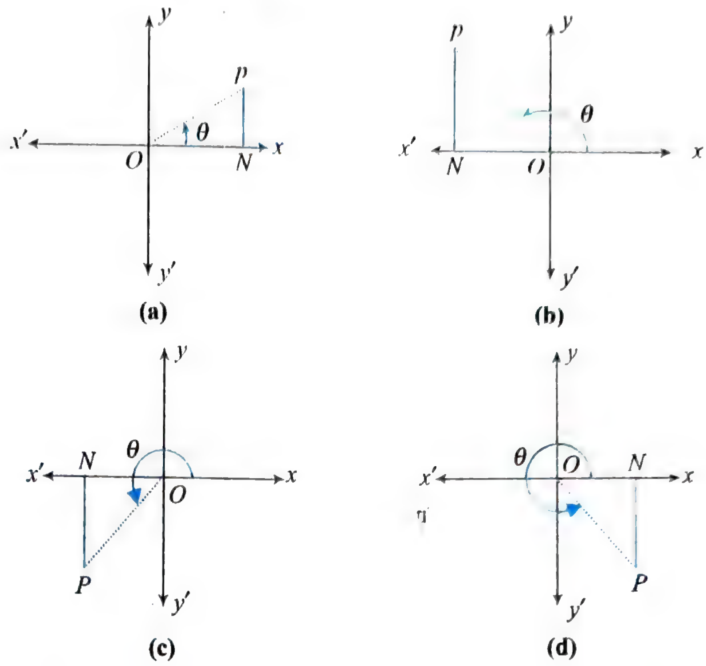
Sol. As we know π rad is equal to 180°, hence,

$$1 \text{ rad} = \frac{180}{\pi} \text{ degree}$$

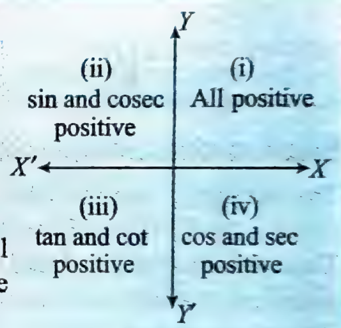
$$\Rightarrow \quad \frac{\pi}{6} \text{ rad} = \frac{\pi}{6} \times \frac{180}{\pi} = 30°$$

FOUR QUADRANTS AND SIGN CONVENTIONS

Consider two mutually perpendicular lines intersecting at O. These two mutually perpendicular lines divide the plane into four parts called quadrants (see Figure).



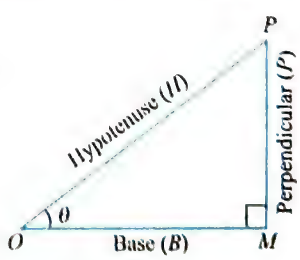
Important Points:



1. To determine the sign of a trigonometrical ratio in any quadrant, OP is taken as positive in all the four quadrants (see Figure).
2. In the first quadrant, all trigonometrical ratios are positive.
3. In the second quadrant, only sin θ and cosec θ are positive.
4. In the third quadrant, only tan θ and cot θ are positive.
5. In the fourth quadrant, only cos θ and sec θ are positive.
6. The values of sin θ and cos θ are such that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$. But tan θ and cot θ can take any real value.

LIMITS OF THE VALUES OF TRIGONOMETRIC RATIOS

In a right angled ΔOPM, perpendicular side is MP is never greater than hypotenuse OP.



As such, is never greater than 1, i.e., sin θ is never greater than 1.

In short, $-1 \le \sin\theta \le 1$ which means: sin θ always lies between -1 and $+1$.

Similarly, base OM is never greater than hypotenuse OP.

As such, OM/OP is never greater than 1,

i.e., cosθ is never greater than 1.

In short, $-1 \le \sin\theta \le 1$ which means $\cos\theta$ lies between -1 and $+1$.

Further, sides MP and OM can have any relative magnitudes depending upon the angle θ and as such $\tan\theta$ and $\cot\theta$ can have any real value.

Commonly Used Values of Trigonometric Functions

Angle (θ)	0°	30°	37°	45°	53°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

Find the approximate values of (i) sin 1° (ii) tan 2° (iii) cos 1°.

(i) $$\sin 1° = \sin\left(1° \times \frac{\pi}{180°}\right) = \sin\frac{\pi}{180°} \approx \frac{\pi}{180°}$$

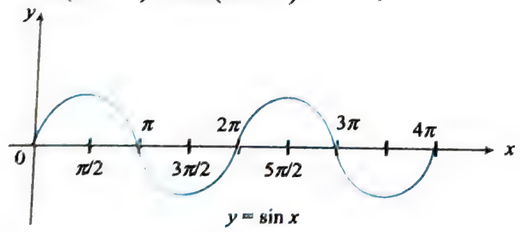
(ii) $$\tan 2° = \tan\left(2° \times \frac{\pi}{180°}\right) = \tan\frac{\pi}{90°} \approx \frac{\pi}{90°}$$

(iii) $$\cos 1° = \cos\left(1° \times \frac{\pi}{180°}\right) = \cos\frac{\pi}{180°} = 1$$

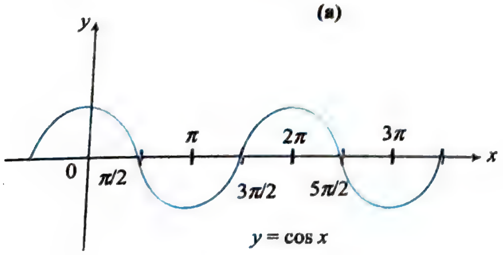
Graphs of Sine and Cosine Functions

The function $y = \sin x$, where x is any dimensionless quantity, is called a sine function. The argument x is usually measured in rad. The function $y = \sin x$ is plotted in figure(a). The maximum positive and negative values of a sine function are $+1$ and -1, respectively. Between $x = 0$ and $x = \pi$, the function is positive, and the peak value of $+1$ occurs at $x = \pi/2$. Similarly, for the interval $x = \pi$ to $x = 2\pi$, the function is negative, and the negative peak occurs at $x = 3\pi/2$. The sine function is a periodic function with a period of 2π. That is, the pattern of the graph repeats itself after an interval of 2π. Mathematically, it may be stated as

$y = \sin x = \sin(2\pi + x) = \sin(4\pi + x) = \cdots \infty$.



(a)



(b)

If the graph of the sine function is displaced to the left through $\pi/2$, we get the graph of the cosine function $y = \cos x$ as shown in Figure. The cosine function is also a periodic function with a period of 2π.

Some Important Trigonometric Formulae

1. (a) $\sin^2\theta + \cos^2\theta = 1$
   (b) $1 + \tan^2\theta = \sec^2\theta$
2. Addition and subtraction formulae:
   (a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
   (b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
   (c) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
3. Multiple formulae:
   (a) $\sin 2A = 2\sin A \cos A$
   (b) $\cos 2A = \cos^2 A - \sin^2 A$
   (c) $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1$
   (d) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Finding the value of sin 15°.

We can write as $\sin 15° = \sin(45° - 30°)$

We have $\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45°.\cos 30° - \cos 45°.\sin 30°$$
$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} \Rightarrow \sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Finding the value of sin 75°.

We can write as $\sin 75° = \sin(45° - 30°)$

We have $\sin(\alpha - \beta) = \sin\alpha.\cos\beta - \cos\alpha.\sin\beta$

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45°.\cos 30° - \cos 45°.\sin 30°$$
$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} \Rightarrow \sin 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

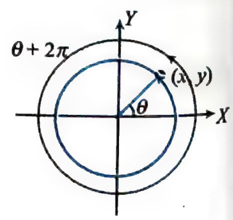
Trigonometrical Ratios of General Angles (Reduction Formulae)

(I) Trigonometric function of an angle $(2n\pi + \theta)$ where $n = 0, 1, 2, 3, \ldots$ will remain same.

$\sin(2n\pi + \theta) = \sin\theta$

$\cos(2n\pi + \theta) = \cos\theta$

$\tan(2n\pi + \theta) = \tan\theta$



Examples:

(a) $\sin 420° = \sin(360° + 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

(b) $\sin 405° = \sin(360° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}$

(c) $\tan 390° = \tan(360° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

(ii) Trigonometric function of an angle $(90° \pm \theta)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to the value of that function in quadrant.

The angle $(90° - \theta)$ lies in the first quadrant and the angle $(90° + \theta)$ lies in the second quadrant

$\sin(90° - \theta) = \cos\theta$, $\cos(90° - \theta) = \sin\theta$

and $\tan(90° - \theta) = \cot\theta$

and $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$

and $\tan(90° - \theta) = -\cot\theta$

Examples:

(a) $\sin 30° = \sin(90° - 60°) = \cos 60°$

(b) $\cos 30° = \cos(90° - 60°) = \sin 60°$

(c) $\sin 120° = \sin(90° + 30°) = \cos 30°$

(d) $\cos 120° = \cos(90° + 30°) = -\sin 30°$

(e) $\tan 120° = \tan(90° + 30°) = -\cot 30°$

(iii) Trigonometric function of an angle $(180° \pm \theta)$ will remain same and the sign of trigonometric function will be according to the value of that function in quadrant.

The angle $(180° - \theta)$ lies in the second quadrant and the angle $(180° + \theta)$ lies in third quadrant.

$\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$

and $\tan(180° - \theta) = -\tan\theta$

and $\sin(180° + \theta) = -\sin\theta$, $\cos(180° + \theta) = -\cos\theta$

and $\tan(180° + \theta) = \tan\theta$

Examples:

(a) $\sin 120° = \sin(180° - 60°) = \sin 60°$

(b) $\cos 120° = \cos(180° - 60°) = -\cos 60°$

(c) $\tan 120° = \tan(180° - 60°) = -\tan 60°$

(d) $\sin 210° = \sin(180° + 30°) = -\sin 30°$

(iv). Trigonometric function of an angle $-\theta$ (negative angles).

The angle $(-\theta)$ lies in the fourth quadrant.

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$

Examples:

(a) $\sin(-30°) = -\sin 30° = -\frac{1}{2}$

(b) $\cos(-30°) = \cos 30° = \frac{\sqrt{3}}{2}$

Inverse Trigonometric Functions

Inverse trigonometric functions are also called anti-trigonometric functions. These are represented by putting a superscript "–1" on the corresponding trigonometric function whose inverse is to be obtained. For example,

inverse of $\sin\theta = x$ means $\theta = \sin^{-1} x$

It is read as "sine inverse x." Just as a trigonometric operation on any angle gives a particular value, an inverse trigonometric operation on any value (or number) returns its corresponding angle.

Properties of Inverse Trigonometric Functions

1. $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1} x) = x$; provided $-\pi/2 \le \theta \le \pi/2$ and $-1 \le x \le 1$
2. $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1} x) = x$; provided $0 \le \theta \le \pi$ and $-1 \le x \le 1$
3. $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1} x) = x$; provided $-\pi/2 \le \theta \le \pi/2$ and $-\infty \le x \le \infty$

ILLUSTRATION 2.9

Find the value of $\sin^{-1} 1$.

Sol. Let $y = \sin^{-1} 1 = \sin^{-1}(\sin \pi/2) = \pi/2$

$[\because \sin \pi/2 = 1$ and $\sin^{-1}(\sin\theta) = \theta$ for $-\pi/2 \le \theta \le \pi/2]$

ILLUSTRATION 2.10

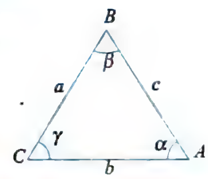
Find the value of $\cos^{-1}(-1/2)$.

Sol. Let $y = \cos^{-1}(-1/2) = \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

$[\because \cos(2\pi/3) = -1/2$ and $\cos^{-1}(\cos\theta) = \theta$ for $0 \le \theta \le \pi]$

Sine and Cosine Rule for Triangles

Sine and cosine rules are very important application in problem solving in physics. If we are given a triangle ABC, we can relate the length of sides and angles between the sides using these laws (figure below).



Sine rule:	Cosine rule:
$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$	$c^2 = a^2 + b^2 - 2ab\cos\gamma$

CONCEPT APPLICATION EXERCISE 2.2

1. Find angle subtended by a circular arc of radius 6 cm and length π cm at its centre, in radian and degree.
2. The two shorter sides of right-angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$.
3. If $\theta = 1/3$, then find the value of $\cos\theta$.
4. Find the value of $\sin(105°)$.

ANSWERS

1. $\frac{\pi}{6}$ rad, 30°	2. $2\sqrt{2}/3$
3. 5/13, 12/13, 5/12	4. $\frac{1}{4}(\sqrt{6} + \sqrt{2})$

BASIC COORDINATE GEOMETRY

If you have to specify the position of a point in space, how will you do it? This is the easiest application of coordinate geometry. We can give, assign, or find out the exact numerical value of the position of points, lines, curves, slopes, etc. This is done with the help of coordinate systems. There are many types of coordinate systems such as rectangular, polar, spherical, cylindrical, etc. Generally the right-handed rectangular coordinate system is used in physics. This system consists of (1) origin and (2) axis or axes.

If the point is on a given line or in a particular direction, only one coordinate is necessary to specify its position; if it is in a plane, two coordinates are required; if it is in space, three coordinates are needed.

ORIGIN

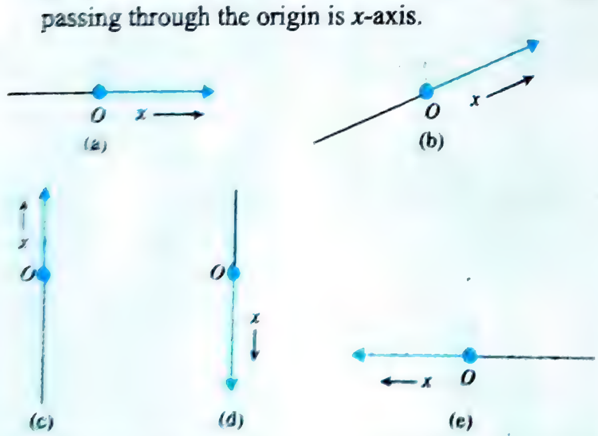
Origin is any fixed point convenient to you. In a room, you can consider any corner as the origin. On a sheet of paper, you can mark any point on it as the origin. All measurements are taken basically with respect to origin.

AXIS

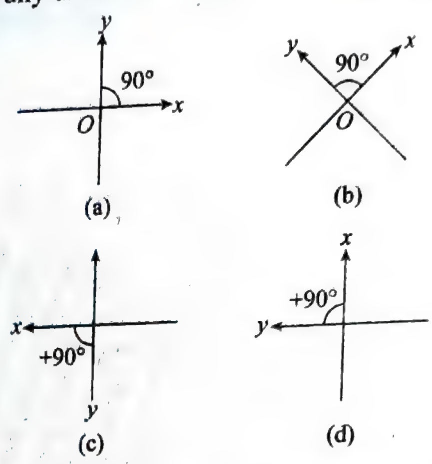
Any fixed direction passing through the origin and convenient to you can be taken as an axis. If the position of a point or positions of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two axes are needed. These are generally called x- and y-axes. If the points are distributed in space, three axes are taken, which are called x-, y-, and z-axes. If x-, y-, and z-axes are mutually perpendicular, the system is called rectangular coordinate system.

Important Points:

1. Origin can be any fixed point convenient to you. It is denoted by O.
2. The x-axis is any fixed direction passing through the origin and convenient to you (Figure). Thus, it is not at all necessary that the (so-called) horizontal line passing through the origin is x-axis.



3. **Unless otherwise explicitly mentioned, all angles are always measured from the direction of the x-axis (called the positive direction of x-axis). Positive angles are measured in anticlockwise direction and negative angles in clockwise direction.**
4. **The y-axis is any fixed direction convenient to you passing through the origin and perpendicular to the x-axis. Perpendicular means making an angle of +90° with the positive direction of the x-axis. Students may feel that once the origin and x-axis have been fixed, the position of the y-axis also gets fixed accordingly. But, it is not the case. The y-axis can be any fixed direction in the plane passing through the origin and x-axis and perpendicular to the x-axis. Thus, x-and y-axes can have any direction as shown in the figure below.**

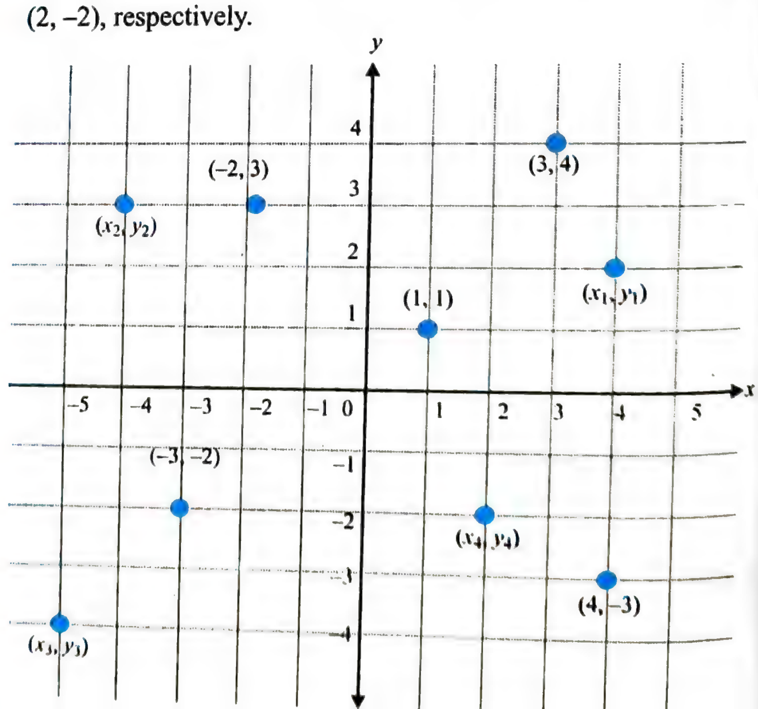


5. Once the origin and x- and y-axes are fixed, the z-axis gets automatically fixed. The convenience of the observerceases to exist. The z-axis is the fixed direction passing through the origin and perpendicular to both x- and y-axes.

POSITION OF A POINT

In case of plane coordinate geometry, i.e., when the position of a point always remains contained in a plane (called x-y plane), the position of the point is specified by its distances from the origin along (or parallel to) the x- and y-axes, as shown in Figure.

You can easily observe that the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) in Figure are (4, 2), (–4, 3), (–5, –4), and (2, –2), respectively.

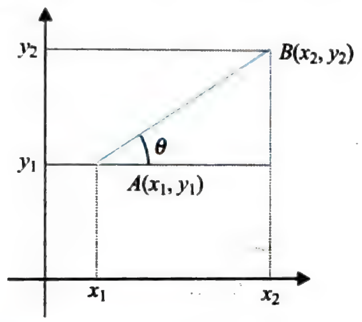


Distance Formulae

1. The distance between two points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. The distance of the point (x_1, y_1) from the origin is $\sqrt{x_1^2 + y_1^2}$.
3. The coordinates of the mid-point of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

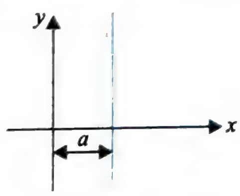
Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by $m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$, where θ is the angle which the line makes with the positive direction of the x-axis (figure below).

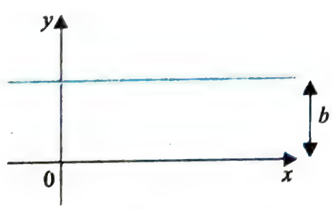


Straight Line Equations

1. $Ax + By + C = 0$ is the general form of the equation of a straight line.
2. The equation of x-axis is $y = 0$.
3. The equation of y-axis is $x = 0$.
4. The equation of a straight line parallel to the y-axis and at a distance a from it is given by $x = a$ (figure below).

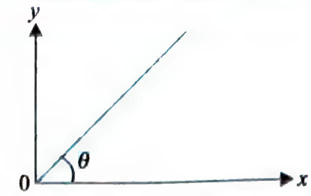


5. The equation of a straight line parallel to the x-axis and at a distance b from it is given by $y = b$ (figure below).



6. $y = mx + c$ is a line which cuts off an intercept c on the y-axis and makes an angle θ with the positive direction of the x-axis in the anticlockwise direction, and $m = \tan\theta$ is called its slope or gradient.

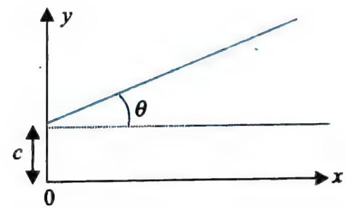
   (a) When $c = 0$, $y = mx$. This is a straight line passing through the origin.



   The graph between x and y will be a straight line as x bears direct dependence on y (figure). Here, m represents the slope of the line. So

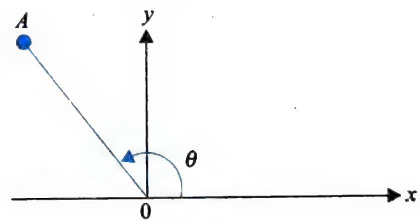
   $$\frac{dy}{dx} = m = \tan\theta$$

   (b) When $c \neq 0$, $y = mx + c$. Graph for this equation is also a straight line but with a positive intercept on the y-axis. When x goes to zero, y accordingly takes the value c. So the straight line will start from $y = c$ instead of the origin (figure). Here also $m = \tan\theta$ is the slope of the straight line.



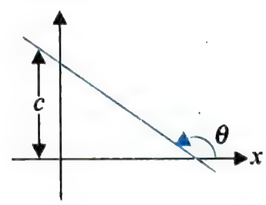
   (c) When $c = 0$, $m < 0$, $y = mx$.

   For $m < 0$, $\theta > 90°$.

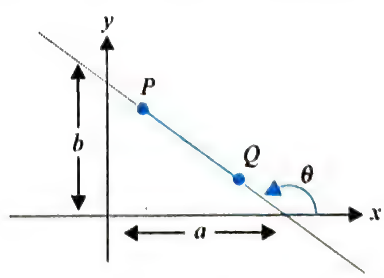


   (d) When $c \neq 0$, $m < 0$, $y = mx + c$.

   For $m < 0$, $\theta > 90°$.



7. $\frac{x}{a} + \frac{y}{b} = 1$ is a line in the intercept form where a and b are intercepts on axes of x and y, respectively.



8. $y - y_1 = m(x - x_1)$ is the equation of a line through a given point (x_1, y_1) and having slope m.

Slope:

The slope m of the line $Ax + By + C = 0$ is given by

$$m = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-A}{B}$$

ILLUSTRATION 2.11

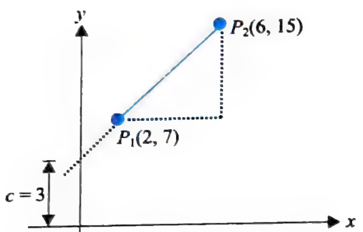
Consider two points $P_1(2, 7)$ and $P_2(6, 15)$. Write the equation and draw a straight line through these points.

Step 1. Obtain the gradient or slope, which is m.

Step 2. c can be found by using the (x, y) values at any given point.

Step 1. Gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{15-7}{6-2} = \dfrac{\text{Height}}{\text{Distance}} = \dfrac{8}{4} = 2$

So, $\quad y = 2x + c. \quad$ ...(i)



Step 2. To find c, put $x = 2, y = 7$ or $x = 6, y = 15$ in Eq. (i).

$7 = 2 \times 2 + c \Rightarrow c = 3$

So, $\quad m = 2, c = 3.$

Hence, the equation becomes $y = 2x + 3$.

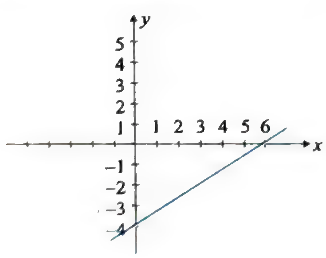
ILLUSTRATION 2.12

Plot the line $2x - 3y = 12$.

Method 1: Given $3y = 2x - 12 \Rightarrow y = \dfrac{2}{3}x - 4$

Therefore, using $y = mx + c$, we get $m = \dfrac{2}{3}$ and $c = -4$.

Here, positive slope (m) means the angle made by the line with the x-axis should be less than 90° and negative c means the line will make in intercept with the negative y-axis.



If $y = 0$, then $\dfrac{2}{3}x = 4 \Rightarrow x = \dfrac{4 \times 3}{2} = 6$

Method 2: Using $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$$\Rightarrow \quad \frac{2x}{12} - \frac{3y}{12} = 1 \quad \Rightarrow \quad \frac{x}{6} + \frac{y}{-4} = 1$$

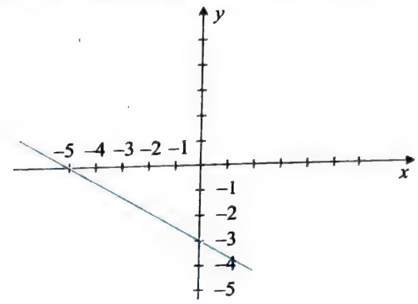
ILLUSTRATION 2.13

Plot the line $-3x - 5y = 15$.

Method 1: Given $5y = -3x - 15$

Using $y = mx + c$, $y = -\dfrac{3}{5}x - 3$

Here the slope is negative, i.e., the line makes an angle greater than 90° with the x-axis. As intercept is negative, it indicates that the line will cut the y-axis on its negative side.



When $x = 0, \quad y = -3.$

When $y = 0, \quad x = -5.$

Method 2: Using $\dfrac{x}{a} + \dfrac{y}{b} = 1;$

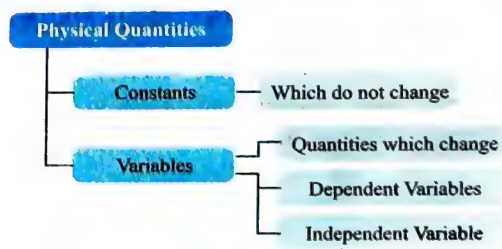
$$\frac{-3x}{15} - \frac{5y}{15} = 1 \quad \Rightarrow \quad \frac{x}{(-5)} + \frac{y}{(-3)} = 1$$

DEPENDENT AND INDEPENDENT VARIABLES

A variable which can have any arbitrary value within specific limits is called as independent variable.

The variables whose value depends upon the numerical values assigned to the independent variable are defined as dependent variables.

The relationship between a dependent and an independent variable is expressed generally by some mathematical relationship called function.



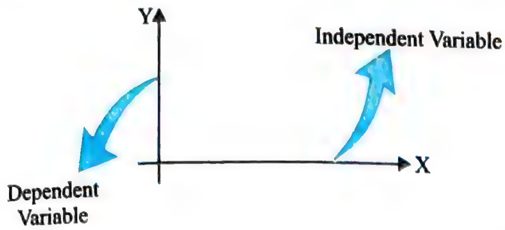
For example, if a car is moving with constant speed say v, the distance d travelled by car in time t is given by

$$d = vt$$

Dependent variable Independent variable

In physics, it is generally required to express certain relations in graphical form.

In plotting of graph, we express the value of dependent variables on Y-axis and independent variables on X-axis.



PLOTTING THE VELOCITY–TIME RELATION OF A PARTICLE MOVING WITH CONSTANT ACCELERATION

Consider equation of motion of a particle which starts moving with initial velocity u and constant acceleration a, we can express the velocity of the particle in relation with tine by the relation:

$$v = u + at$$

Here v depends upon t,it means v dependent variable and t is independent variable

We can write $v = u + at = at + u$ ...(i)

The acceleration of the particle is constant , it means eq.(i) is a linear equation. If we plot velocity time graph it should be straight line.

Comparing eq.(i) with $y = mx + c$ ...(ii)

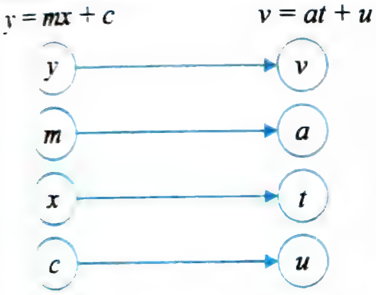
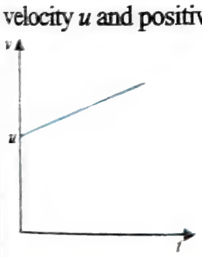
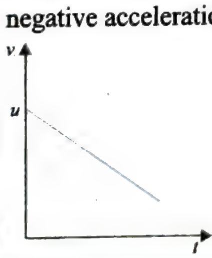
$y = mx + c$ $v = at + u$ 	If velocity–time graph is drawn, • The slope of velocity–time graph is the acceleration. • Intercept of the graph is initial velocity of the particle.
If a particle moves with initial velocity u and positive acceleration a 	If a particle moves with initial velocity u and negative acceleration 

ILLUSTRATION 2.14

If a particle starts moving with initial velocity $25\,\text{m}\,\text{s}^{-1}$ and retardation $2\ \text{m s}^{-2}$. Draw the velocity–time graph.

Sol. For a particle moving with constant acceleration, we can define the velocity in relation with time by the equation: $v = u + at = at + u$

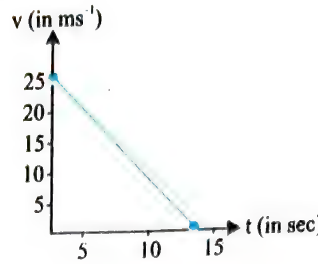
We are given initial velocity $u = 25\ \text{m s}^{-1}$ and acceleration $a = -2\ \text{m s}^{-2}$

Hence we can write $v = -2t + 25$

Comparing with $y = mx + c$: Slope of the line $m = -2$ and $c = 25$

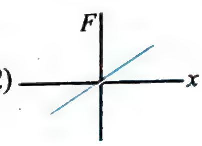
Velocity will be zero if $0 = -2t + 25 \Rightarrow 2t = 25 \Rightarrow t = 12.5$ sec

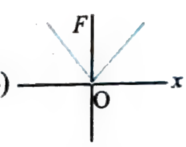
It means the line should cut x-axis(time axis) at t = 12.5 sec and y-axis (velocity axis) on $v = 25\ \text{m s}^{-1}$. Now we can plot the straight line as shown in figure.



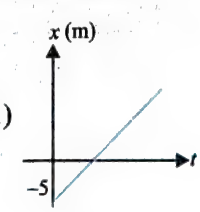
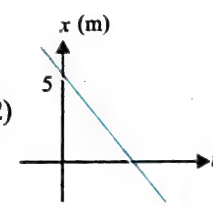
CONCEPT APPLICATION EXERCISE 2.3

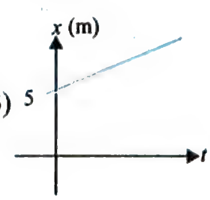
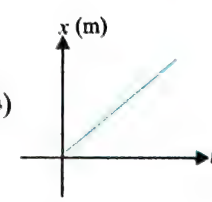
1. Plot the lines: (a) $3x + 2y = 0$, (b) $x - 3y + 6 = 0$.
2. If a particle starts moving with initial velocity $u = 1\ \text{m s}^{-1}$ and acceleration $a = 2\ \text{m s}^{-2}$, the velocity of the particle at any time is given by $v = u + at = 1 + 2t$. Plot the velocity–time graph of the particle.
3. The spring force is given by $F = -kx$, where k is a constant and x is the deformation of spring .The F–x graph is

(1)  (2) 

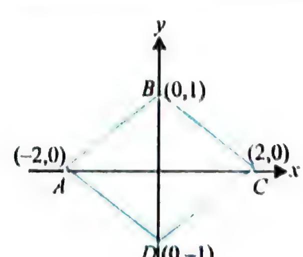
(3)  (4) 

4. A particle starts moving with constant, velocity $v = 2$ m/s, from position $x = 5$ m. The position–time graph will be

(1)  (2) 

(3)  (4) 

5. A parallelogram $ABCD$ is shown in figure



	Column I		Column II
i.	Equation of side AB	a.	$2y + x = 2$
ii.	Equation of side BC	b.	$2y - x = 2$
iii.	Equation of side CD	c.	$2y + x = -2$
iv.	Equation of side DA	d.	$2y - x = -2$

Correct matching is

(1) i → b; ii → a; iii → d; iv → c

(2) i → a; ii → b; iii → d; iv → c

(3) i → b; ii → d; iii → c; iv → a

(4) i → c; ii → a; iii → d; iv → b

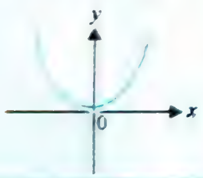
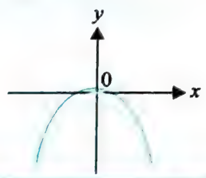
ANSWERS

3. (3) 4. (3) 5. (1)

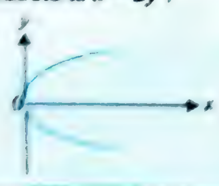
PARABOLA: QUADRATIC EQUATIONS

Let us now discuss graphs of quadratic equations. For the equation $y = ax^2 + bx + c$ (where a, b, and c are constants), the graph between x and y is a parabola. As long as $a \neq 0$, this equation represents a quadratic function. So what is the simplest quadratic equation? It is $y = ax^2$ (obtained by putting $b = 0$, $c = 0$), which is the equation of a parabola.

- The graph for $y = ax^2$ will be a symmetric parabola about y-axis. The orientation of the parabola is decided by the sign of a.

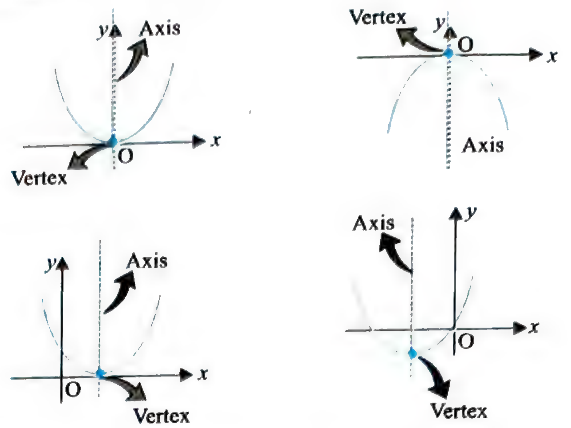
When a is positive	When a is negative
The equation of the parabola is $y = ax^2$.	The equation of the parabola will be $y = ax^2$.
	

- If we exchange x and y in this equation, i.e., $x = ay^2$, then the axis of symmetry changes and becomes x-axis. As this orientation changes as per the sign of a, the orientation becomes opposite when a is negative.

When a is positive	When a is negative
The equation of the parabola is $x = ay^2$.	The equation of the parabola will be $x = ay^2$.
	

Important Points about Parabola

Axis and vertex of parabola: The curve has an axis of symmetry which is called the axis of parabola. The point of intersection of the parabola with its axis is termed the vertex of the parabola.



The graph of the quadratic equation $y = ax^2 + bx + c$ has the same shape as that of the parabola $y = ax^2$. But the difference is:

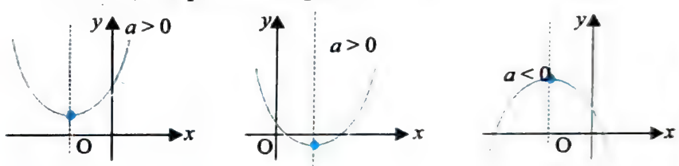
- The vertex does not pass through the origin.
- The axis of parabola is parallel to y-axis.

We have general equation of parabola: $y = ax^2 + bx + c$.

For graphs opening up, the vertex is a minimum (low point). For graphs opening down, the vertex is a maximum (high point).

The general form of parabola is $y = ax^2 + bx + c$.

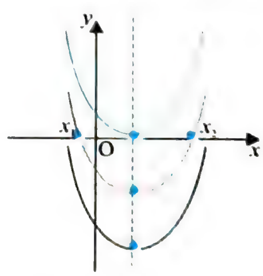
- The axis of parabola is parallel to y-axis.
- If $a > 0$, the parabola opens up.
- If $a < 0$, the parabola opens down.



Finding Vertex of Parabola

For finding the x-coordinate of the vertex of the parabola, let us consider the figure below. In the given figure, the axis of the dotted parabola and both the bold parabolas is same, hence the x-coordinate of vertex of the all parabolas should be same.

The dotted parabola touches the x-axis where $y = 0$.



$$y = ax^2 + bx + c = 0$$

$$\Rightarrow \quad ax^2 + bx + c = 0$$

On solving this quadratic equation, we get $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If the parabola touches x-axis $b^2 - 4ac = 0$; hence $x = -b/2a$.

The x-coordinate of vertex of both bold parabolas will be also $-b/2a$.

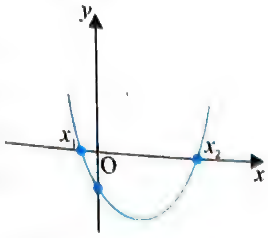
Steps in Plotting Parabola

- The general form of parabola is $y = ax^2 + bx + c$.
- Orientation of graph depends upon sign of a.

(i) when a is +ve, the graph will open up.

(ii) when a is –ve, the graph will open down.

- To find the x-coordinate of the vertex, use the equation $x = -b/2a$.
- Then substitute the value of x back into the equation of the parabola and solve for y. Point (x, y) so obtained represents vertex.



- The parabola cuts x-axis where $y = 0$. Substituting $y = 0$ in parabola equation, we will get $ax^2 + bx + c = 0$. If we solve this quadratic, we will get the points where the parabola cuts x-axis.

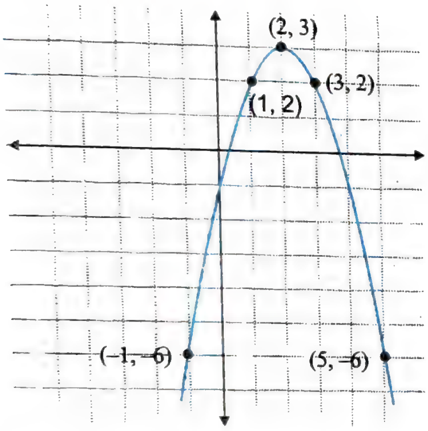
$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

- The parabola cuts y-axis where $x = 0$, $y = a(0)^2 + b(0) + c = c$ or $y = c$.

ILLUSTRATION 2.15

Plot a graph for the equation $y = -x^2 + 4x - 1$.

Sol. $a = -1$, $b = 4$, $c = -1$. As a is negative, the parabola should open down.



Vertex: $x = -b/2a = 2$. Putting this value of x, we get $y = 3$. Hence, the vertex of the parabola is (2, 3).

Assume two values of x as follows and find the corresponding values of y.

x	1	–1
y	2	–6

Points obtained are (1, 2) and (–1, –6).

Points obtained are (2, 3) (vertex), [1, 2], and [–1, –6].

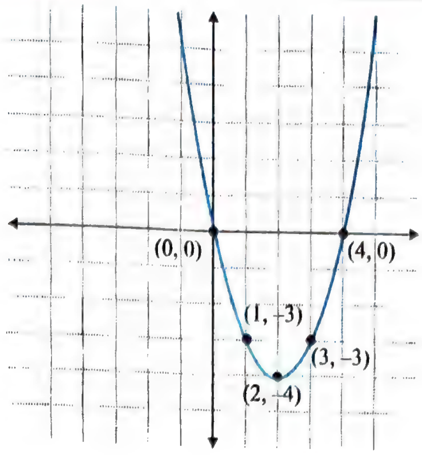
Symmetry of parabola: Mirror image points of (1, 2) and (–1, –6) are (3, 2) and (5, –6), respectively.

Now, sketch the parabola as shown in the figure above.

ILLUSTRATION 2.16

Plot a graph for the equation $y = x^2 - 4x$.

Sol. $a = 1$, $b = -4$. As a is positive, the parabola should open up. As $c = 0$, the parabola passes through the origin.



Vertex: $x = -b/2a = 2$, so $y = -4$.

$\Rightarrow$ Vertex = (2, –4). Assuming two values of x,

x	1	0
y	–3	0

Points obtained: (2, –4), (1, –3), (0, 0).

Symmetry of parabola: Mirror image points of (1, –3) and (0, 0) are (3, –3) and (4, 0), respectively.

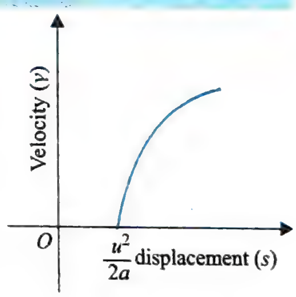
Now, sketch the parabola as shown in the figure above.

ILLUSTRATION 2.17

A particle starts with some initial velocity with an acceleration along the direction of motion. Draw a graph depicting the variation of velocity (v) along y-axis with the variation of displacement (s) along x-axis.

Sol. For uniformly accelerated motion, the relation between velocity (v) and displacement (s) is given by $v^2 = u^2 + 2as$.

Now, the above equation should be transformed to a suitable form, before the exact shape can be known.



$$\therefore \quad (v-0)^2 = 2a\left(s - \frac{u^2}{2a}\right)$$

which is of the form $(y - k)^2 = 4a(x - h)$

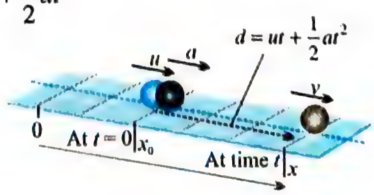
The vertex is at (h, k), i.e., $\left(\frac{u^2}{2a}, 0\right)$ and axis coincides with x (i.e., s)-axis.

Applications of Parabola in Plotting Position–Time Graph

Consider a particle starts moving from $x = 0$ with initial velocity u and constant acceleration a along x-axis.

We can relate position and time by an equation of motion,

$$x = ut + \frac{1}{2}at^2$$



If the particle starts moving from $x = x_0$, the distance travelled by the particle in time t is

$$x = x_0 + d \Rightarrow x = x_0 + \left(ut + \frac{1}{2}at^2\right)$$

Hence, the position of the particle at time t, $x = x_0 + ut + \frac{1}{2}at^2$

Hence, we can write the position of the particle at time t,

This is relation between position x of the particle in time t.

$$x = x_0 + ut + \frac{1}{2}at^2 \qquad \ldots\text{(i)}$$

Here t is independent variable and position x is dependent variable.

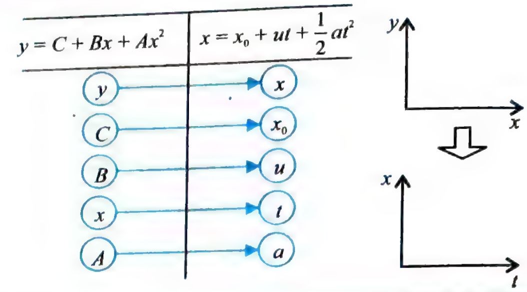
In this equation, the maximum power of t is 2. So we can say the position of the particle is the quadratic function of time.

If we plot position–time (x–t) graph, the graph should be a parabola.

We have general equation of parabola,

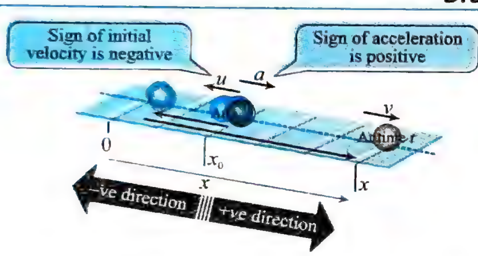
$$y = Ax^2 + Bx + C \text{ or } y = C + Bx + Ax^2 \qquad \ldots\text{(ii)}$$

If we compare equations (i) and (ii),



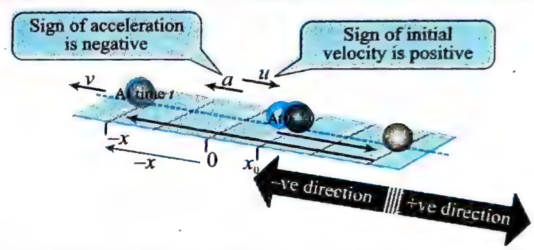
$y = Ax^2 + Bx + C$ or $y = C + Bx + Ax^2$	$x = x_0 + ut + \frac{1}{2}at^2$
If the coefficient of x^2, A, is positive, the parabola opens up.	If the coefficient of t^2, a—the acceleration of a particle, is positive, the parabola opens up.
If the coefficient of x^2, A, is negative, the parabola opens down.	If the coefficient of t^2, a—the acceleration of a particle, is negative, the parabola opens down.

Drawing position–time graph



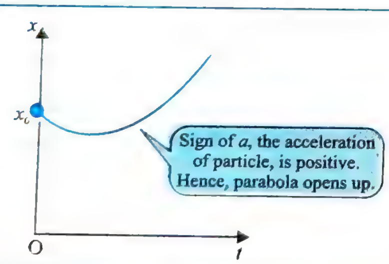
The position of the particle in relation with time as

$$x = x_0 - ut + \frac{1}{2}at^2$$



The position of the particle in relation with time as

$$x = x_0 + ut - \frac{1}{2}at^2$$



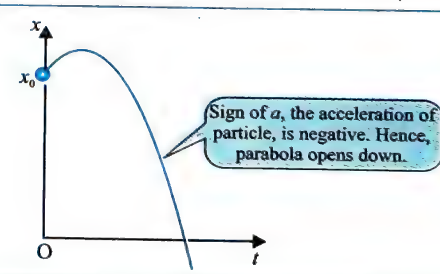


ILLUSTRATION 2.18

A particle starts with uniform acceleration. Draw a graph taking the displacement (s) of the particle along y-axis and time (t) along x-axis. What is the curve known as?

Sol. While drawing the graph between any two quantities, the first step will be to establish a relation between them, in terms of known constants. Thus, for a uniformly accelerated motion, the relation between (s) and (t) is $s = ut + \frac{1}{2}at^2$

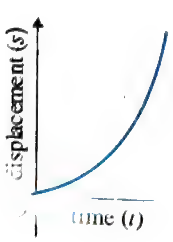
But since $u = 0$,

$$\therefore \quad s = \frac{1}{2}at^2$$

s is taken along y-axis; replace s by y and, t being along x-axis, replace t by x.

Therefore, $y = \frac{ax^2}{2}$ or $x^2 = \frac{2y}{a}$

which is of the form $x^2 = 4ay$.



Therefore, the graph will be parabola with its axis as y i.e., (s) axis. Notice that the portion of graph to the left of the displacement axis (i.e., towards negative time) is omitted, as our point of interest lies in only the time elapsed after the beginning of motion.

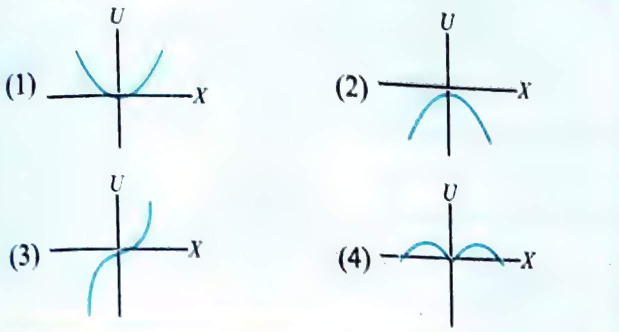
CONCEPT APPLICATION EXERCISE 2.4

1. Find the vertex of the following quadratic equations and plot the graph:
   (a) $y = x^2 - 8x$ (b) $y = -2x^2 + 3$
   (c) $y = x^2 - 6x + 4$
2. If a particle starts moving along x-axis from the origin with initial velocity $u = 1\ \text{m s}^{-1}$ and acceleration $a = 2\ \text{m s}^{-2}$, the relationship between displacement and time is

$$x = ut + \frac{1}{2}at^2 = 1 \times t + \frac{1}{2} \times 2 \times t^2 = t + t^2$$

Draw the displacement (x)–time (t) graph.

3. A body is attached to a spring whose other end is fixed. If the spring is elongated by x, its potential energy is $U = 5x^2$, where x is in metre and U is in joule. U–x graph is



ANSWERS

3. (1)

FUNCTION

Function is a rule of relationship between two variables in which one is assumed to be a dependent variable and the other independent variable:

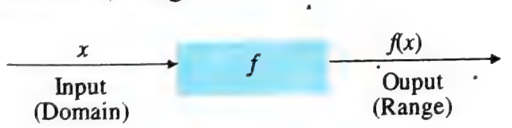
Example 1: The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent variable and temperature is the dependent variable.

Example 2: The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each of the above examples, value of one variable quantity (dependent variable), which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as $y = f(x)$. Here f represents the function, x the independent variable, and y the dependent variable.

All possible values of independent variables (x) are called *domain* of function. All possible values of dependent variable (y) are called *range* of function.

Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (see figure below).



We usually denote functions in one of the two ways:

1. By giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function.
2. By giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

Strictly speaking, we should call the function f and not $f(x)$, $y = \sin x$. Here the function is sine and x is the independent variable.

ILLUSTRATION 2.19

Suppose that the function F is defined for all real numbers r by the formula $f(r) = 2(r - 1) + 3$. Evaluate F at the input values $0, 2, x + 2$, and $f(2)$.

Sol. In each case, we substitute the given input value for r into the formula for F:

$$f(0) = 2(0 - 1) + 3 = -2 + 3 = 1$$
$$f(2) = 2(2 - 1) + 3 = 2 + 3 = 5$$
$$f(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$$
$$f(f(2)) = f(5) = 2(5 - 1) + 3 = 11$$

ILLUSTRATION 2.20

A function $f(x)$ is defined as $f(x) = x^2 + 3$. Find $f(0)$, $F(1)$, $f(x^2)$, $f(x + 1)$ and $f(f(1))$.

Sol. $f(0) = 0^2 + 3 = 3$

$$f(1) = 1^2 + 3 = 4;\ f(x^2) = (x^2)^2 + 3 = x^4 + 3$$
$$f(x + 1) = (x + 1)^2 + 3 = x^2 + 2x + 4$$
$$f(f(1)) = f(4) = 4^2 + 3 = 19$$

DIFFERENTIATION

The purpose of differential calculus is to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which the first quantity depends) varies independently. In our day-to-day life, we often face such situations, e.g., growth of plants, expansion of solids on heating, variation in the velocity of a uniformly accelerated object, growth in the population of a country, etc.

Quantity: Anything that can be measured is called a quantity.

Constants and variables: A quantity whose value remains constant throughout the mathematical operation is called a constant, e.g., integers, fractions, π, e, etc. On the other hand, a quantity which can have any numerical value within certain specific limits is called a variable. A variable is usually represented by u, v, w, x, y, z, etc.

Dependent and independent variables: A variable which can have any arbitrary value within specific limits is called an independent variable whereas a variable whose value depends upon the numerical values assigned to the independent variable is defined as a dependant variable

LIMIT OF A FUNCTION

Let us consider that y is a function of the variable x as given by:

$$y = \frac{x^2 - 4}{x - 2}$$

The given function y is defined for all values of x but not for $x = 2$. For, if we set $x = 2$, we get $y = \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0}$ = not defined,

i.e., for the value of the given function y cannot be found.

Let us calculate the value of function y for the value of x, slightly smaller than 2 to the values slightly greater than 2:

Value of x	Value of $y=\dfrac{x^2-4}{x-2}$
$x=1.8$	$y=\dfrac{(1.8)^2-4}{1.8-2}=3.8$
$x=1.9$	$y=\dfrac{(1.9)^2-4}{1.9-2}=3.9$
$x=1.95$	$y=\dfrac{(1.95)^2-4}{1.95-2}=3.95$
$x=1.99$	$y=\dfrac{(1.99)^2-4}{1.99-2}=3.99$
$x=2.01$	$y=\dfrac{(2.01)^2-4}{2.01-2}=4.01$
$x=2.05$	$y=\dfrac{(2.05)^2-4}{2.05-2}=4.05$
$x=2.1$	$y=\dfrac{(2.1)^2-4}{2.1-2}=4.1$
$x=2.2$	$y=\dfrac{(2.2)^2-4}{2.2-2}=4.2$

It is clear from table, the value of the given function y approaches 4, when x tends to or approaches 2, both from values smaller than 2 or from values greater than 2. The phrase 'x tends to 2' or 'x approaches 2' is mathematically represented as '$x-2$'.

Symbolically, we write $y=\lim\limits_{x\to 2}\left(\dfrac{x^2-4}{x-2}\right)=4$

The limit of the function y in the given case can also be found as below:

$$y=\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)=\lim_{x\to 2}\left(\frac{(x+2)\,(x-2)}{x-2}\right)$$

In the above expression, x is not equal to 2. It only tends to 2. Likewise, the factor $(x-2)$ is not zero. Therefore, the factor $(x-2)$ occurring in numerator and denominator can be cancelled.

Hence, $y=\lim\limits_{x\to 2}(x+2)=2+2=4$

Differential Coefficient or Derivative of a Function

Suppose y be a function of x, i.e., $y=f(x)$. ...(i)

The value of the function or the dependent variable y depends on the value of the independent variable x. If we change the value of x to $x+\Delta x$, then the value of the function will also change. Let it become $y+\Delta y$. Hence,

$$y+\Delta y=f(x+\Delta x) \qquad \text{...(ii)}$$

Subtracting Eq. (i) from Eq. (ii), we get

$$y+\Delta y-y=f(x+\Delta x)-f(x)$$

or $$\Delta y=f(x+\Delta x)-f(x) \qquad \text{...(iii)}$$

Equation (iii) provides change in the value of function y, when the value of the variable x is changed from x to $x+\Delta x$.

Dividing both sides of Eq. (iii) by Δx, we get

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \text{...(iv)}$$

Equation (iv) gives the average rate of change of the functi when the value of the variable x changes in the interval betwe x and $x+\Delta x$. Taking limits on both sides of Eq. (iv), when approaches zero, we get

$$\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \text{...(v)}$$

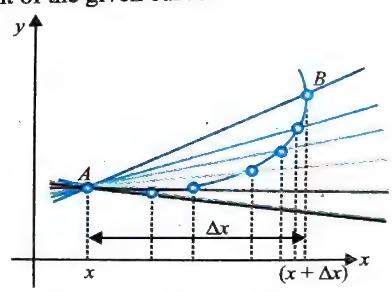
If the above limit exists uniquely and finitely for all valu of x in the given interval, then it is called differential coefficie or derivative of $f(x)$ or y with respect to x. It is represented $\dfrac{d}{dx}[f(x)]$ or $\dfrac{dy}{dx}$.

Hence, $$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \text{...(vi)}$$

It is clear that Eq. (v) or Eq. (vi) provides instantaneous ra of change of the function y with respect to variable x. Hence, th differentiation of a function with respect to a variable implie instantaneous rate of change of the function with respect to th variable.

Graphical Representation of Derivative of a Function

A secant is a line drawn through two points on a curve. The li AB is a secant of the given curve.



Slope of the line AB, $m=\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{\Delta y}{\Delta x}$

If we keep up decreasing the interval Δx, the secant of give graph will about to touch at a single point A.

A tangent is a line which just touches a curve at a point. Now we can say that a tangent as the limit of a secant of the curv taken as the separation between the points tends to zero.

We can say when Δx tends to zero, the secant on the curve wi change to tangent at a point.

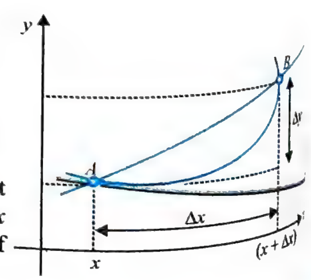
Slope of the secant AB, $m=\dfrac{\Delta y}{\Delta x}$

We define slope of the

tangent at A, $m=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}$

$\Rightarrow \qquad m=\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x}=\dfrac{dy}{dx}$

This slope of tangent at a point is written with a symbol dy/dx which is also the derivative of y with respect to x.



Physical meaning of dy/dx: The concept of differentiation is made use of in physics in determining the instantaneous rate of change of a physical quantity w.r.t. some other quantity, which varies in a continuous manner.

Defining Instantaneous Velocity and Instantaneous Acceleration

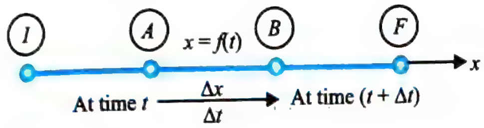
Suppose a particle is moving along x-axis. The position (x) of the particle is change with time according to relation $x = f(t)$

We call $f(t)$ as the function of time, if here t is independent variable and x is dependent variable.

If we calculate average velocity of the particle for the motion from A to B, we write

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{x_B - x_A}{t_B - t_A}$$

$$\Rightarrow \quad v_{av} = \frac{f(t+\Delta t) - f(t)}{(t+\Delta t) - t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

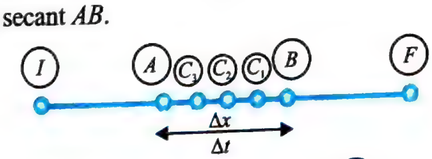


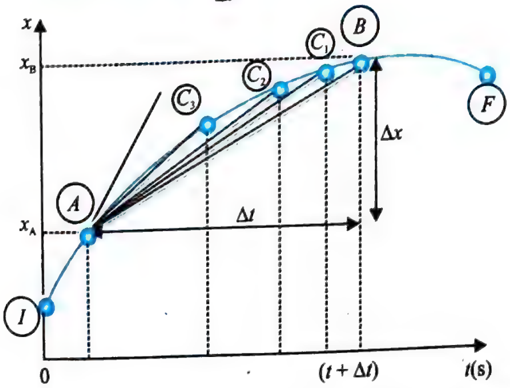
Now what happens if Δt approaches zero or in other words Δt is infinitesimally small. When we say Δt approaches zero, it does not mean that Δt is equal to zero. It means that Δt is very close to zero.

If Δt is very small, then the distance covered during this time will also be very small. So Δx also approaches zero and points A and B will lie very close to each other.

Let us write $\Delta t = dt$ when Δt tends to 0 and $\Delta x = dx$ when Δx tends to 0. In this situation, the average velocity becomes $v_{av} = \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$. If Δt is very small, at this situation, average velocity becomes instantaneous velocity.

Let us plot the position of the particle at different time. The graph we get is called position–time graph or displacement–time graph. The average velocity of a particle is $v_{avg} = \Delta x/\Delta t$. It is the slope of secant AB.





As points A and B are very close to each other, we can neglect any change in velocity from A to B. We can assume that velocity remains constant for the motion from A to B. Then the average velocity becomes instantaneous velocity or we call if simply velocity.

So, $\quad v = v_{av}$ (when $\Delta t \to 0$)

The slope of this tangent line represents the velocity of the particle at point A. It is the instantaneous velocity at this position.

In calculus notation, this limit is called the derivative of x with respect to t, written as dx/dt.

Hence, instantaneous velocity, $v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \Rightarrow v = \frac{dx}{dt}$

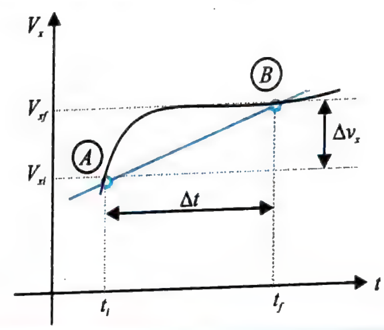
Now we can say the time rate of change of position is equal to instantaneous velocity or simply velocity.

The average acceleration a of the particle is defined as the change in velocity Δv_x divided by the time interval Δt during which that change occurs.

If $\Delta t \to 0$, then instantaneous acceleration or acceleration $a = a_{av}$.

$$a = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

The slope of the tangent line in velocity–time graph represents the acceleration of the car at point A.



Important Points:

- dx does not mean d times x.
- dy does not mean d times y.
- $\frac{d}{dx} f(x)$ does not mean $\frac{d}{dx}$ times $f(x)$.
- The derivative of constant function is 0, i.e., $\frac{d}{dx}(c) = 0$.

Notation

There are many ways to denote the derivative of a function $y = f(x)$. Besides $f'(x)$, the most common notations are as follows:

y'	"y prime"	Nice and brief but does not name the independent variable.
$\frac{dy}{dx}$	"dy by dx"	Names the variables and uses d for derivative.
$\frac{df}{dx}$	"df by dx"	Emphasizes the function's name.
$\frac{d}{dx} f(x)$	"d by dx of f"	Emphasizes the idea that differentiation is an operation performed on f.
$D_x f$	"dx of f"	A common operator notation.

$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives, i.e., dy/dt.

PROPERTIES OF DERIVATIVES

Derivative of a Constant Times a Function

The derivative of a constant times a function is the constant times the derivative of the function. That is,

$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)] = k\frac{df}{dx}$, where k is a constant.

Derivative of the Sum of Two Functions

The derivative of the sum of two functions is equal to the sum of their derivatives. That is, $\frac{d}{dx}[f(x)+g(x)] = \frac{df}{dx}+\frac{dg}{dx}$

Linear Property of Differentiation

The above two rules may be combined to provide the linear property of differentiation. Mathematically, it may be expressed as $\frac{d}{dx}[af(x)+bg(x)] = a\frac{df}{dx}+b\frac{dg}{dx}$, where a and b are constants.

DERIVATIVE OF A POWER FUNCTION

$y = f(x) = x^n$ is called power function.

Before learning the differentiation of power function, let us learn some algebraic expansions.

$$(x+a)^2 = x^2 + 2x.a + a^2$$
$$(x+a)^3 = x^3 + 3x^2.a + 3x.a^2 + a^3$$
$$(x+a)^4 = x^4 + 4x^3.a + 6x^2.a^2 + 4x.a^3 + a^4$$

Let $y = f(x) = x^2$. Find the value of $\frac{dy}{dx}$ or $\frac{d}{dx}d(x)$ or $f'(x)$.

We know $\frac{dy}{dx} = \lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Here $y = f(x) = x^2$:

$$\frac{d}{dx}(x^2) = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x^2+2x(\Delta x)+(\Delta x)^2)-x^2}{\Delta x}$$
$$= \lim_{\Delta x\to 0}(2x+(\Delta x)) = 2x \quad \text{...(i)}$$

Here $y = f(x) = x^3$:

$$\frac{d}{dx}(x^3) = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x^3+3x^2(\Delta x)+3x(\Delta x)^2+(\Delta x)^3)-x^3}{\Delta x}$$
$$= \lim_{\Delta x\to 0}(3x^2+2x+(\Delta x)+(\Delta x)^2) = 3x^2 \quad \text{...(ii)}$$

Here $y = f(x) = x^4$:

$$\frac{d}{dx}(x^4) = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^4 - x^4}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x^4+4x^3(\Delta x)+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x)^4)-x^4}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\left(4x^3+6x^2(\Delta x)+4x(\Delta x)^2+(\Delta x)^3\right) = 4x^3 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we observe a pattern:

$2x \quad 3x^2 \quad 4x^3 \quad 5x^4 \quad 6x^5.....$

In general, we can write: $\frac{d}{dx}(x^n) = nx^{n-1}$

Example: If we are given $y = x^{100}$, then

$$\frac{dy}{dx} = \frac{dx^{100}}{dx} = 100\,x^{(100-1)} = 100x^{99}$$

Example: If we are given $y = \frac{1}{x^n}$, then $y = x^{-n}$, then

$$\frac{dy}{dx} = \frac{d(x^{-n})}{dx} = (-n)x^{(-n-1)} = -nx^{-(n+1)}$$
$$\Rightarrow \quad \frac{dy}{dx} = -\frac{n}{x^{(n+1)}}$$

Example: If we are given $y = 7x^5$, then

$$\frac{d}{dx}7x^5 = 7\left(\frac{d}{dx}x^5\right) = 7.5x^4 = 35x^4$$

Important Points:

- The derivative of power function: $y = x^n$ is $\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1}$; n is any integer or rational number.
- Constant multiple rule: $\frac{d}{dx}cx^n = c\left(\frac{d}{dx}x^n\right) = cnx^{n-1}$

ILLUSTRATION 2.21

If we are given some expression $y = ax^2 + bx + c$ where a, b and c are constants, find dy/dx.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(ax^2+bx+c) = \frac{d(ax^2)}{dx}+\frac{d(bx)}{dx}+\frac{d(c)}{dx}$

$$= a\frac{dx^2}{dx}+b\frac{dx}{dx}+0 = a.2x^{2-1}+b.1.x^{1-1} = a.2x+b.1.x^0$$
$$= \frac{dy}{dx} = 2ax+b$$

ILLUSTRATION 2.22

If we are given some expression $y = x^2+\frac{1}{x^2}$. Find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(x^2+x^{-2}) = \frac{d(x^2)}{dx}+\frac{d(x^{-2})}{dx}$

$$= 2x+(-2)x^{(-2-1)} = 2x-2x^{-3} \Rightarrow \frac{dy}{dx} = 2x-\frac{2}{x^3}$$

ILLUSTRATION 2.23

Differentiate the following w.r.t x: $y = 4x^3 + 3x^2 + x + 7$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(4x^3 + 3x^2 + x + 7)$

$$= \frac{d}{dx}(4x^3) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(7)$$

$$= 4\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x^2) + 1 + 0$$

$$= 4(3x^{3-1}) + 3(2x^{2-1}) + 1 \Rightarrow \frac{dy}{dx} = 12x^2 + 6x + 1$$

ILLUSTRATION 2.24

Differentiate the following w.r.t x: $x^3 + 4x^{3/2} - 3x$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(x^3 + 4x^{3/2} - 3x)$

$$= \frac{d}{dx}(x^3) + 4.\frac{d}{dx}(x^{3/2}) - 3.\frac{d}{dx}(x)$$

$$= 3x^{3-1} + 4.\frac{3}{2}.x^{\left[\frac{3}{2}-1\right]} - 3.1 = 3x^2 + 6x^{\frac{1}{2}} - 3$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 + 6x^{1/2} - 3$$

ILLUSTRATION 2.25

Find the slope of the tangent on the curve $y = x^2 + 3x + 4$ at (–1, 2).

Sol. This slope of tangent at a point is the value of dy/dx at that point.

Given curve is $y + x^2 + 3x + 4$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x)^2 + 3\frac{d}{dx}(x) + \frac{d}{dx}(4) = [2x] + 3[1] + [0]$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + 3$$

Hence $\left[\frac{dy}{dx}\right]_{at\ (-1,2)} = 2(-1) + 3 = 1$

i.e., slope of tangent on the curve at (–1, 2) is 1

ILLUSTRATION 2.26

The area A of a dot of a blot of ink is growing such that after t seconds, $A = 3t^2 + \frac{t}{5} + 7$. Calculate the rate of increase of area after 5 seconds.

Sol. Since the area (A) of the blot is increasing with time, dA/dt denotes the rate of increase of area of time t.

Clearly, $\frac{dA}{dt} = \frac{d}{dt}(A) = \frac{d}{dt}\left(3t^2 + \frac{t}{5} + 7\right)$

$$= \frac{d}{dt}(3t^2) + \frac{d}{dt}\left(\frac{t}{5}\right) + \frac{d}{dt}(7) = 6t + \frac{1}{5}$$

When $t = 5$, $\frac{dA}{dt} = 6 \times 5 + \frac{1}{5} = 30.2$

Thus, after 5 seconds, area of the blot is increasing at the rate of 30.2 units/s.

DERIVATIVE OF FUNCTION SIN x

Let $\quad y = \sin x$ ...(i)

Suppose that x gets a small increment Δx, so that the corresponding small increment in y is equal to Δy. Then,

$$y + \Delta y = \sin(x + \Delta x) \quad \text{...(ii)}$$

Subtracting equation (i) from (ii), we have

$$(y + \Delta y) - y = \sin(x + \Delta x) - \sin x \quad \text{...(iii)}$$

Now, $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Therefore, equation (iii) becomes

$$\Delta y = 2\cos\left(\frac{x + \Delta x + x}{2}\right)\sin\left(\frac{x + \Delta x - x}{2}\right)$$

$$= 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

Dividing both sides by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x} = \left(x + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

Taking limits of both sides as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\cos\left(x + \frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]$$

$$= \lim_{\Delta x \to 0}\left[\cos\left(x + \frac{\Delta x}{2}\right) \times \frac{\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]$$

$$\left(\text{if angle is small, we can write } \sin\frac{\Delta x}{2} = \frac{\Delta x}{2}\right)$$

Hence, $\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \cos\left(x + \frac{0}{2}\right) \times 1 = \cos x$

or $\quad \frac{d}{dx}(\sin x) = \cos x$ ...(iv)

DERIVATIVE OF FUNCTION COS x

Let $\quad y = \cos x$ ...(i)

Suppose that x gets a small increment Δx, so that the corresponding small increment in y is equal to Δy. Then,

$$y + \Delta y = \cos(x + \Delta x) \quad \text{...(ii)}$$

Subtracting equation (i) from equation (ii), we have

$$(y + \Delta y) - y = \cos(x + \Delta x) - \cos x \quad \text{...(iii)}$$

Now, $\quad \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Therefore, equation (iii) becomes

$$\Delta y = 2\sin\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{x-x-\Delta x}{2}\right)$$

$$= 2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(-\frac{\Delta x}{2}\right)$$

or $$\Delta y = -2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}$$

Dividing both sides by Δx, we have

$$\frac{\Delta y}{\Delta x} = -\frac{2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x} = -\sin\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

Taking limits of both sides as $\Delta x \to 0$, we have

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left[-\sin\left(x+\frac{\Delta x}{2}\right)\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\right]$$

$$= -\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right)\times\frac{\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$\left(\text{if angle is small, we can write } \frac{\Delta x}{2} = \frac{\Delta x}{2}\right)$$

Hence, $$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx} = -\sin\left(x+\frac{0}{2}\right)\times 1 = -\sin x$$

or $$\frac{d}{dx}(\cos x) = -\sin x \quad \text{...(iv)}$$

DIFFERENTIATION OF COMMONLY USED FUNCTIONS

	Function	Derivative
(i)	$y=$ constant	$\frac{dy}{dt}=0$
(ii)	$y=x^n$	$\frac{dy}{dx}=nx^{n-1}$
(iii)	$y=\sin x$	$\frac{dy}{dx}=\cos x$
(iv)	$y=\cos x$	$\frac{dy}{dx}=-\sin x$
(v)	$y=\tan x$	$\frac{dy}{dx}=\sec^2 x$
(vi)	$y=e^x$	$\frac{dy}{dx}=e^x$
(vii)	$y=\ln x$	$\frac{dy}{dx}=\frac{1}{x}$

PRODUCT RULE

The derivative of the product of two functions $f(x)$ and $g(x)$ is given by

$$\frac{d}{dx}[f(x)g(x)] = f\frac{dg}{dx}+g\frac{df}{dx}$$

ILLUSTRATION 2.27

Find the derivatives of $y = (x^2+1)(x^3+3)$.

Sol. From the product rule with $u = x^2+1$ and $v = x^3+3$, we find

$$\frac{d}{dx}[(x^2+1)(x^3+3)] = (x^2+1)\frac{d}{dx}(x^3+3)+(x^3+3)\frac{d}{dx}(x^2+1)$$

$$= (x^2+1)(3x^2)+(x^3+3)(2x)$$

$$= 3x^4+3x^2+2x^4+6x$$

$$= 5x^4+3x^2+6x$$

The above illustration can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check:

$$y = (x^2+1)(x^3+3) = x^5+x^3+3x^2+3$$

$$\frac{dy}{dx} = 5x^4+3x^2+6x$$

This is in agreement with our first calculation.

ILLUSTRATION 2.28

If $y = [3x+2][2x-1]$, then find $\frac{dy}{dx}$.

Sol. Here, $u = 3x+2$, $v = 2x-1$.

Differentiating both sides, we get

$$\frac{dy}{dx} = \frac{d}{dx}[3x+2][2x-1]$$

Using product rule, we get

$$\frac{d[uv]}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}$$

$$\frac{dy}{dx} = [3x+2]\frac{d}{dx}[2x-1]+[2x-1]\frac{d}{dx}[3x+2]$$

$$= [3x+2][2]+[2x-1]3$$

$$= 6x+4+6x-3 = 12x+1$$

ILLUSTRATION 2.29

If $y = [2x^3+3][2x^{-3}+1]$, then find $\frac{dy}{dx}$.

Sol. Here, $u = 2x^3+3$, $v = 2x^{-3}+1$.

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(2x^3+3)(2x^{-3}+1)]$$

Using product rule, $\frac{d(uv)}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}$

$$\frac{dy}{dx} = [2x^3+3]\frac{d}{dx}[2x^{-3}+1]+[2x^{-3}+1]\frac{d}{dx}[2x^3+3]$$

$$= [2x^3+3][-6x^{-4}]+[2x^{-3}+1][6x^2]$$

$$= [2x^3+3]\left[\frac{-6}{x^4}\right]+\left[\frac{2}{x^3}+1\right][6x^2] = -\frac{18}{x^4}+6x^2$$

ILLUSTRATION 2.30

If $y = e^x \log x$. Find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(e^x \log x)$

$$= e^x \frac{d(\log x)}{dx} + \log x . \frac{de^x}{dx}$$

$$= e^x . \frac{1}{x} + \log x . e^x$$

$$= \frac{dy}{dx} = e^x \left(\frac{1}{x} + \log x\right)$$

QUOTIENT RULE

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

ILLUSTRATION 2.31

Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$.

Sol. We apply the quotient rule with $u = t^2 - 1$ and $v = t^2 + 1$.

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - v(du/dt)}{v^2}$$

$$= \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

ILLUSTRATION 2.32

If $y = \left[\frac{x^2+1}{x+1}\right]$, then find $\frac{dy}{dx}$.

Sol. Here, $u(x) = x^2 + 1$, $v(x) = x + 1$.

Using quotient rule, we get

$$\frac{dy}{dx} = \frac{(x+1)\frac{d(x^2+1)}{dx} - (x^2+1)\frac{d(x+1)}{dx}}{(x+1)^2}$$

$$= \frac{(x+1)2x - (x^2+1)1}{(x+1)^2}$$

$$= \frac{2x^2 + 2x - x^2 - 1}{(x+1)^2} = \frac{x^2 + 2x - 1}{(x+1)^2}$$

ILLUSTRATION 2.33

If $y = \frac{(x^2 + 2x)}{(3x - 4)}$, then find $\frac{dy}{dx}$.

Sol. Here, $u(x) = x^2 + 2x$, $v(x) = 3x - 4$.

Using quotient rule, we get

$$\frac{dy}{dx} = \frac{(3x-4)\frac{d(x^2+2x)}{dx} - (x^2+2x)\frac{d(3x-4)}{dx}}{(3x-4)^2}$$

$$= \frac{(3x-4)(2x+2) - (x^2+2x)3}{(3x-4)^2} = \frac{3x^2 - 8x - 8}{(3x-4)^2}$$

ILLUSTRATION 2.34

If $y = \frac{\sin x}{x + \cos x}$, then find $\frac{dy}{dx}$.

Sol. Here, $u(x) = \sin x$, $v(x) = x + \cos x$.

$$\frac{dy}{dx} = \frac{(x+\cos x)\frac{d(\sin x)}{dx} - \sin x\frac{d(x+\cos x)}{dx}}{(x+\cos x)^2}$$

$$= \frac{(x+\cos x)\cos x - \sin x(1 - \sin x)}{(x+\cos x)^2}$$

$$= \frac{x\cos x + \cos^2 x - \sin x + \sin^2 x)}{(x+\cos x)^2}$$

$$= \frac{x\cos x - \sin x + \sin^2 x + \cos^2 x}{(x+\cos x)^2}$$

$$= \frac{x\cos x - \sin x + 1}{(x+\cos x)^2}$$

CHAIN RULE OR "OUTSIDE INSIDE" RULE

This rule is very useful in physical applications. Suppose that f is a function of x, which in turn is a function of t. The derivative of f with respect to t is equal to the product of the two derivatives, df/dx and dx/dt, i.e., $\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$. It sometimes helps to think about the chain rule the following way: if $y = f(g(x))$, $dy/dx = f'[g(x)] \cdot g'(x)$.

In words: To find dy/dx, differentiate the "outside" function f and leave the "inside" $g(x)$ alone; then multiply by the derivative of the inside.

Important Points

	Function	Derivative
(i)	$y = u^n$	$\frac{dy}{dx} = nu^{n-1}\left(\frac{du}{dx}\right)$
(ii)	$y = \sin u$	$\frac{dy}{dx} = \cos u\left(\frac{du}{dx}\right)$
(iii)	$y = \cos u$	$\frac{dy}{dx} = -\sin u\left(\frac{du}{dx}\right)$

(iv)	$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\left(\frac{du}{dx}\right)$
(v)	$y = e^u$	$\frac{dy}{dx} = e^u\left(\frac{du}{dx}\right)$
(vi)	$y = \ln u$	$\frac{dy}{dx} = \frac{1}{u}\left(\frac{du}{dx}\right)$

- If $y = \sin(ax + b)$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[(ax+b)^n\right] = n(ax+b)^{n-1} \times a = na(ax+b)^{n-1}.$$

- If $y = \sin(ax + b)$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(ax+b)\right] = \cos(ax+b).\frac{d(ax+b)}{dx}$$
$$= \cos(ax+b).a = a\cos(ax+b)$$

- If $y = \cos(ax + b)$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(ax+b)\right] = -\sin(ax+b).\frac{d(ax+b)}{dx}$$
$$= -\sin(ax+b).a = -a\sin(ax+b)$$

ILLUSTRATION 2.35

If $y = 4e^{x^2-2x}$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = 4.\frac{d}{dx}e^{(x^2-2x)} = 4e^{(x^2-2x)} \times \frac{d}{dx}(x^2-2x)$

$$= 4e^{x^2-2x} \times (2x-2) \Rightarrow \frac{dy}{dx} = 8(x-1)e^{x^2-2x}$$

ILLUSTRATION 2.36

If $y = (x^2 + 1)^{1/2}$, find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{d}{dx}(x^2+1)^{1/2}$

$$= \frac{1}{2}(x^2+1)^{\left(\frac{1}{2}-1\right)} \times \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x \Rightarrow \frac{dy}{dx} = \frac{x}{(x^2+1)^{1/2}}$$

ILLUSTRATION 2.37

Find the derivative of $y = \sin(x^2 - 4)$.

Sol. We now know how to differentiate $\sin x$ and $x^2 - 4$, but how do we differentiate a composite like $\sin(x^2 - 4)$? The answer is, with the chain rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The chain rule is probably the most widely used differentiation rule in mathematics.

Let $u = x^2 - 4$

Then $y = \sin u$

We know the differentiation of u w.r.t x and differentiation of y w.r.t. u.

$$\frac{du}{dx} = 2x \text{ and } \frac{dy}{du} = \cos u$$

and we can write $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$.

Hence, $\frac{dy}{dx} = (\cos u)\cdot(2x) = 2x\cdot\cos(x^2-4)$

ILLUSTRATION 2.38

If $y = \cos^2 x$, then find $\frac{dy}{dx}$.

Sol. We know that $\cos 2x = 2\cos^2 x - 1$

$$\Rightarrow \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\cos^2 x] = \frac{d}{dx}\left[\frac{1+\cos 2x}{2}\right]$$
$$= \frac{d}{dx}(1/2) + \frac{1}{2}\frac{d}{dx}(\cos 2x)$$
$$= \frac{1}{2}(-\sin 2x)2 = -\sin 2x$$

Alternative method: $y = \cos^2 x = (\cos x)^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d[(\cos x)^2]}{dx} = 2(\cos x)^{2-1}(-\sin x)$$
$$= -2\cos x\sin x = -\sin 2x$$

ILLUSTRATION 2.39

If $y = \cos x^3$, then find $\frac{dy}{dx}$.

Sol. This type of questions are done by substitution.

Put $x^3 = u$, and differentiate u w.r.t. x. So,

$$3x^2 = \frac{du}{dx} \quad \text{...(i)}$$

Also, $y = \cos u$.

Now, differentiating y w.r.t. u, we get

$$\frac{dy}{du} = -\sin u = -\sin x^3 \quad \text{...(ii)}$$

Using Eqs. (i) and (ii), we get $\frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{dx} = -3x^2 \sin x^3$

ILLUSTRATION 2.40

If $x = at^4$, $y = bt^3$, then find $\frac{dy}{dx}$.

Sol. This is called implicit differentiation or indirect differentiation. Here, both the given terms are differentiated independently w.r.t. a third variable and then combined together. Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = 4at^3, \quad \frac{dy}{dt} = 3bt^2$$

Dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$, we get

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{3bt^2}{4at^3}=\frac{3b}{4at}$$

DOUBLE DIFFERENTIATION

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')'=f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y=f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

Another notation is $f''(x)=D_2f(x)$.

Interpretation of Double Derivative

We can interpret $f''(x)$ as the slope of the curve $y=f'(x)$ at point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y=f(x)$.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If $s=s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t)=s'(t)=\frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called acceleration $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function.

$$a(t)=v'(t)=s''(t)$$

or in Leibniz notation, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$

ILLUSTRATION 2.41

If $f(x)=x\cos x$, find $f''(x)$.

Sol. Using the product rule, we have

$$f'(x)=x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)=-x\sin x+\cos x$$

To find $f''(x)$, we differentiate $f'(x)$:

$$f''(x)=\frac{d}{dx}(-x\sin x+\cos x)$$

$$=-x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(-x)+\frac{d}{dx}(\cos x)$$

$$=-x\cos x-\sin x-\sin x=-x\cos x-2\sin x$$

APPLICATIONS OF DERIVATIVE IN PHYSICS

The position (x) of the particle is change with time according to relation $x=f(t)$. The time rate of change of position is equal to instantaneous velocity or simply velocity $v=dx/dt$. The rate of change of velocity with respect to time is equal to instantaneous acceleration or simply acceleration $a=dv/dt$. Hence, acceleration

$$a=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2}$$

We can say acceleration is the second derivative of position with respect to time.

ILLUSTRATION 2.42

The displacement in metre of a particle moving with uniform acceleration after t second is given by $s=27t+4.9t^2$. Find the magnitudes of initial velocity, acceleration and final velocity at $t=3$ seconds.

Sol. If v is the velocity at any instant, then

$$v=\frac{ds}{dt}=\frac{d}{dt}(s)=\frac{d}{dt}(27t+4.9t^2)$$

$$=\frac{d}{dt}(27t)+\frac{d}{dt}(4.9t^2)=27+4.9\times 2t=27+9.8t$$

When $t=0$, $v_0=$ initial velocity $=27$ m/s

When $t=3$ s,

$$v_3=\text{final velocity}=27+9.8\times 3=27+29.4=56.4 \text{ m/s}$$

If a is the acceleration, then

$$a=\frac{du}{dt}=\frac{d}{dt}(v)=\frac{d}{dt}(27+9.8t)=9.8 \text{ i.e., } a=9.8 \text{ m/s}^2$$

ILLUSTRATION 2.43

A toy train is moving along x-axis in such a way that its co-ordinate (x) varies with time (t) according to the expression $x=2-5t+6t^2+9t^3$, where x is in meters and t is in seconds. What is the initial velocity and acceleration of the toy train?

Sol. The time rate of change of position is equal to instantaneous velocity or velocity $v=\frac{dx}{dt}$.

Given position time relation $x=2-5t+6t^2+9t^2$

Velocity of the toy train $v=\frac{dx}{dt}$

$$\Rightarrow \quad v=\frac{d}{dt}(2-5t+6t^2+9t^3)$$

$$=\frac{d(2)}{dt}-5\frac{d(t)}{dt}+6\frac{d(t^2)}{dt}+9\frac{d(t^3)}{dt}$$

$$=-5+6(2t)+9(3t^2)=-5+12t+27t^2$$

Hence, velocity of the toy train, $v=-5+12t+27t^2$

The rate of change of velocity with respect to time is equal to instantaneous acceleration or acceleration,

$$a=\frac{dv}{dt}=\frac{d^2x}{dt^2}$$

Hence, acceleration of the toy train $a=\frac{dv}{dt}$

$$a=\frac{d}{dt}\left(-5+12t+27t^2\right)$$

$$=-\frac{d(5)}{dt}+12\frac{d(t)}{dt}+27\frac{d(t^2)}{dt}$$

$$=0+12[1]+27[2t]=12+54t$$

Hence, acceleration of the toy train $a=12+54t$

Now we have velocity and acceleration of the toy train in relation with time.

Velocity $v = -5 + 12t + 27t^2$ and acceleration $a = 12 + 54t$

'Initial' means at $t = 0$, for initial velocity and acceleration of the toy train we put $t = 0$ in the above equations of v and a.

$$v|_{(at\ t=0)} = -5 + 12t + 27t^2 = -5 \text{ m/s}$$

$$a|_{(at\ t=0)} = 12 + 54t = 12 \text{ m/s}^2$$

ILLUSTRATION 2.44

A ball is thrown up along an inclined plane. It goes up with a velocity which is decreasing continuously, then the ball comes to a stop momentarily. Thereafter it starts coming down with an increasing velocity. Its coordinate x as a function of time t come out to be: $x(t) = 18 + (12)t - (2)t^2$. Where x is measured along the inclined path and the positive x direction is up the slope. Determine an expression for the velocity component $v_x(t)$ as function of time. Also determine its velocity at $t = 2$ s.

Sol. The time rate of change of position is equal to instantaneous velocity or velocity $v = \dfrac{dx}{dt}$.

Given position time relation $x(t) = 18 + (12)t - (2)t^2$

$$v_x(t) = \frac{d}{dt}(18 + 12t - 2t^2)$$

$$= 12 - 2 \times 2t$$

$$\Rightarrow \quad v_x(t) = 12 - 4t$$

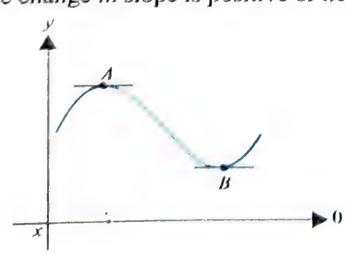
Now, to find velocity of the ball at $t = 2$s, substitute the value of t in $v_x(t)$

$$v|_{at\ t=2\text{ sec}} = 12 - (4 \times 2) = 12 - 8 = 4 \text{ m/s}$$

Maximum and Minimum Values of a Function

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, the first-order derivative becomes zero. As at these points the function does not offer any variation, the derivative ends up with zero to indicate no variation. The important thing here is that the first derivative offers no identification points of maxima and minima. This job is completed by the second derivative.

From the graph, we can easily find out that the slope of the curve at points A and B is zero. So dy/dx must be zero, i.e., $dy/dx = 0$. Now to distinguish A and B as the points of maxima or minima, we need to check the trend of the slope at the points of consideration, i.e., whether the change in slope is positive or negative.

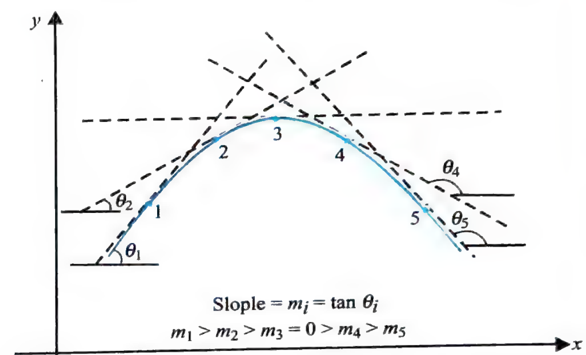


Maxima

Just before the maximum, the slope is positive; at the maximum, it is zero; and just after the maximum; it is negative. Thus, dy/dx decreases at a maximum and hence the rate of change of dy/dx is negative at a maximum, i.e., $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) < 0$ at maximum.

The quantity $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is the rate of change of the slope. It is written as d^2y/dx^2.

Conditions for maxima are: (a) $\dfrac{dy}{dx} = 0$ and (b) $\dfrac{d^2y}{dx^2} < 0$.

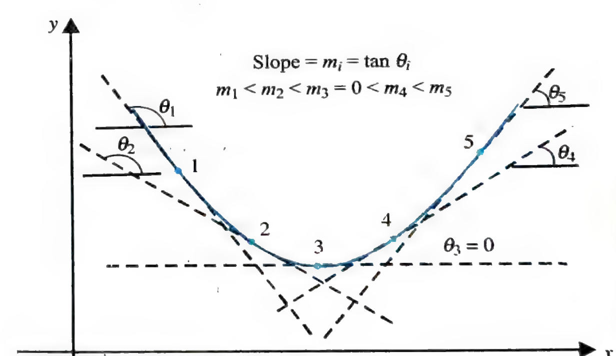


For maxima, as x increases the slope decreases

Minima

Similarly, at a minimum the slope changes from negative to positive. Hence, with the increase of x, the slope increases which means the rate of change of slope with respect to x is positive.

Hence, $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) > 0$.



For minima, as x increases slope increases.

Conditions for minima are: (a) $\dfrac{dy}{dx} = 0$ and (b) $\dfrac{d^2y}{dx^2} > 0$.

Quite often it is known from the physical situation whether the quantity is a maximum or minimum. The test on d^2y/dx^2 may then be omitted.

ILLUSTRATION 2.45

Find the minimum and maximum values of the function $y = x^3 - 3x^2 + 6$. Also find the values of x at which these occur.

Sol. Given $y = x^3 - 3x^2 + 6$

Differentiating y w.r.t. x, $\dfrac{dy}{dx} = 3x^2 - 6x$

Putting $dy/dx = 0$, we get the values at which the function is maximum or minimum. So

$$3x^2 - 6x = 0$$

$$\Rightarrow \quad x(3x-6) = 0 \quad \Rightarrow \quad x = 0, +2$$

To distinguish the values of x as the point of maximum or minimum, we need second derivative of the function.

$$\therefore \quad \frac{d^2y}{dx^2} = 6x - 6; \quad \text{Now} \left(\frac{d^2y}{dx^2}\right)_{x=0} = -6 < 0.$$

So, $x = 0$ is a point of maximum.

$$\text{Similarly, } \left(\frac{d^2y}{dx^2}\right)_{x=+2} = 6 > 0$$

So, $x = +2$ is a point of minimum.

Hence, the maximum value of y is $0^3 - 3 \times 0 + 6 = 6$ and the minimum value of y is $(2)^3 - 3(2)^2 + 6 = 2$.

ILLUSTRATION 2.46

The particle's position as a function of time is given as $x = 5t^2 - 9t + 3$. Find out the maximum value of position co-ordinate? Also, plot the graph.

Sol. We are given $x = 5t^2 - 9t + 3$, the position is changing with x.

Differentiate x w.r.t t, we get $\dfrac{dx}{dt} = 10t - 9 = 0$

$\therefore \quad t = 9/10 = 0.9$

Check whether maxima or minima exists.

We need to do double differentiation test: $\dfrac{d^2x}{dt^2} = 10 > 0$

Therefore, there exists a minima at $t = 0.9$

Now, check for the limiting values.

When $t = 0$, $x = 3$ and when $t = \infty$, $x = \infty$.

So, the maximum position co-ordinate does not exist.

ILLUSTRATION 2.47

The velocity v of a particle is given by the equation $v = 6t^2 - 6t^3$, where v is in m s^{-1}, t is the instant of time in seconds while 6 and 6 are suitable dimensional constants. At what values of t will the velocity be maximum and minimum? Determine these maximum and minimum values of the velocity.

Sol. Given $v = 6t^2 - 6t^3$. Differentiating v w.r.t. t, we have

$$\frac{dv}{dt} = 12t - 18t^2.$$

Putting $dv/dt = 0$, we will get the values of t at which v is maximum or minimum. Therefore,

$$12t - 18t^2 = 0 \quad \Rightarrow \quad t = 0, 2/3 \text{ s}$$

To the distinguish between points of maxima and minima, we need the second derivative of v.

$$\frac{d^2v}{dt^2} = 12 - 36t$$

$$\text{Now} \quad \left.\frac{d^2v}{dt^2}\right|_{t=0} = 12 > 0$$

So, $t = 0$ is a point of minima.

$$\left.\frac{d^2v}{dt^2}\right|_{t=2/3\text{s}} = 12 - 36 \times \frac{2}{3} = -12 < 0$$

So, $t = 2/3$ s is a point of maxima.

Hence, the minimum value of v is 0 m s^{-1} (by putting $t = 0$ s in v).

The maximum value of v is

$$6 \times \frac{4}{9} - 6 \times \frac{8}{27} = \frac{8}{3} - \frac{16}{9} = \frac{8}{9}\,\text{m s}^{-1} \left(\text{by putting } t = \frac{2}{3}\text{s in } v\right)$$

ILLUSTRATION 2.48

The position of a particle as a function of time is given as $x = 5t^2 - 9t + 3$. Here x is in metre and t is in sec. Find the maximum/minimum value of position of the particle and plot the graph.

Sol. Given function $x = 5t^2 - 9t + 3$

1st derivative, $\dfrac{dx}{dt} = 10t - 9 = 0 \qquad \therefore t = \dfrac{9}{10} = 0.9\text{sec}$

2nd derivative test, $\dfrac{d^2x}{dt^2} = 10 > 0$

As second derivative is positive, hence there exists a minima at $t = 0.9$ sec.

Now, check for the limiting values.

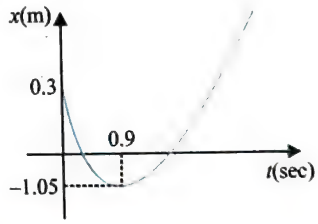
When $t = 0$; $x = 3$ m and as $t \to \infty$, $x \to \infty$.

It means the maximum position coordinate does not exist.

For minimum position, putting $t = 0.9$ sec in the equation

$$x = 5(0.9)^2 - 9(0.9) + 3 = -1.05 \text{ m}$$

Now let us plot the graph



CONCEPT APPLICATION EXERCISE 2.5

1. Differentiate the following w.r.t. x.
   (a) 9 (b) π^4
   (c) $2e^3$ (d) $x^2 + 5$
   (e) $(x + 5)^{-1/2}$ (f) $5x^{3/2}$
   (g) $\sqrt{x+3}$ (h) $(2x^2 + 9)^3$

2. Differentiate the following w.r.t. x.
   (a) $(x^2 + 3x)(2x + 7)$ (b) $(3x^2 + 2)(4x - 3x^3)$
   (c) $\sqrt{x}\,(x^3 + x^2 - 3x)$ (d) $\sin x \log x$

3. Differentiate the following w.r.t. x.
   (a) $\tan^3 x$ (b) $\tan x^2$
   (c) $\sin^2 \sqrt{x}$

4. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, find dy/dx.

5. A particle starts from rest and its angular displacement (in rad) is given by $\theta = \frac{t^2}{20} + \frac{t}{5}$. Calculate the angular velocity at the end of $t = 4$ s.
6. A metallic disc is being heated. Its area A (in m^2) at any time t (in second) is given by $A = 5t^2 + 4t + 8$. Calculate the rate of increase in area at $t = 3$ s.
7. If $y = \sin x$, then $\frac{d^2y}{dx^2}$ will be ________.
8. If $y = x^3$, then $\frac{d^2y}{dx^2}$ is ________.
9. If $y = 2\sin(\omega t + \phi)$ where ω and ϕ constant, then dy/dt will be ________.

ANSWERS

1. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** $2x$

(e) $-\frac{1}{2}(x+5)^{-3/2}$ **(f)** $\frac{15}{2}x^{1/2}$ **(g)** $\frac{1}{2}(x+3)^{-1/2}$

(h) $12(2x^2+9)^2x$

2. **(a)** $6x^2 + 26x + 21$ **(b)** $-45x^2 + 18x^2 + 8$

(c) $\frac{1}{2\sqrt{x}}(7x^3 + 5x^2 - 9x)$ **(d)** $\frac{\sin x}{x} + \cos x \log x$

3. **(a)** $3(\tan^2 x)(\sec^2 x)$ **(b)** $\sec^2(x^2)2x$

(c) $\frac{1}{2\sqrt{x}}\sin(2\sqrt{x})$

4. $\tan(\theta/2)$ **5.** 0.6 rad s^{-} **6.** 34 m^2/s **7.** $-\sin x$

8. $6x$ **9.** $2\omega\cos(\omega t + \phi)$

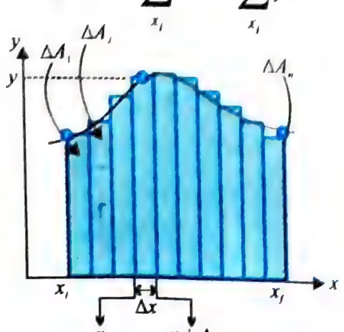
ELEMENTARY INTEGRATION

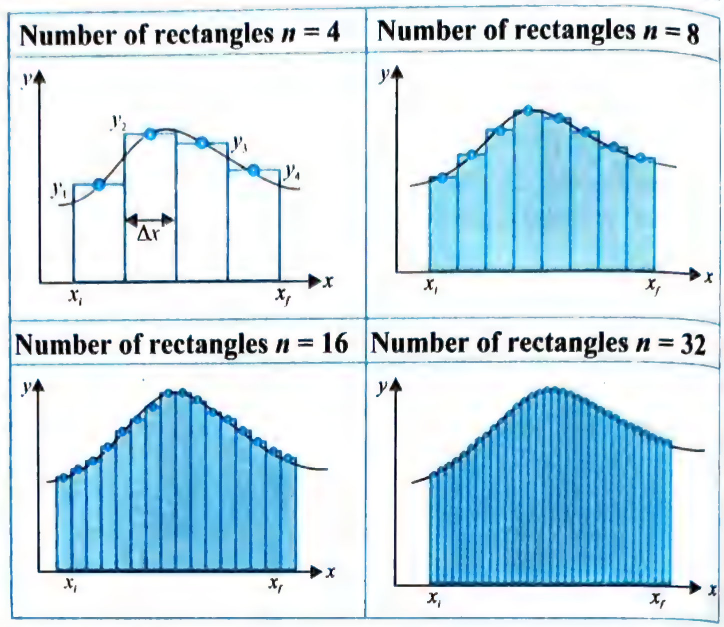
Consider a function $y = f(x)$ is plotted, the graph of the function is a curve as shown in the figure below. Suppose that we are interested in finding the area under the curve from x_i to x_f. Since it is not a regular geometrical figure (such as a rectangle, a square, a circle, etc.), no formula for evaluating the area is known to us. However, it is possible to calculate this area by making use of integral calculus. For this, we divide the whole area into a number of elementary rectangular strips, say each of width Δx as shown in the figure.

$$A = \Delta A_1 + \Delta A_2 + \ldots = \sum_{x_i}^{x_f} \Delta A$$

The area of the strip between x and $x + \Delta x$, $\Delta A = y\Delta x$

$$A = \Delta A_1 + \Delta A_2 + \ldots = \sum_{x_i}^{x_f} \Delta A = \sum_{x_i}^{x_f} y\,\Delta x$$





Now, if we increase the number of strips, the width Δx of an elementary strip will also decrease. In the limiting case, when $n \to \infty$, $\Delta x \to 0$, we can say Δx is very small or $\Delta x = dx$. It follows that in that event, the above summation will be exactly equal to the area under the curve from x_i to x_f.

$$A = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} y\,\Delta x = \int_{x_i}^{x_f} y\,dx$$

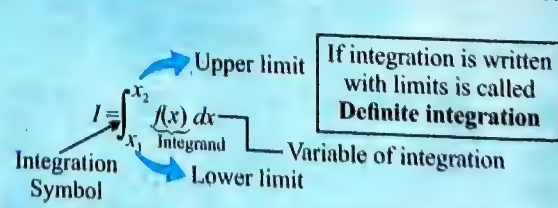
For getting area under the curve, here we added the areas of strips. Hence, integration of function gives the area between the limits x_i and x_f

$$A = \int_{x_i}^{x_f} y\,dx = \int_{x_i}^{x_f} f(x)\,dx$$

This expression is read as integration of y between the limits x_i and x_f. Basically, integration is summation. The symbol $\int$ for integration owes its origin to letter S (for summation). The letter S, when stretched, takes the shape of symbol $\int$.

Important Points:

- The integration of a function $f(x)$ can be written as:



The result of a definite integration is a fixed value as area of a curve $y = f(x)$ between the limits x_i and x_f is always a fixed or definite quantity.

- If the above expression is given without limits, the integration is called 'indefinite integration': $I = \int f(x)dx$
In this case, we cannot get the definite value of area under the curve. Hence, in case of 'indefinite integration', we do not get a finite value of result.

Integration as Inverse of Differentiation

Integration is the process of finding the function whose derivative is given. For this reason, the process of integration is called inverse process of differentiation. For understanding this, let us consider a function, $y = f(x)$. We can write

$$\frac{dy}{dx} = f'(x) \text{ or } dy = f'(x)dx \quad \text{...(i)}$$

Differentiation means to divide the function into infinite number of small elements. We cannot add, like in algebra, as it involves infinite terms. Further each term is infinitely small. Integration is the method of summation of an infinite series in which each term tends to zero. In fact integration is just an inverse process of differentiation.

Integrating equation (i) both sides, we get

$$\int dy = \int f'(x)dx \Rightarrow y = f(x)$$

Here we can observe if we integrate $f'(x)$, we get $f(x)$. In fact integration of a functions is called the antiderivative of the function $f(x)$.

It means if we integrate the differentiation of a function, we will get the original function.

Consider a function $f(x)$, whose derivative w.r.t. x another function $f'(x)$ i.e., $\frac{d}{dx}[f(x)] = f'(x)$

Symbolically, it is written as $\int f'(x)dx = f(x)+c$

Here, $f'(x)$ is called integrand, $f'(x)dx$ is called element of integration and the symbol $\int$ is the sign for integration.

Let us proceed to obtain integral of x^n w.r.t. x. We know

$$\frac{d}{dx}(x^{n+1}+C) = (n+1)x^n \text{ (here } C \text{ is a constant)}$$

Since the process of integration is the inverse of differentiation,

$$\int (n+1)x^n dx = x^{n+1} + C$$

or $$(n+1)\int x^n dx = x^{n+1} + C$$

or $$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ (here } c \text{ is another constant)}$$

The above formula holds for all values of n, except $n = -1$. In similar way, we can write other formulas of integration.

The differentiation of $\sin x + c$: $\frac{d}{dx}(\sin x+c) = \cos x$	Hence the integration of $\cos x$: $\int(\cos x)dx = \sin x + c$
The differentiation of $\cos x + c$: $\frac{d}{dx}(\cos x+c) = \ \sin x$	Hence the integration of $\sin x$: $\int(\sin x)dx = -\cos x + c$

Finding Integration of Constant Function

We are given $y = k$(a constant)

Let us calculate $\int y\, dx$. We can write $\int k\,dx = k\int dx$

We know $\int (x^n)dx = \frac{x^{(n+1)}}{(n+1)} + c \qquad n \neq -1$

We can write, $k\int dx = k\int (x^0)dx = k\left[\frac{x^{(0+1)}}{(0+1)}\right] + c = kx + c$

Hence $k\int dx = kx + c$

ILLUSTRATION 2.49

Integrate w.r.t. x:

1. $y = x^9$ **2.** $y = x^{3/2}$ **3.** $y = x^{-7}$ **4.** $y = 6$

Sol.

1. $\int x^9\, dx = \frac{x^{9+1}}{9+1} + c = \frac{1}{10}x^{10} + c$

2. $\int x^{3/2} dx = \frac{x^{3/2+1}}{\frac{3}{2}+1} + c = \frac{2}{5}x^{5/2} + c$

3. $\int x^{-7} dx = \frac{x^{-7+1}}{-7+1} + c = -\frac{1}{6}x^{-6} + c$

4. As $k\int dx = kx + c$, hence, $6\int dx = 6x + c$

Some Important Properties of Integration

- The integral of the product of a constant and a function of x is equal to the product of the constant and integral of that function. Mathematically,

$$\int cu\,dx = c\int u\,dx,$$

where c is a constant and u is a function of x.

- The integral of the sum (or difference) of a number of functions is equal to the sum (or difference) of their integrals. Mathematically,

$$\int (u \pm v \pm w)dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx$$

ILLUSTRATION 2.50

Integrate $\int (2\cos x + 6x^2)dx$

Sol. We know that the integral of the sum of a number of functions is equal to the sum of their integrals.

Let $I = \int (2\cos x + 6x^2)dx$

$$= \int (2\cos x)dx + \int (6x^2)dx = 2\int (\cos x)dx + 6\int (x^2)dx$$

$$\Rightarrow \quad I = 2[\sin x] + 6\left[\frac{x^{(2+1)}}{(2+1)}\right] + c = 2\sin x + 2x^3 + c$$

Standard Formulae for Integration

1. $\int cu\,dx = c\int u\,dx$

2. $\int (u \pm v \pm w)dx = \int u\,dx \pm \int v\,dx \pm \int w\,dx$

3. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1}$ + constant, provided $n \neq -1$

4. $\int c\, dx = cx$ + constant, where c is a constant

5. $\int cx^n\, dx = c\dfrac{x^{n+1}}{n+1}$ + constant

6. $\int x^{-1}\, dx = \log_e x$ + constant

7. $\int e^x\, dx = e^x$ + constant

8. $\int e^{ax+b}\, dx = \dfrac{e^{ax+b}}{\dfrac{d}{dx}(ax+b)} = \dfrac{e^{ax+b}}{a}$ + constant

9. $\int \sin x\, dx = -\cos x$ + constant

10. $\int \sin ax\, dx = -\dfrac{\cos ax}{a}$ + constant

11. $\int \cos x\, dx = \sin x$ + constant

12. $\int \cos ax\, dx = \dfrac{\sin ax}{a}$ + constant

13. $\int \sec^2 x\, dx = \tan x$ + constant

14. $\int \text{cosec}^2 x\, dx = -\cot x$ + constant

15. $\int \sin(ax+b)\, dx = \dfrac{\cos(ax+b)}{a}$ + constant

16. $\int \cos(ax+b)\, dx = \dfrac{\sin(ax+b)}{a}$ + constant

17. $\int (ax+b)^n\, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)}$ + constant, provided $n \neq -1$

18. $\int \dfrac{a}{ax+b}\, dx = \log_e(ax+b)$ + constant

19. $\int e^{ax+b}\, dx = \dfrac{e^{ax+b}}{a}$ + constant

20. $\int \sec^2(ax+b)\, dx = \dfrac{\tan(ax+b)}{a}$ + constant

21. $\int \text{cosec}^2(ax+b)\, dx = -\dfrac{\cot(ax+b)}{a}$ + constant

ILLUSTRATION 2.51

Integrate the following w.r.t. x.

1. x^3 2. $x - \dfrac{1}{x}$ 3. $e^{2x} + \dfrac{1}{x^2}$

4. $\dfrac{1}{2x+3}$ 5. $\cos(4x+3)$ 6. $\cos^2 x$

Sol.

1. $y = x^3$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx = \int x^3\, dx = \frac{x^4}{4} + c \quad \left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right]$$

2. $y = x - \dfrac{1}{x}$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx = \int\left(x - \frac{1}{x}\right) dx \quad \left[\because \int \frac{1}{x}\, dx = \ln x + c\right]$$

$$= \frac{x^2}{2} - \ln x + c$$

3. $y = e^{2x} + \dfrac{1}{x^2}$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx = \int\left(e^{2x} + \frac{1}{x^2}\right) dx$$

$$= \frac{e^{2x}}{2} + \frac{x^{-2+1}}{-2+1} + c \left[\because \int e^{ax} = \frac{e^{ax}}{a} + c\right]$$

$$= \frac{e^{2x}}{2} - \frac{1}{x} + c$$

4. $y = \dfrac{1}{2x+3}$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx = \int \frac{dx}{2x+3} = \frac{\ln|2x+3|}{2} + c$$

5. $y = \cos(4x+3)$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx = \int \cos(4x+3)\, dx = \frac{\sin(4x+3)}{4} + c$$

6. $y = \cos^2 x$

Integrating both sides w.r.t. x, we get

$$I = \int y\, dx$$

$$= \int \cos^2 x\, dx = \int\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \int \frac{1}{2}\, dx + \int \frac{\cos 2x}{2}\, dx$$

$$= \frac{x}{2} + \frac{\sin 2x}{2\times 2} + c = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$$

DEFINITE INTEGRAL OF A FUNCTION

The definite integral of a function is one of the important concepts of mathematics directly applied in physics, engineering, and other disciplines. In physics, the calculation of certain physical quantities such as displacement, velocity, center of mass, moment of inertia, work, impulse, etc., includes the evaluation of a definite integral.

The integration of a function $f(x)$ w.r.t. x when x varies from a to b, called definite integral, is written as $\int_a^b f(x)dx$, where $f(x)$ is called the integrand, dx is called the derivative with respect to x, and a and b are the lower and upper limits of the integral, respectively. If the integral of $f(x)$ w.r.t. x is $g(x)$, then the value of the above integral is $[g(b) - g(a)]$.

Note: Indefinite integrals do not find direct application in physics unless we determine the value of the constant of integration. Also, the moment we assign a value to this constant, the integral no longer remains indefinite but becomes definite.

Algebraic Method to Evaluate Definite Integral

Algebraically, the definite integral between limits a and b can be calculated in the same way as we determine the indefinite integral with a little difference as illustrated below:

$$\int_a^b f(x)dx = I(x)\Big|_a^b = I(b) - I(a)$$

where $I(x)$ is the indefinite integral of the function $f(x)$.

Properties of Definite Integral

1. **Linearity property**

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

where α and β are constants.

2. **Additive property**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx$$

where c is an intermediate limit between a and b.

3. A definite integral carried out in a direction of decreasing x is equal in magnitude but opposite in sign to the integral carried out in the direction of increasing x between the same pair of limits, i.e., $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$.

Illustration 2.52

1. Solve:

(a) $\int_0^3 (ax^2 + bx + c)\,dx$ (b) $\int_{-1}^1 e^x\,dx$

(c) $\int_{-\pi/2}^{\pi/2} \cos x\,dx$ (d) $\int_0^{10} \sec^2(3x+6)\,dx$

Sol.

1. $I = \int_0^3 (ax^2 + bx + c)\,dx$

$$= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\Bigg|_0^3 = \frac{a(3)^3}{3} + \frac{b(3)^2}{2} + c(3) - 0$$

$$= 9a + \frac{9}{2}b + 3c \quad \left[\because \int_a^b f(x)dx = I(x)\Big|_a^b = I(b) - I(a)\right]$$

2. $I = \int_{-1}^1 e^x\,dx = e^x\Big|_{-1}^1 = e^1 - e^{-1} = e - \frac{1}{e}$

$$\left[\because \int_a^b f(x)dx = I(x)\Big|_a^b = I(b) - I(a)\right]$$

3. $I = \int_{-\pi/2}^{\pi/2} \cos x\,dx$

$$= \sin x\Big|_{-\pi/2}^{\pi/2} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 1 + 1 = 2$$

$$\left[\because \int_a^b f(x)dx = I(x)\Big|_a^b = I(b) - I(a)\right]$$

4. $I = \int_0^{10} \sec^2(3x+6)dx$

$$= \frac{\tan(3x+6)}{3}\Bigg|_0^{10} = \frac{1}{3}[\tan(36°) - \tan(6°)]$$

$$\left[\because \int_a^b f(x)dx = I(x)\Big|_a^b = I(b) - I(a)\right]$$

Illustration 2.53

Solve the integral $I = \int_0^\pi \sin^2 x\,dx$.

Sol. $I = \int_0^\pi \sin^2 x\,dx$

$$\int_0^\pi \sin^2 x\,dx = \int_0^\pi \left(\frac{1-\cos 2x}{2}\right)dx \left(\because \sin^2 x = \frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\int_0^\pi dx - \int_0^\pi (\cos 2x)\,dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi$$

$$= \frac{1}{2}\left[(\pi - 0) - \frac{(\sin 2\pi - \sin 0)}{2}\right] = \frac{\pi}{2}$$

Illustration 2.54

Solve the integral $I = \int_\infty^R \frac{GMm}{x^2}\,dx$.

Sol. $I = \int_\infty^R \frac{GMm}{x^2}\,dx = GMm\int_\infty^R \frac{dx}{x^2} = GMm\left[-\frac{1}{x}\right]_\infty^R$

$$= GMm\left[-\frac{1}{R} + \frac{1}{\infty}\right] = \frac{-GMm}{R}$$

The above expression represents gravitational potential energy which is obtained by integrating GMm/x^2 between the limits infinity (∞) and radius of the earth (R).

Rule of Substitution

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$$

1. Substitute $u = g(x)$, $du = g'(x)\,dx$.
2. Evaluate by finding an antiderivative $F(u)$ of $f(u)$ (any one will do.)

Illustration 2.55

Evaluate $\int \sqrt{1+y^2} \cdot 2y\,dy$

Sol. Let $I = \int \sqrt{1+y^2} \cdot 2y\,dy$

Let $u = 1 + y^2$, then $du = 2y\,dy$.

$$I = \int u^{1/2} du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} \quad \text{[Integrate, using rule no. 3 with } n = 1/2\text{]}$$

$$= \frac{2}{3}u^{3/2} + C \quad \text{[Simpler form]}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \quad \text{[Replace } u \text{ by } 1 + y^2\text{]}$$

Illustration 2.56

Evaluate: $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Sol. **Method 1**

Let $I = \int \frac{2z dz}{\sqrt[3]{z^2+1}}$.

Let $u = z^2 + 1$, then $du = 2z\,dz$.

$$I = \int u^{-1/3} du \quad \text{[In the form of } u^n du\text{]}$$

$$= \frac{u^{2/3}}{2/3} + C \quad \text{[Integrate with respect to } u\text{]}$$

$$= \frac{3}{2}\, u^{2/3} + C$$

$$= (z^2 + 1)^{2/3} + C \quad \text{[Replace } u \text{ by } z^2 + 1\text{]}$$

Method 2

Let $u = \sqrt[3]{z^2+1} \Rightarrow u^3 = z^2 + 1$

Then $3u^2\,du = 2z\,dz$

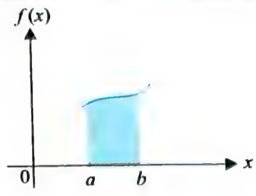
$$I = \int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2 du}{u} = 3 \cdot \int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \quad \text{[Integrate with respect to } u\text{]}$$

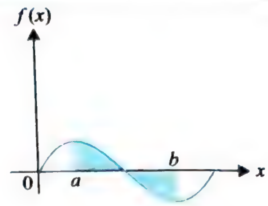
$$= \frac{3}{2}\,(z^2 + 1)^{2/3} + C \quad \text{[Replace } u \text{ by } (z^2 + 1)^{1/3}\text{]}$$

Geometrical Significance of a Definite Integral

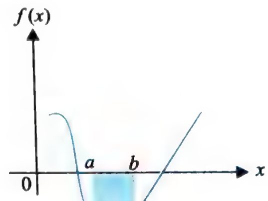
The definition and geometrical significance of a definite integral coincide with each other. Consider a function $y = f(x)$ which is represented graphically as shown in figure (a)–(c). The area bounded by this curve, x-axis, and the two lines $x = a$ and $x = b$ defines the definite integral of the function $f(x)$ as $D = \int_a^b f(x)\,dx$, where a is called the lower limit and b is called the upper limit. More precisely, we can say that the definite integral as defined by the above equation signifies the area under the curve between the limits a and b.



(a) $D > 0$



(b) $D = 0$



(c) $D < 1$

Note the following points from figure.

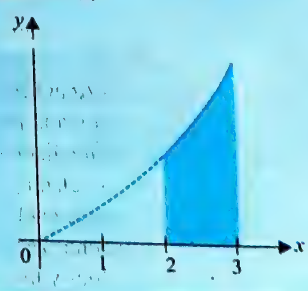
- If the curve lies above x-axis, the area enclosed under the curve is positive, i.e., $D > 0$.
- If the curve lies below x-axis, the area enclosed by the curve is negative, i.e., $D < 0$.
- If the curve is symmetrically distributed about x-axis, then the positive and negative areas enclosed by the curve will be cancelled, as shown in figure, i.e., $D = 0$.

Geometrical Method to Evaluate Definite Integral

The definite integral of a function can be determined if we can geometrically calculate the area enclosed by the curve. Though we are introducing this method as an alternative procedure to evaluate a definite integral, there exist certain situations in which it is the only possible method to evaluate the definite integral. For example, when the function $f(x)$ is very complicated and has no mathematical form, this method is useful.

Illustration 2.57

Calculate the area enclosed under the curve $f(x) = x^2$ between the limits $x = 2$ and $x = 3$.



Sol. $\int_2^3 f(x)dx = \int_2^3 x^2 dx$

$$= \left.\frac{x^3}{3}\right|_2^3 = \frac{(3)^3}{3} - \frac{(2)^3}{3} = \frac{19}{3} = 6.333$$

CONCEPT APPLICATION EXERCISE 2.6

1. Calculate:

(a) $\int (x^2+2)dx$ (b) $\int \left(x^3-\frac{1}{x}+3x\right)dx$

(c) $\int (x^{3/2}-x^{1/2}+5x)dx$ (d) $\int (e^{2x}+5)dx$

2. Evaluate:

(a) $\int_1^2 x^3\,dx$ (b) $\int_u^v Mv\,dv$

(c) $\int_3^4 \left(\frac{1}{x}\right)dx$ (d) $\int_4^9 \sqrt{x}\,dx$

(e) $\int_0^{\pi/4} \cos 2x\,dx$

ANSWERS

1. (a) $\frac{x^3}{3}+2x+c$ (b) $\frac{x^4}{4}-\log x+\frac{3x^2}{2}+c$

(c) $\frac{2x^{5/2}}{5}-\frac{2x^{3/2}}{3}+\frac{5}{2}x^2+c$ (d) $\frac{e^{2x}}{2}+5x+c$

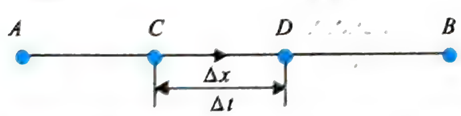
2. (a) $\frac{15}{4}$ (b) $\frac{M}{2}(v^2-u^2)$

(c) $\log_e(4/3)$ (d) 38/3 (e) $\frac{1}{2}$

APPLICATIONS OF INTEGRATION IN ONE-DIMENSIONAL MOTION

Let a particle move from point A to point B. Suppose the particle takes some finite time Δt to cover a finite distance Δx between points C and D. It is not known how the particle travelled from C to D. Particle may have travelled with uniform velocity or with variable velocity. Acceleration may or may not have been constant during the motion from C to D. But certainly we can write average velocity from C to D.

$$\vec{v}_{av}=\frac{\Delta\vec{x}}{\Delta t}$$



Now what happens if Δt approaches zero or, in other words, Δt is infinitesimally small. When we say Δt approaches zero, it does not mean that Δt is equal to zero. It means that Δt is very very close to zero. If Δt is very small, then the distance covered during this time will also be very small. So Δx also approaches zero and then points C and D will lie very close to each other.

Let us write $\Delta t = dt$ when Δt approaches zero and $\Delta x = dx$ when Δx approaches zero. Then average velocity becomes

$$v_{av}=\vec{v}_{av}=\frac{\Delta\vec{x}}{\Delta t}=\frac{d\vec{x}}{dt}$$

We can also write $\vec{v}_{av}=\lim_{\Delta t\to 0}\left(\frac{\Delta\vec{x}}{\Delta t}\right)=\frac{d\vec{x}}{dt}$

When Δt approaches zero, the average velocity becomes instantaneous velocity. We can explain this as follows: since points C and D are very close to each other, we can neglect any change in velocity from C to D, if there is any. We can assume that velocity remains constant for the motion from C to D. Then the average velocity becomes *instantaneous velocity* or simply *velocity*. So $\vec{v}=\vec{v}_{av}$ when $\Delta t\to 0$. Or we can write simply instantaneous velocity as $\vec{v}=d\vec{x}/dt$. It can be stated as follows: The time rate of change of position is equal to instantaneous velocity or simply velocity.

Similarly, we can define *instantaneous acceleration* or simply *acceleration* as the time rate of change of velocity. So

$$a=\frac{dv}{dt}=v\frac{dv}{dx}$$

We have seen that by using differentiation we can find velocity by differentiating position and acceleration by differentiating velocity. We can do reverse also. We can find velocity from acceleration and position from velocity. Here we will have to use integration.

Velocity is given by the integration of acceleration with respect to time. So, $v=\int a\,dt + C$.

Position is given by the integration of velocity with respect to time.

So $\quad x=\int v\,dt + C$

Here C is known as the constant of integration. Its value depends upon the given conditions. Its value can be different for different conditions.

DERIVATIONS OF EQUATIONS OF MOTIONS BY CALCULUS METHOD

Let a particle start moving with velocity u at time $t=0$ along a straight line. The particle has a constant acceleration a. Let at $t=t$, its velocity becomes v and it covers a displacement of s during this time.

$$v=\int a\,dt+C_1 \quad \text{or} \quad v=at+C_1$$

At $t=0$, $v=u$. This gives $C_1=u$.

Putting C_1, we get $v=u+at$ ...(i)

$$x=\int v\,dt+C_2$$
$$=\int (u+at)\,dt+C_2$$
$$=\int u\,dt+\int at\,dt+C_2=ut+\frac{1}{2}at^2+C_2$$

At $t=0$, $x=0$. This gives $C_2=0$.

Putting C_2, we get $x=ut+\frac{1}{2}at^2$ ...(ii)

$$v\frac{dv}{dx}=a$$

$$\Rightarrow \quad \int v\,dv=\int a\,dx+C_3 \quad\Rightarrow\quad \frac{v^2}{2}=ax+C_3$$

At $x=0$, $v=u$. This gives $C_3=u^2/2$.

$$\frac{v^2}{2}=ax+\frac{u^2}{2}\Rightarrow v^2=u^2+2ax \quad \text{...(iii)}$$

ILLUSTRATION 2.50

Sita is driving along a straight highway in her car. At time $t=0$, when Sita is moving at 10 m s^{-1} in the positive x-direction, she passes a signpost at $x=50$ m. Here acceleration is a function of time:

$$a=2.0\text{ m s}^{-2}-\left(\frac{1}{10}\text{ m s}^{-3}\right)t$$

(a) Derive expressions for her velocity and position as functions of time.
(b) At what time is her velocity greatest?
(c) What is the maximum velocity?
(d) Where is the car when it reaches the maximum velocity?

Sol.

(a) At time $t = 0$, Sita's position is $x_0 = 50$, and her velocity is $v_0 = 10\ \text{m s}^{-1}$. Since we know that acceleration a can be given as $a = dv/dt$. Hence, $dv = a\,dt$.

Integrating both sides, we get

$$\int_{v_0}^{v} dv = \int_0^t a\,dt.$$

$$[v - v_0] = \int_0^t \left[2 - \frac{1}{10}t\right] dt$$

$$v - 10 = \left[2t - \frac{t^2}{20}\right]_0^t$$

$$v = 2t - \frac{t^2}{20} + 10\,(\text{m s}^{-1}) = 10 + 2t - \frac{t^2}{20}(\text{m s}^{-1}) \qquad \text{...(i)}$$

We know that velocity in relation with position is given by

$$v = \frac{dx}{dt}$$

$$dx = v.dt$$

Integrating both sides, we get

$$\int_{x_0}^{x} dx = \int_0^t \left(2t - \frac{t^2}{20} + 10\right) dt$$

$$[x]_{50}^{x} = \left[2\frac{t^2}{2} - \frac{t^3}{60} + 10t\right]_0^t$$

$$x = 50 + 10t + t^2 - \frac{t^3}{60}(\text{m}) \qquad \text{...(ii)}$$

(b) The maximum value of v occurs when v stops increasing and begins to decrease. At this instant, $dv/dt = a = 0$. Setting the expression for acceleration equal to zero, we obtain

$$0 = 2.0 - (0.10)t, \Rightarrow t = \frac{20}{0.10} = 20\ \text{s}$$

a is positive between $t = 0$ and 20 s and negative after that. It is zero at $t = 20$ s, the time at which v is maximum. The car speeds up after $t = 2$ s (because v and a have the same sign) and slows down after $t = 20$ s (because v and a have opposite signs).

(c) We find the maximum velocity by substituting $t = 20$ s (when velocity is maximum) into the general velocity equation:

$$v_{max} = 10 + 2 \times 20 - \frac{(20)^2}{20} = 30\ \text{m s}^{-1}$$

(d) The maximum value of v occurs at time $t = 20$ s. We obtain the position of the car (that is, the value of x) at that time by substituting $t = 20$ s into the general expression for x.

$$x = 50 + 10 \times 20 + (20)^2 - \frac{(20)^3}{60} = 517\ \text{m}$$

ILLUSTRATION 2.59

Let the instantaneous velocity of a rocket, just after launching, be given by the expression $v = 2t + 3t^2$ (where v is in m s^{-1} and t is in seconds). Find out the distance travelled by the rocket from $t = 2$ s to $t = 3$ s.

Sol. To find the distance travelled, we need to integrate v. [The limits of integration will be from 2 s to 3 s as we have to find the distance travelled between $t = 2$ s and $t = 3$ s.

$$x = \int_2^3 v\,dt = \int_2^3 (2t + 3t^2)\,dt$$

$$= \left|\frac{2t^2}{2} + \frac{3t^3}{3}\right|_2^3 = \left|t^2 + t^3\right|_2^3 = 24\ \text{m}$$

ILLUSTRATION 2.60

A particle moves with a constant acceleration $a = 2\ \text{m s}^{-2}$ along a straight line. If it moves with an initial velocity of $5\ \text{m s}^{-1}$, then obtain an expression for its instantaneous velocity.

Sol. We know that acceleration is the time rate of change of velocity, i.e., $a = dv/dt$ and differentiation is the inverse operation of integration. So by integrating acceleration, we can obtain the expression of velocity.

So, $\quad v = \int a\,dt = 2\int dt = 2t + c \qquad \text{...(i)}$

where c is the constant of integration and its value can be obtained from the initial conditions.

At $t = 0$, $v = 5\ \text{m s}^{-1}$. Putting these in Eq. (i), we have

$$5 = 2 \times 0 + c$$

$$\Rightarrow \quad c = 5\ \text{m s}^{-1}$$

Therefore, $v = 2t + 5$ is the required expression for the instantaneous velocity.

ILLUSTRATION 2.61

In the previous problem, if the particle occupies a position $x = 7$ m at $t = 1$ s, then obtain an expression for the instantaneous displacement of the particle.

Sol. Again, we can use the idea that displacement is the integration of velocity w.r.t. time.

So, $\quad x = \int v\,dt = \int (2t + 5)dt = \frac{2t^2}{2} + 5t + c = t^2 + 5t + c$

where c is the constant of integration. Its value can be determined by using the given condition. (As particular details have been given about the particle.)

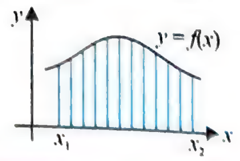
At $t = 1$ s, $x = 7$ m $= 7 = 1^2 + 5 \times 1 + c \Rightarrow c = 1$ m

Hence, the expression becomes $x = t^2 + 5t + 1$

Graphical Interpretation of Integration

We know the area under the graph gives the value of integration.

Area under graph, $A = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx$



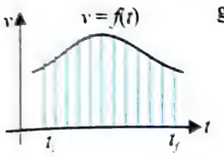
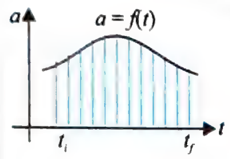
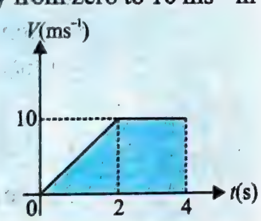
We know $v=\dfrac{dx}{dt} \Rightarrow dx = v\,dt$ Integrating both sides, $\int_{x_i}^{x_f} dx = \int_{t_i}^{t_f} v\,dt$	We know $a=\dfrac{dv}{dt} \Rightarrow dv = a\,dt$ Integrating both sides, $\int_{v_i}^{v_f} dv = \int_{t_i}^{t_f} a\,dt$
$x\Big\|_{x_i}^{x_f} = (x_f - x_i) = \int_{t_i}^{t_f} v\,dt$ Displacement; Area under velocity–time graph  It means area under velocity–time graph gives the displacement.	$v\Big\|_{v_i}^{v_f} = (v_f - v_i) = \int_{t_i}^{t_f} a\,dt$ Change in velocity; Area under acceleration–time graph  It means area under acceleration–time graph gives the change in velocity.

ILLUSTRATION 2.62

The speed of a car increases uniformly from zero to 10 ms^{-1} in 2s; and then remains constant.

(a) Find the distance traveled by the car in the first two seconds.

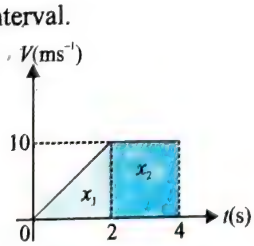
(b) Find the distance traveled by the car in the next two seconds.



Sol. The geometrical significance of the integral implies that area under the *vt* graph within the given time interval gives distance covered by the car during that interval.

(a) Distance covered in the first 2 s = Area of triangle Distance covered up to 2 sec,

$$X_1 = \frac{1}{2}(2)(10) = 10 \text{ m.}$$



(b) Distance covered in next two seconds = Area of the rectangle Distance covered during the time interval 2 sec to 4 sec, $x_2 = (4-2)(10) = 20$ m.

ILLUSTRATION 2.63

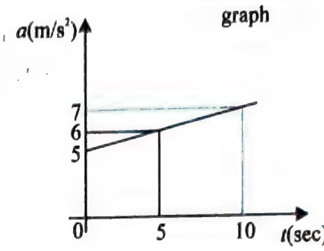
A particle starts moving at $t = 0$ along *x*-axis from $x = 0$, with initial velocity 10 m/s. The acceleration of the particle is linearly increasing as shown by its acceleration–time graph. Find the change in velocity of the particle during the time interval $t = 5$ sec to 10 sec. Also find the final velocity of the particle at time $t = 10$ sec.

Sol.

We know change in velocity:

$$(v_f - v_i) = \int_{t_i}^{t_f} a\,dt$$

Change in velocity; Area under acceleration-time graph



The change in velocity of the particle during the time interval $t = 5$ sec to $t = 10$ sec

= Area of graph between $t = 5$ sec and t = 10 sec

$$\Delta v = \frac{(6+7)}{2} \times (10-5) = \frac{13}{2} \times 5 = \frac{65}{2} \text{ m/s}$$

The change in velocity of the particle during the time interval $t = 0$ to $t = 10$ sec.

= Area of graph between $t = 0$ and $t = 10$ sec

$$\Delta v = \frac{(5+7)}{2} \times (10-0) = \frac{12}{2} \times 10 = 60 \text{ m/s}$$

The velocity of the particle at $t = 0$ is 10 m/s and change in velocity during this interval is 60 m/s. Hence, velocity of the particle at $t = 10$ sec: $v_{t=10\text{ sec}} = 10 + 60 = 70$ m/s.

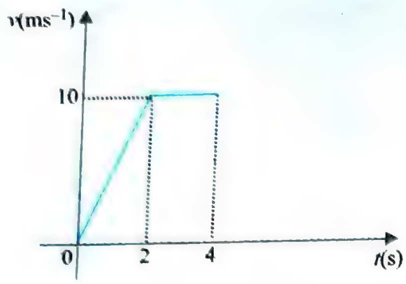
CONCEPT APPLICATION EXERCISE 2.7

1. The displacement of a particle is given by $y = (6t^2 + 3t + 4)$ m, where t is in seconds. Calculate the instantaneous speed of the particle.
2. The velocity of a particle is given by $v = 12 + 3(t + 7t^2)$. What is the acceleration of the particle?
3. A particle starts from origin with uniform acceleration. Its displacement after t seconds is given in meter by the relation $x = 2 + 5t + 7t^2$. Calculate the magnitude of its
   (a) Initial velocity **(b)** Velocity at $t = 4$ s
   (c) Uniform acceleration
   (d) Displacement at $t = 5$ s
4. The acceleration of a particle is given by $a = t^3 - 3t^2 + 5$, where a is in ms^{-2} and t is in second. At $t = 1$ s, the displacement and velocity are 8.30 m and 6.25 ms^{-1}, respectively. Calculate the displacement and velocity at $t = 2$ s.
5. A particle starts moving along the x-axis from $t = 0$, its position varying with time as $x = 2t^3 - 3t^2 + 1$.
   (a) At what time instant is its velocity zero?
   (b) What is the velocity when it passes through the origin?
6. A particle moves along the x-axis obeying the equation $x = t(t-1)(t-2)$, where x is in meter and t is in second
   (a) Find the initial velocity of the particle.
   (b) Find the initial acceleration of the particle.
   (c) Find the time when the displacement of the particle is zero.

(d) Find the displacement when the velocity of the particle is zero.

(e) Find the acceleration of the particle when its velocity is zero.

7. The speed of a car increases uniformly from zero to 10 m s^{-1} in 2 s and then remains constant (figure).

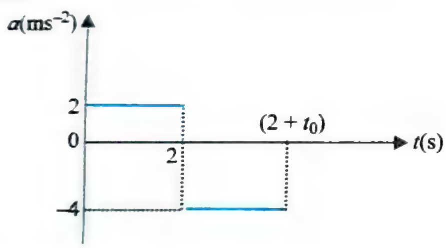


(a) Find the distance travelled by the car in the first two seconds.

(b) Find the distance travelled by the car in the next two seconds.

(c) Find the total distance travelled in 4 s.

8. A car accelerates from rest with 2 m s^{-2} for 2 s and then decelerates constantly with 4 m s^{-2} for t_0 second to come to rest. The graph for the motion is shown in figure.



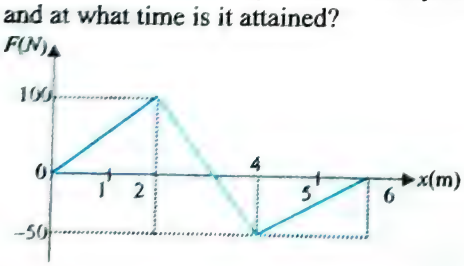
(a) Find the maximum speed attained by the car.

(b) Find the value of t_0.

9. A stationary particle of mass $m = 1.5$ kg is acted upon by a variable force. The variation of force with respect to displacement is plotted in the figure below.

(a) Calculate the velocity acquired by the particle after getting displaced through 6 m.

(b) What is the maximum speed attained by the particle and at what time is it attained?



10. The displacement of a body at any time t after starting is given by $s = 15t - 0.4t^2$. Find the time when the velocity of the body will be 7 m s^{-1}.

11. A particle moves along a straight line such that its displacement at any time t is given by $s = t^3 - 6t^2 + 3t + 4$ m. Find the velocity when the acceleration is 0.

12. The displacement x of a particle moving in one dimension under the action of a constant force is related to time t by the equation $t = \sqrt{x} + 3$, where x is in meter and t is in second. Find the displacement of the particle when its velocity is zero.

13. The acceleration of a motorcycle is given by $a_x(t) = At - Bt^2$, where $A = 1.50$ m s^{-3} and $B = 0.120$ m s^{-4}. The motorcycle is at rest at the origin at time $t = 0$.

(a) Find its position and velocity as functions of time.

(b) Calculate the maximum velocity it attains.

14. The acceleration of a particle varies with time t seconds according to the relation $a = 6t + 6$ m s^{-2}. Find velocity and position as functions of time. It is given that the particle starts from origin at $t = 0$ with velocity 2 m s^{-1}.

ANSWERS

1. $12t + 3$m s^{-1} **2.** $3 + 42t$

3. (a) 5 m s^{-1} **(b)** 61 m s^{-1} **(c)** 14 m s^{-2} **(d)** 202 m

4. $v = 8$ m s^{-1}, $x = 15.6$ m

5. (a) $t = 1$ s, $t = 0$ s **(b)** $t = 0$ m s^{-1}

6. (a) 2 m s^{-1} **(b)** –6 m s^{-2} **(c)** $t = 0$ s, $t = 1$ s, $t = 2$ s

(d) $\frac{-2}{3\sqrt{3}}$ m **(e)** $2\sqrt{3}$ m s^{-2}

7. (a) 10 m **(b)** 20 m **(c)** 30 m

8. (a) 4 m s^{-1} **(b)** 1 s

9. (a) $\frac{20}{\sqrt{3}}$ m s^{-1} **(b)** $\frac{20}{\sqrt{3}}\sqrt{5}$ m s^{-1}

10. 10 s **11.** –9 m s^{-1} **12.** 0

13. (a) $v_x = (0.75 \text{ m s}^{-3})t^3 - (0.04 \text{ m s}^{-4})t^3$

$x = (0.25 \text{ m s}^{-3})t^3 - (0.010 \text{ m s}^{-4})t^4$

(b) 39.1 m s^{-1}

14. $v = 3t^2 + 6t + 2$, $x = t^3 + 3t^2 + 2t$

Solved Examples

EXAMPLE 2.1

The side of a square increases uniformly at rate of 1 cm/s. At what rate is the area increasing when the side of the square is 6 m?

Sol. Let x be the side of the square.

Area of the square, i.e., $A = x^2$

Rate of change of area, i.e., $\frac{dA}{dt} = \frac{d}{dt}(x^2) = 2x\frac{dx}{dt}$

According to the given condition, $\frac{dx}{dt} = 1$ cm/s

When $x = 6$ m $= 600$ cm,

$$\frac{dx}{dt} = 2x\frac{dx}{dt} = 2 \times 600\,\text{cm} \times 1\,\text{cm/s} = 1200 \text{ cm}^2/\text{s}$$

Thus, the area is increasing at the rate of 1200 cm/s.

EXAMPLE 2.2

If the area of a circle increases at a uniform rate, prove that the rate of increase of the perimeter varies inversely as the radius.

Sol. Let x be the radius of the circle. If A is its area, $A = \pi x^2$.

Rate of increase of area, i.e.,

$$\frac{dA}{dt} = \frac{d}{dt}(\pi x^2) = 2\pi x\frac{dx}{dt}$$

Further, as the perimeter (P) of the circle is $2\pi x$, rate of increase of perimeter $= \frac{dP}{dt} = \frac{d}{dt}(2\pi x) = 2\pi\frac{dx}{dt}$

Thus, $$\frac{\frac{dP}{dt}}{\frac{dA}{dt}} = \frac{2\pi\frac{dx}{dt}}{2\pi x\frac{dx}{dt}} = \frac{1}{x}$$

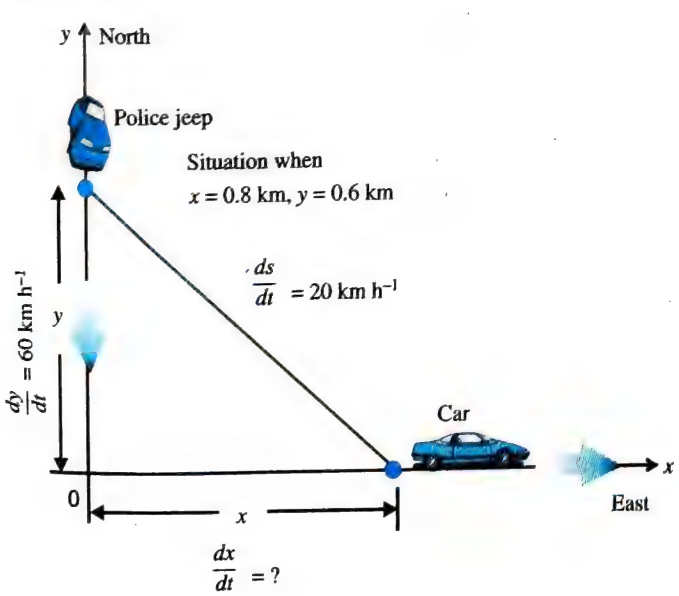
or, $$\frac{dP}{dt} = \frac{\frac{dA}{dt}}{x}$$

or, $$\frac{dP}{dt} \propto \frac{1}{x}\left(\text{as } \frac{dA}{dt} \text{ is a constant}\right)$$

EXAMPLE 2.3

A police jeep, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east, the police determine with radar that the distance between them and the car is increasing at 20 km h^{-1}. If the jeep is moving at 60 km h^{-1} at the instant of measurement, what is the speed of the car?

Sol. We draw a diagram of the car and jeep in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway. Let x be the position of car at time t.



y = position of jeep at time t,

s = distance between can and jeep at time t.

We assume x, y, and s to be differentiable functions of t.

$x = 0.8$ km, $y = 0.6$ km, $\frac{dy}{dt} = -60$ km h^{-1}

$\frac{ds}{dt} = 20$ km h^{-1} (dy/dt is negative because y is decreasing.)

The variables are related as:

$$s^2 = x^2 + y^2 \quad \text{...(i) (Pythagorean theorem)}$$

Differentiate Eq. (i) with respect to t, we get

$$\frac{ds^2}{dt} = \frac{dx^2}{dt} + \frac{dy^2}{dt} = \frac{ds^2}{ds}\cdot\frac{ds}{dt} = 2s\frac{ds}{dt}$$

Similarly, $$\frac{dx^2}{dt} = \frac{dx^2}{dx}\cdot\frac{dx}{dt} = 2x\frac{dx}{dt}$$

and similarly $$\frac{dy^2}{dt} = \frac{dy^2}{dy}\cdot\frac{dy}{dt} = 2y\frac{dy}{dt}$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Chain rule, $$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Evaluate, with $x = 0.8$ km, $y = 0.6$ km, $dy/dt = -60$ km h^{-1}, $ds/dt = 20$ km h^{-1}, and solve for dx/dt.

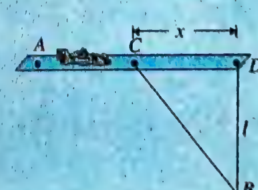
$$20 = \frac{1}{\sqrt{\underbrace{(0.8)^2 + (0.6)^2}_{1}}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\Rightarrow \quad 20 = 0.8\frac{dx}{dt} - 36 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

At the moment given in question, the car's speed is 70 km h^{-1}.

EXAMPLE 2.4

From point A located on a highway as shown in figure, one has to get by car as soon as possible to point B located in the field at a distance l from the highway. It is known that the car moves in the field η time slower on the highway. At what distance from point D one must turn off the highway?



Sol. Suppose at x distance from D the car turns off the highway. Let v be the speed of car on highway, then its speed on field will be v/η.

Time of motion = Time to travel on highway + Time to travel on field

Time to travel on highway, $T_{\text{highway}} = \frac{AD - x}{v}$

Time to travel on field, $T_{\text{field}} = \dfrac{\sqrt{x^2+l^2}}{v/\eta}$

Here total time, $T = \left(\dfrac{AD-x}{v}\right) + \dfrac{\sqrt{x^2+l^2}}{v/\eta}$ ...(i)

If T to be minimum, $\dfrac{dT}{dx} = 0$

$$\frac{dt}{dx} = \frac{d}{dx}\left[\frac{AD-x}{v} + \frac{\eta}{v}(x^2+l^2)^{1/2}\right] \quad [AD \text{ is constant}]$$

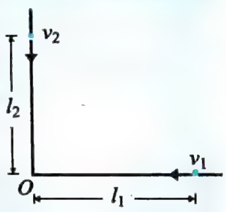
$$0 = -\frac{1}{v} + \frac{\eta}{v} \times \frac{1}{2}(x^2+l^2)^{-1/2} \times 2x$$

$$\eta^2 x^2 = x^2 + l^2 \Rightarrow x = \frac{l}{\sqrt{\eta^2 - 1}}$$

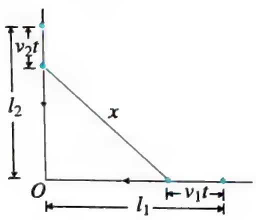
Here we need not to do double derivative test as we know the maximum time for this case is not defined.

EXAMPLE 2.5

Two particles, 1 and 2 move with constant velocities v_1 and v_2 along two mutually perpendicular straight lines towards the intersection point O. At the moment $t = 0$ the particles were located at the distance l_1 and l_2 from the point O. How soon will the distance between the particles become the smallest? What is it equal to?



Sol. The situation is shown in the figure. After time t, let separation between them is x. Then



$$x^2 = (l_1 - v_1 t)^2 + (l_2 - v_2 t)^2 \quad \text{...(i)}$$

x to be minimum, $\dfrac{dx}{dt} = 0$

Differentiating equation (i) w.r.t. time, we have

$$\frac{dx^2}{dt} = \frac{d}{dt}[(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2]$$

$$2x\frac{dx}{dt} = 2(l_1 - v_1 t) \times (-v_1) + 2(l_2 - v_2 t) \times (-v_2)$$

or $$0 = (l_1 - v_1 t)v_1 + (l_2 - v_2 t)v_2 = (l_1 v_1 + l_2 v_2) - t(v_1^2 + v_2^2)$$

which gives $t = \left(\dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)$

Now $x_{\min} = \sqrt{(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2}$ ...(ii)

Substituting value of t in equation (ii), we get

$$x_{\min} = \sqrt{\left[l_1 - v_1\left(\frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)\right]^2 + \left[l_1 - v_2\left(\frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)\right]^2}$$

After solving, we get $x_{\min} = \left(\dfrac{l_1 v_2 - l_2 v_1}{\sqrt{v_1^2 + v_2^2}}\right)$

EXAMPLE 2.6

Two bodies start moving in the same straight line at the same instant of time from the same origin. The first body moves with a constant velocity of 40 ms^{-1}, and the second starts from rest with a constant acceleration of 4 ms^{-2}. Find the time that elapses before the second catches the first body. Find also the greatest distance between them prior to it and time at which this occurs.

Sol. The distance travelled by first body in time t, $s_1 = 40t$

The distance travelled by second body in time t,

$$s_2 = \frac{1}{2} \times 4 \times t^2 = 2t^2$$

The separation between them at any time t,

$$s = s_1 - s_2 = 40t - 2t^2 \quad \text{...(i)}$$

When second body catches first body, $s_1 - s_2 = 0$

$$s_1 - s_2 = 40t - 2t^2 = 0$$

which gives $t = 0$, 20 s

At $t = 0$, both start moving from same point. Hence, at $t = 20$ s, the second body will catch the first body.

For s to be minimum or maximum, $ds/dt = 0$.

$$\therefore \quad \frac{ds}{dt} = \frac{d}{dt}(40t - 2t^2) = 40 - 4t \quad \text{...(ii)}$$

If $\dfrac{ds}{dt} = 0$, then $\dfrac{d}{dt}(40t - 2t^2) = 0$

or $40 - 2 \times 2t = 0$ or $t = 10$ s

Now check s is minimum or maximum doing double derivative test. Differentiating equation (ii) again $\dfrac{d^2s}{dt^2} = -4$.

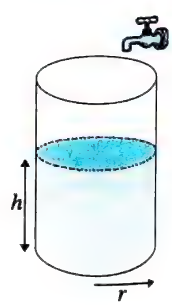
The double derivative of s w.r.t t is negative. Hence, we will get maximum value of the function at $t = 10$ s.

The greatest distance travelled them putting $t = 10$ s in Eq. (i), we get $s = 40 \times 10 - 2 \times 10^2 = 200$ m

EXAMPLE 2.7

Water pours out at the rate of Q from a tap, into a cylindrical vessel of radius r. Find the rate at which the height of water level rises when the height is h.

Sol. If V be the volume of liquid in the cylinder, at a height h of the water level, then $V = \pi r^2 h$.



Differentiating both sides w.r.t time t, we get

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\Rightarrow \quad Q = \pi r^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{Q}{\pi r^2}$$

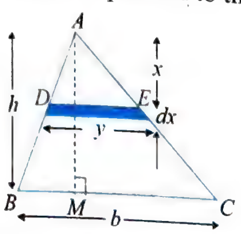
Note that dV/dt represents the rate at which the volume of liquid in the cylinder increases, which is same as the rate of pouring of water through the tap.

EXAMPLE 2.8

Using the method of integration, show that the area of triangle of base b and altitude h is $\frac{1}{2}bh$.

Sol. Let ABC be the triangle with base $BC = b$ and altitude $AM = h$.

Let us consider a thin strip DE on the triangle of thickness dx, at a distance x from the vertex A, parallel to the base BC.



If y be the length of the strip DE, then from similar triangles ABC and ADE, we have $\frac{y}{x} = \frac{b}{h}$.

$$\Rightarrow \quad y = \frac{bx}{h}$$

Therefore, the area of the rectangular strip DE is given by

$$dA = y\,dx = \frac{bx\,dx}{h}$$

The complete area of the triangle can be obtained by summing up (integration) the areas of individual strips such as DE.

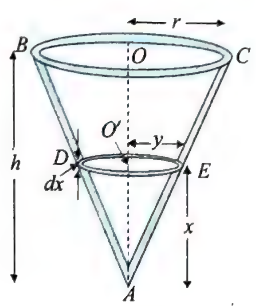
$$A = \sum dA = \int_0^h \frac{bx\,dx}{h} = \frac{b}{h}\left[\frac{x^2}{2}\right]_0^h = \frac{b}{2h}(h^2 - 0) = \frac{1}{2}bh$$

EXAMPLE 2.9

Using the method of integration, show that the volume of a right circular cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.

Sol. Let ABC be the right circular cone. Consider a circular section of the cone DE, with plane parallel to its base, of thickness dx, at a distance of x from the apex A.

If y be its radius, then from similar triangles, AOC and $AO'E$, we have



$$\frac{y}{x} = \frac{r}{h} \quad \Rightarrow y = \frac{rx}{h}$$

Therefore, area of the circular section $DE = \pi y^2 = \frac{\pi r^2 x^2}{h^2}$

Therefore, volume of the circular section $DE = \frac{\pi r^2 x^2 dx}{h^2}$

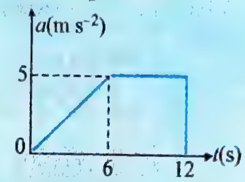
i.e., $\quad dV = \frac{\pi r^2 x^2 dx}{h^2}$

Now, the total volume of the cone can be obtained as the summation (integration) of the volumes of each circular sections such as DE, i.e.,

$$V = \sum dV = \int_0^h \frac{\pi r^2 x^2 dx}{h^2} = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{3h^2}[h^3 - 0] = \frac{\pi r^2 h}{3}$$

EXAMPLE 2.10

An experiment on the take off performance of an aeroplane shows that the acceleration varies as shown in and that it takes 12 s to take off from a rest position.



(a) Write the acceleration vs. time, velocity vs. time and position vs. time relations for complete journey.

(b) Plot velocity vs. time relation for the motion.

(c) Find the distance along the run way covered by the aeroplane.

Sol. **For time $t = 0$ to 6 s**

We can write equation for this motion, it is $y = mx$ type line where m is the slope of the line.

Here $a = \frac{(5-0)}{(6-0)} \cdot t \qquad \Rightarrow a = \frac{5}{6}t \ (\text{m s}^{-2}) \qquad$ ...(i)

(For $0 \le t < 6$ s)

From 6 s to 12 s acceleration is constant. we can write equation for this motion, it is y = constant.

Hence $\quad a = 5 \ (\text{m s}^{-2}) \qquad$ (For 6 s $\le t <$ 12 s)

For time $t = 0$ to 6 s

From Eq. (i), $a = \frac{dv}{dt} = \frac{5}{6}t \Rightarrow dv = \frac{5}{6}t\,dt$

Integrating both sides, we get $\int_0^v dv = \frac{5}{6}\int_0^t t\,dt$

$$[v]_0^v = \frac{5}{6}\left[\frac{t^2}{2}\right]_0^t \text{ or } (v - 0) = \frac{5}{12}(t^2 - 0) \text{ or } v = \frac{5}{12}t^2$$

Here velocity at $t = 6$ s is $v_{(t=6s)} = \frac{5}{12} \times 6^2 = 15 \text{ m s}^{-1}$

For time $t = 6$ s to 12 s

The acceleration is constant.

$$a = 5 \ (\text{m s}^{-2}) \text{ or } \frac{dv}{dt} = 5 \quad \text{or} \quad dv = 5\,dt$$

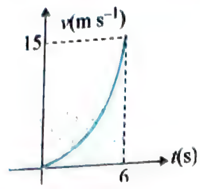
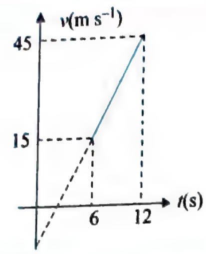
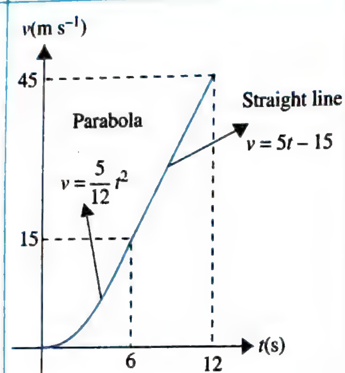
Integrating both sides, $\int_{15}^{v} dv = \int_6^t 5 \cdot dt$

$$[v]_{15}^{v} = 5[t]_6^t$$

$$v - 15 = 5\,[t - 6]$$

$$\Rightarrow \quad v = 5t - 15$$

Velocity–time relation will be a straight line of type $y = mx + c$.

For time t = 0 to 6 s $v = \frac{5}{12}t^2$ Velocity time graph will be parabola passing through origin.	
For time t = 6 s to 12 s $v = 5t - 15$ At t = 12 s $v = 5(12) - 15 = 45\ \text{m s}^{-1}$	
Velocity–time graph of complete motion	

Position–time relations

For time t = 0 to 6 s

As we know $v = \frac{dx}{dt} = \frac{5}{12}t^2$

Hence, $\frac{5}{12}t^2 = \frac{dx}{dt} \Rightarrow dx = \frac{5}{12}\int t^2 dt$

Integrating both sides, $\int_0^x dx = \frac{5}{12}\int_0^t t^2 dt$

$$(x-0) = \frac{5}{12}\left[\frac{t^3}{3}\right]_0^t \quad \Rightarrow \quad x = \frac{5}{36}t^3$$

Position or the distance travelled from 0 to 6 s

$$x_{(t=6\,sec)} = \frac{5}{36}\times(6)^3 = 30\ \text{m}$$

For time t = 6 s to 12 s

$$v = 5t - 15 \quad \Rightarrow \quad \frac{dx}{dt} = 5t - 5$$

or $\quad dx = (5t - 15)dt$

Integrating both sides

$$\int_{30}^{x} dx = \int_6^t (5t-15)\,dt$$

$$[x]_{30}^{x} = 5\int_6^t t\,dt - 15\int_6^t dt$$

$$(x-30) = 5\left[\frac{t^2}{2}\right]_6^t - 15[t]_6^t$$

$$x - 30 = \frac{5}{2}[t^2 - 6^2] - 15[t-6]$$

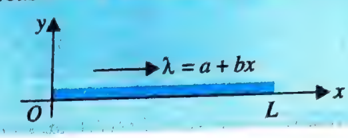
$$x = \frac{5}{2}t^2 - 15t + 30$$

At t = 12 s,

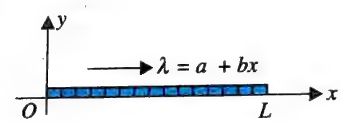
$$x_{(t=12\,s)} = \frac{5}{2}(12)^2 - 15\times12 + 30 = 210\ \text{m}$$

EXAMPLE 2.11

You are given a rod of length L. The linear mass density is λ such that $\lambda = a + bx$. Here a and b are constants and the mass of the rod increases as x decreases. Find the mass of the rod.



Sol. We are not calculated the mass of the rod simply multiplying linear mass density λ will the length of the rod as λ is not constant.



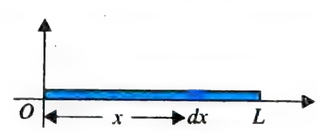
Hence, we have to divide entire rod length into a number of elements and add the mass of all elements.

$$M = dm_1 + dm_2 + dm_3 + \cdots$$

This step can be written in terms of integration.

$$M = \int dm$$

If we take an element dx on the rod at a distance x from the left end of the rod.



Mass of this element $dm = \lambda \,.\, dx$

$$dm = (a + bx)\,dx$$

Hence, total mass,

$$M = \int dm = \int_0^L (a+bx)\,dx = \int_0^L a\,dx + \int_0^L bx\,dx$$

$$= a\int_0^L dx + b\int_0^L x\,dx = a[x]_o^L + b\left[\frac{x^2}{2}\right]_0^L = aL + \frac{bL^2}{2}$$

EXAMPLE 2.12

The work done for a small displacement dx is given by $\frac{GMm}{x^2}dx$.

Calculate the total work done by using the method of integration when displacement varies from $x = R$ to $x = \infty$.

Sol. If W is the work done when displacement varies from $x=R$ to $x=\infty$, then

$$W=\int_R^\infty \frac{GMm}{x^2}dx = GMm\int_R^\infty \frac{1}{x^2}dx = GMm\int_R^\infty x^{-2}dx$$

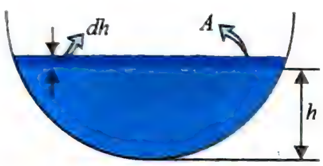
$$= GMm\left|\frac{x^{-2+1}}{-1}\right|_R^\infty = -GMm\left|\frac{1}{x}\right|_R^\infty$$

$$= -GMm\left|\frac{1}{\infty}-\frac{1}{R}\right| = -GMm\left|0-\frac{1}{R}\right| = \frac{GMm}{R}$$

EXAMPLE 2.13

Spirit is kept in a bowl whose top surface is exposed to atmosphere. The sprit evaporates at a rate that is proportional to the surface area of the liquid. Initially, the height of liquid in the bowl is H_0. It becomes $H_0/2$ in time t_0. How much more time will be needed for the height of liquid to become $H_0/4$?

Sol. Let the exposed area of the bowl be A, at this time the depth of the sprit in bowl be h. Let the depth decrease by dh in further interval dt



Volume of the sprit evaporated $= Adh$

According to the problem sprit evaporates at a rate that is proportional to the surface area of the liquid $\frac{Adh}{dt}\propto A$

$$\Rightarrow \quad \frac{Adh}{dt}=-kA \quad \text{or} \quad dh=-k\,dt \qquad \text{...(i)}$$

Where k is a positive constant. We have introduced a negative sign because h is decreasing with time and dh/dt is a negative quantity.

Integrating equation (i) both sides with proper limits, we get

$$\int_{H_0}^{h} dh=-k\int_0^t dt \Rightarrow [h]_{H_0}^{h} = -k[t]_0^t$$

$$h-H_0=-kt \qquad \text{or} \qquad h=H_0-kt$$

Now $h=H_0/2$ at $t=t_0$.

$$\therefore \quad \frac{H_0}{2}=H_0-kt_0 \Rightarrow \frac{H_0}{2}=kt_0 \qquad \text{...(ii)}$$

Let depth be $H_0/4$ at time t. Then

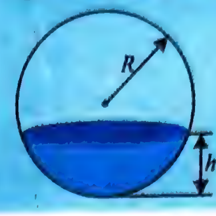
$$\frac{H_0}{4}=H_0-kt_0 \Rightarrow \frac{3H_0}{4}=kt \qquad \text{...(iii)}$$

From (ii) and (iii), we get $t=\frac{3}{2}t_0$

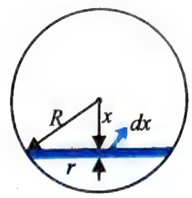
Hence, the time needed, $\Delta t=\frac{3}{2}t_0-t_0=\frac{t_0}{2}$

EXAMPLE 2.14

Find the volume of a segment of height h of a sphere of radius R.



Sol. Let us consider a disc-shaped element of thickness dx at a distance x from the centre.



The radius of elemental disc, $r^2=R^2-x^2$

Volume of elemental disc $dV=\pi r^2\,dx=\pi(R^2-x^2)dx$

$\therefore$ Required volume is

$$V=\int_{x=R-h}^{x=R} dV$$

$$=\pi\int_{(R-h)}^{R}(R^2-x^2)\,dx$$

$$=\left[\pi\left[R^2x\right]_{R-h}^{R}-\left[\frac{x^3}{3}\right]_{R-h}^{R}\right]$$

$$=\pi\left[R^2(R-R+h)-\frac{1}{3}R^3+\frac{1}{3}(R-h)^3\right]$$

$$=\pi\left[R^2h-\frac{R^3}{3}+\frac{1}{3}(R^3-h^3-3R^2h+3Rh^2)\right]$$

$$\Rightarrow \quad V=\frac{\pi}{3}h^2[3R-h]$$

3 Vectors

VECTORS AND SCALARS

Certain physical quantities are completely described by a numerical value alone (with units specified) and are added according to the ordinary rules of algebra. As an example, the mass of a system is described by saying that it is 5 kg. If two bodies, one having a mass of 5 kg and other having a mass of 2 kg, are added together to make a composite system, the total mass of the system becomes 5 kg + 2 kg = 7 kg. Such quantities are called **scalars**.

When we say, "a car is moving," we can raise two basic questions. First, in which direction is the car moving? Second, how fast is the car moving? This means that we need both magnitude (fastness = speed) and direction of motion of the car to express its motion by a vector quantity known as "velocity." Hence, the physical quantities such as position, velocity, acceleration, force, etc., which need both magnitude and direction to express them completely are termed as **vectors**.

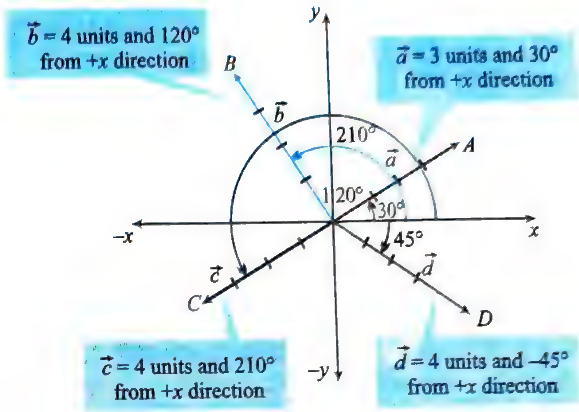
In Greek language, the word "vector" means "carrier" which signifies a directional nature.

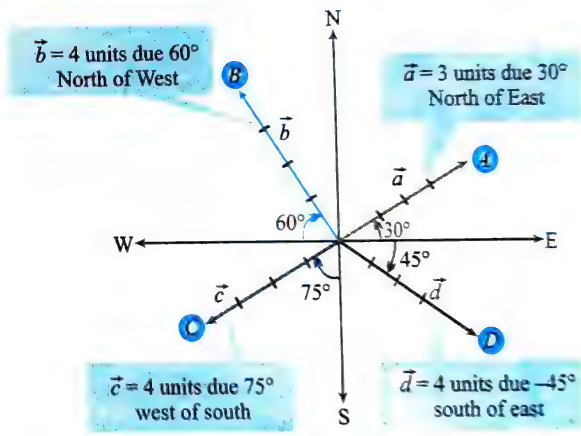
If a physical quantity in addition to magnitude (a) has a specified direction and (b) obeys the law of parallelogram of addition, then only it is said to be a vector. If any of the above conditions is not satisfied, the physical quantity cannot be a vector.

If a physical quantity is a vector, it has a direction, but the converse may or may not be true, i.e., if a physical quantity has a direction, it may or may not be a vector, e.g., time, pressure, surface tension, current, etc., have directions but are not vectors because they do not obey parallelogram law of addition.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$.

Example of vector quantity: displacement, velocity, acceleration, force, etc.





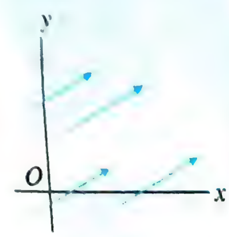
REPRESENTATION OF VECTORS

A vector is represented by a straight arrow. The tail is the staring point, the head is the ending point and the line measures the direction. The length of the line gives the value or magnitude of the vector. To express them mathematically, we need to draw the origin and the coordinate systems.

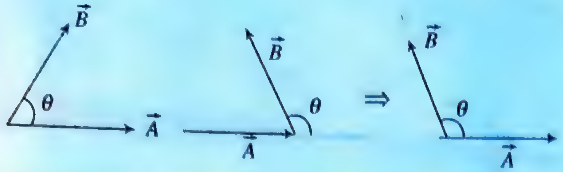
Mathematically, vector is represented by $\vec{A}$. Sometimes, it is represented by bold letter **A**.

Important Points:

- **Equality of two vectors:** For many purposes, two vectors $\vec{A}$ and $\vec{B}$ may be defined to be equal if they have the same magnitude and if they point in the same direction. That is, $\vec{A} = \vec{B}$ only if $A = B$ and if $\vec{A}$ and $\vec{B}$ point in the same direction along parallel lines. For example, all the vectors shown in Figure are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.



- Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e. $0 \le \theta \le \pi$).



- Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}$ $(= \lambda \vec{A})$ whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times $|\vec{A}|$.

ADDITION OF VECTORS

If you pour 2 liters of milk in a pot containing 1 liter of water, the net (resultant) amount of milk and water in the pot is $(2 + 1) = 3$ liters. If we remove (subtract) 1 liter of mixture from the pot we will have a net $(3 - 1) = 2$ liters of mixture in it.

Let us now repeat a similar experiment with a vector, say a *force*. If you push a stationary object with a force of 10 N and simultaneously another person pushes it with a force of same magnitude in opposite direction, the object does not move. That is because the net force acting on the object is zero, but according to the scalar addition, it should be $(10 + 10) = 20$ N. Hence, vectors cannot be operated like scalars. The vectors obey different algebra which is known as "vector algebra."

Notice that $\vec{A} + \vec{B} = \vec{C}$ is very different from $A + B = C$. The first equation is a vector sum, which must be handled carefully, such as with the graphical method. The second equation is a simple algebraic addition of numbers that is handled with the normal rules of arithmetic.

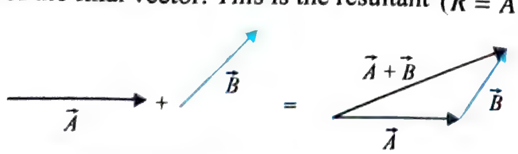
TRIANGLE LAW OF VECTOR ADDITION

Triangle law of vector addition states that when two vectors are represented by two sides of a triangle in magnitude and direction taken in same order, then the third side of that triangle represents in magnitude and direction the resultant of the vectors. We can now make a general rule for adding two vectors $\vec{A}$ and $\vec{B}$ by triangle law.

Step 1: Draw vectors $\vec{A}$ and $\vec{B}$ with proper length and direction.

Step 2: After drawing the first vector $(\vec{A})$, place the next vector $(\vec{B})$ so its tail (non-pointed end) is attached to the head (arrow end) of the first vector.

Step 3: Draw a vector connecting the tail of the first vector with the head of the final vector. This is the resultant $(\vec{R} = \vec{A} + \vec{B})$.

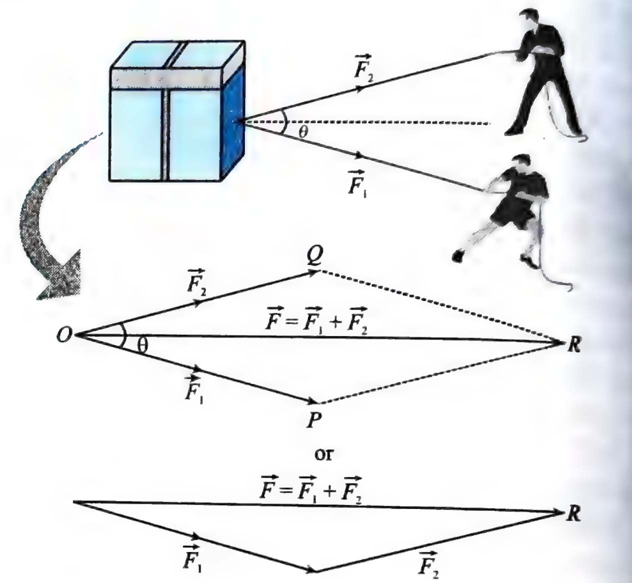


Place the tail of vector B at the tip of the vector A. The resultant starts where the vector A began and goes to the tip of the vector B.

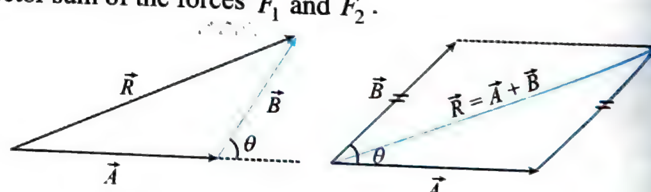
This technique for adding vectors is often called the "head to tail method" or "triangle law of vector addition".

PARALLELOGRAM LAW OF VECTOR ADDITION

Let two men pull a box simultaneously by two strings on a smooth horizontal floor making an angle θ between the strings as shown in the figure. If we pull the box by the string 1, it moves along the string 1 while when we pull the box by the string 2 along, it moves along the string 2. But when the box is pulled by both strings together, it does not move along any particular string. Rather it moves along a line between the strings (line of action of the forces $\vec{F}_1$ and $\vec{F}_2$). The proper line of motion of the box is shown as a dotted line which can be experimentally calculated. In fact, it is an experimental evidence that the box moves along the line which passes through the diagonal OR of the parallelogram $OPRQ$ as shown in the figure.



If the adjacent sides OP and OQ of the parallelogram represent the forces $\vec{F}_1$ and $\vec{F}_2$, respectively, it is experimentally verified that the diagonal OR of the parallelogram passing through the point of application O of the forces represents the net effect or vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$.

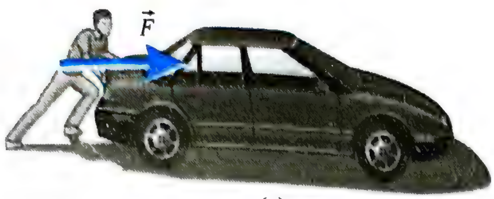


(a) (b)

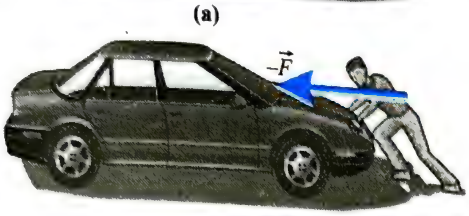
SUBTRACTION OF VECTORS

The subtraction of one vector from another is based on the following fact, that *when a vector is multiplied by –1, the magnitude of the vector remains the same, but the direction of the vector is reversed.*

In figure (a), a man is pushing on his stalled car with 500 N of force, and is trying to move it eastward. The force vector $\vec{F}$ that he applies to the car is 500 N, due east. The force vector for a man pushing on a car with same magnitude of force in a direction due west is $-\vec{F}$. The presence of the (-1) factor reverses the direction of the vector, but does not change its magnitude.

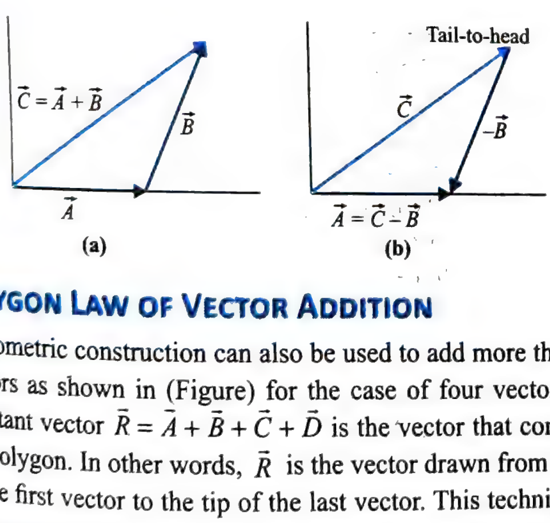


(a)



(b)

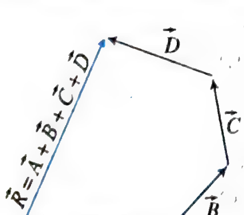
In practice, the vector subtraction is carried out exactly like vector addition, except the fact that one of the vectors added is multiplied by a scalar factor of –1. To see why, look at the two vectors $\vec{A}$ and $\vec{B}$ in figure (a). These vectors add together to give a third vector $\vec{C}$, thus $\vec{C} = \vec{A} + \vec{B}$. Therefore, we can calculate vector $\vec{A}$ as $\vec{A} = \vec{C} - \vec{B}$, which is an example of vector subtraction. However, we can also write this result as $\vec{A} = \vec{C} + (-\vec{B})$ and treat it as vector addition. Figure (b) shows how to calculate vector $\vec{A}$ by adding the vectors $\vec{C}$ and $-\vec{B}$. Notice that vectors $\vec{C}$ and $-\vec{B}$ are arranged tail to head and that any suitable method of vector addition can be employed to determine $\vec{A}$.



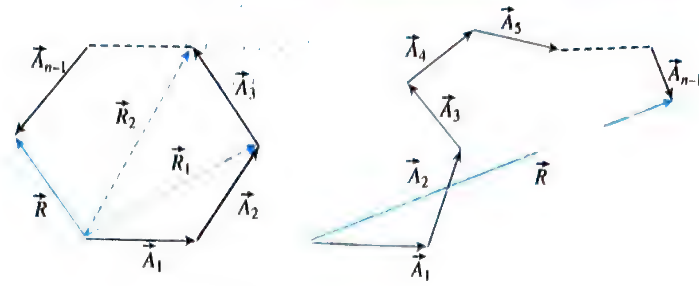
(a) (b)

Polygon Law of Vector Addition

A geometric construction can also be used to add more than two vectors as shown in (Figure) for the case of four vectors. The resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ is the vector that completes the polygon. In other words, $\vec{R}$ is the vector drawn from the tail of the first vector to the tip of the last vector. This technique for adding vectors is often called the **head to tail method**.



If n number of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_{n-1}$ are represented by a polygon of n sides, following the logic of triangle law of vectors, we have $\vec{A}_1 + \vec{A}_2 = \vec{R}_1$ (figure below).



$$\vec{R}_1 + \vec{A}_3 = \vec{R}_2$$

....

....

....

$$\vec{R}_{n+s} + \vec{A}_{n-1} = \vec{R}$$

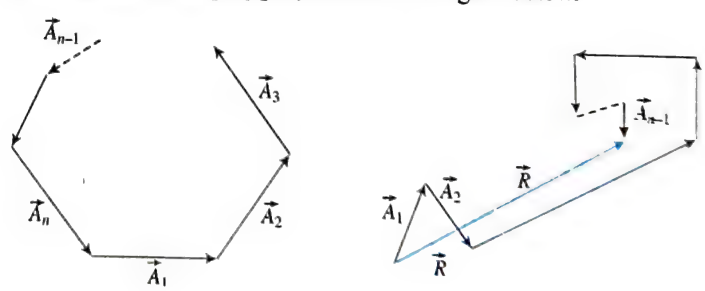
Summing up both sides, we have

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \cdots + \vec{A}_{n-1} = \vec{R}$$

This tells us that if $(n-1)$ sides of a polygon of n sides represent $(n-1)$ vectors, i.e., $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_{n-1}$, respectively, in same cyclic order (sense), the closing side, that is, nth side of the polygon taken in the reverse order (sense), represents the resultant $\vec{R}$ of all the above vectors. This is known as **polygon law of vectors.**

If we assume $-\vec{R} = \vec{A}_n$ substituting $-\vec{R} = \vec{A}_n$ in the previous expressions, we have $\vec{A}_1 + \vec{A}_2 + \vec{A}_n = 0$.

That means if n vectors are represented by the sides of a polygon of n sides in cyclic manner, the resultant of these vectors is a zero vector. Triangle and rectangle are the polygons of sides three and four, respectively. The non-coplanar vectors can be represented by a polygon of sides which cannot lie in a single plane. Sometimes, for coplanar vectors represented by a polygon, the resultant (closing side of the polygon) may cross the component vectors (other sides of the polygon) as shown in figure below.



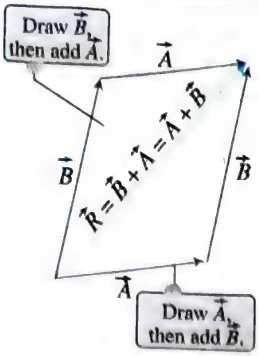
- Resultant of two unequal vectors cannot be zero.
- Resultant of three co-planar vectors may or may not be zero.
- Resultant of three non-co-planar vectors cannot be zero.

Some Properties of Vector Addition

Commutative Law of Addition

When two vectors are added, the sum is independent of the order of addition. This property, which can be seen from the geometric construction in Figure, is known as the commutative law of addition:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$



Associative Law of Addition

When three or more vectors are added, their sum is independent of the way in which the individual vectors are grouped together. A geometric proof of this rule for three vectors is given in figure. This property is called the associative law of addition:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

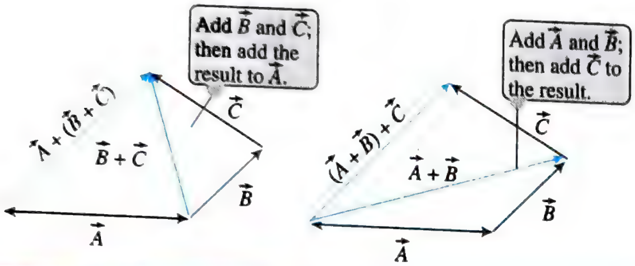
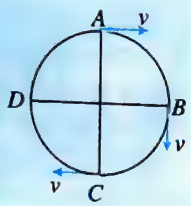


ILLUSTRATION 3.1

A car is moving round a circular track with a constant speed v of 20 m s^{-1} (as shown in Figure). At different times, the car is at A, B and C, respectively. Find the velocity change (a) from A to C, and (b) from A to B.

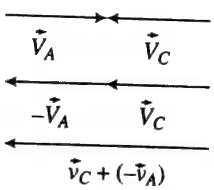


Sol.

(a) Change in velocity, as the particle moves from A to C, is the difference of final velocity vector and initial velocity vector.

$$\Delta\vec{v}_{AC} = \vec{v}_C - \vec{v}_A = \vec{v}_C + (-\vec{v}_A)$$

(If we take left direction as positive, the right direction will be negative.)



$$\Delta v_{AC} = 20 + 20 = 40 \text{ m s}^{-1}$$

As Δv_{AC} is positive, hence, change in the velocity should be in the direction of left, i.e., in the direction of $\vec{v}_C$.

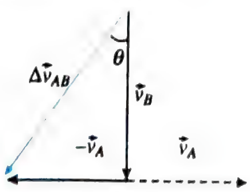
$$\Delta\vec{v}_{AC} = 40 \text{ m s}^{-1} \text{ in the direction of } \vec{v}_C$$

(b) Change in velocity, as the particle moves from A to B,

$$\Delta\vec{v}_{AB} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

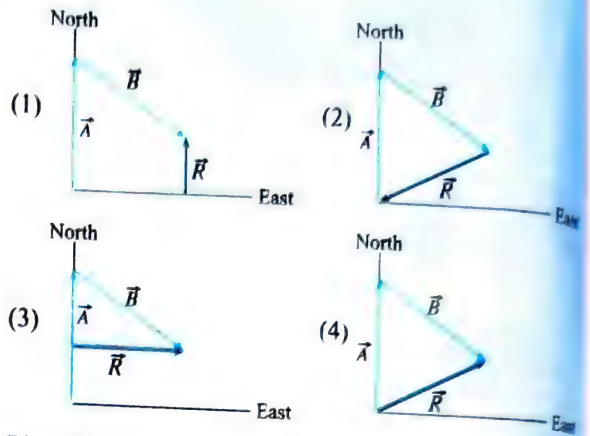
$$\Delta v_{AB} = \sqrt{20^2 + 20^2} = \sqrt{800} \text{ m s}^{-1}$$

$$\tan\theta = \frac{20}{20} = 1;\ \theta = 45^\circ$$

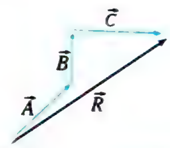


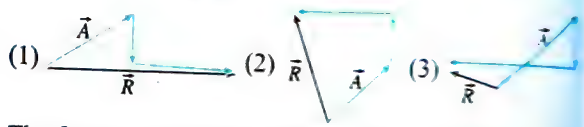
CONCEPT APPLICATION EXERCISE 3.1

1. During a relay race, runner A runs a certain distance due north and then hands off the baton to runner B, who runs for the same distance in a direction south of east. The two displacement vectors $\vec{A}$ and $\vec{B}$ can be added together to give a resultant vector $\vec{R}$. Which drawing correctly shows the resultant vector?

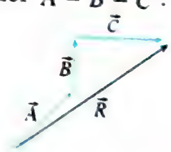


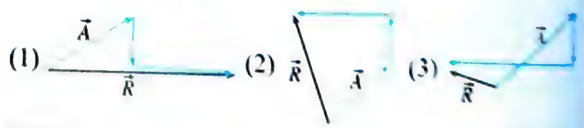
2. The first drawing shows three displacement vectors $\vec{A}, \vec{B}$, and $\vec{C}$, which are added in a tail-to-head fashion. The resultant vector is labeled $\vec{R}$. Which of the following drawings shows the correct resultant vector for $\vec{A} + \vec{B} - \vec{C}$?





3. The first drawing shows the sum of three displacement vectors, $\vec{A}, \vec{B}$, and $\vec{C}$. The resultant vector is labeled $\vec{R}$. Which of the following drawings shows the correct resultant vector for $\vec{A} - \vec{B} - \vec{C}$?





4. Which of the following displacement vectors (if any) are equal?

Variable	Magnitude	Direction
$\vec{A}$	100 m	30° north of east
$\vec{B}$	100 m	30° south of west
$\vec{C}$	100 m	60° east of north

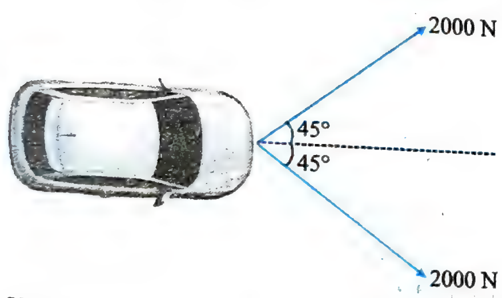
5. A force vector $\vec{F}_1$ points due east and has a magnitude of 200 N. A second force $\vec{F}_2$ is added to $\vec{F}_1$. The resultant of the two vectors has a magnitude of 400 N and points along the east/west line. Find the magnitude and direction of $\vec{F}_2$. Note that there are two answers.

6. Consider the following four force vectors:
   (a) $\vec{F}_1 = 50.0$ N, due east
   (b) $\vec{F}_2 = 10.0$ N, due east
   (c) $\vec{F}_3 = 40.0$ N, due west
   (d) $\vec{F}_4 = 30.0$ N, due west

   Which two vectors add together to give a resultant with the smallest magnitude, and which two vectors add to give a resultant with the largest magnitude? In each case specify the magnitude and direction of the resultant.

7. Vector $\vec{A}$ has a magnitude of 10 units and points due west, while vector $\vec{B}$ has the same magnitude and points due south. Find the magnitude and direction of (a) $\vec{A}+\vec{B}$ and (b) $\vec{A}-\vec{B}$. Specify the directions relative to due west.

8. A car is being pulled out of the mud by two forces that are applied by the two ropes shown in the drawing. The dashed line in the drawing bisects the 90° angle. The magnitude of the force applied by each rope is 2000 N. Arrange the force vectors tail to head and use the graphical technique to answer the following questions:



   (a) How much force would a single rope need to apply to accomplish the same effect as the two forces are added together?
   (b) How would the single rope be directed relative to the dashed line?

ANSWERS

1. (4) **2.** (2) **3.** (3) **4.** $\vec{A} = \vec{C}$

5. Force due east = 200 N
Force due west = 600 N

6. $F_1 + F_3 = 10$ N, due east
$F_3 + F_4 = 70$ N, due west

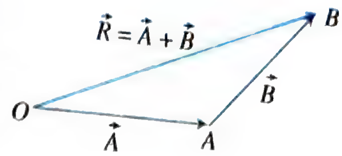
7. (a) 45° south of west
(b) 45° north of west

8. (a) $2000\sqrt{2}$ N
(b) Along the dashed line

VECTOR ADDITION BY ANALYTICAL METHOD

Triangle Law of Vector Addition of Two Vectors

If two non-zero vectors are represented by two sides of a triangle taken in the same order, then the resultant is given by the closing side of triangle in opposite order, i.e., $\vec{R} = \vec{A} + \vec{B}$



$\because \quad \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

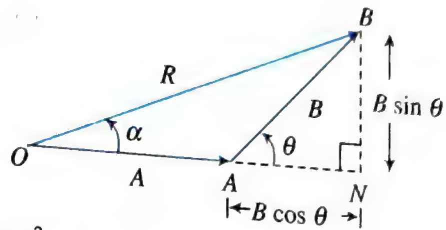
Magnitude of resultant vector: In ΔABN

$$\cos\theta = \frac{AN}{B}$$

$\therefore \quad AN = B\cos\theta$

$$\sin\theta = \frac{BN}{B}$$

$\therefore \quad BN = B\sin\theta$



In ΔOBN, we have $OB^2 = ON^2 + BN^2$

$$\Rightarrow \quad R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$
$$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$
$$= A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$
$$= A^2 + B^2 + 2AB\cos\theta$$
$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Direction of resultant vectors: If θ is the angle between $\vec{A}$ and $\vec{B}$, then

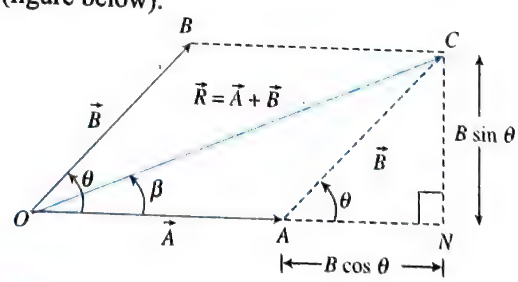
$$|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

$$\tan\alpha = \frac{BN}{ON} = \frac{BN}{OA + AN} = \frac{B\sin\theta}{A + B\cos\theta}$$

Parallelogram Law of Vector Addition of Two Vectors

If two non-zero vectors are represented by two adjacent sides of a parallelogram, then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of two vectors (figure below).



Magnitude

Since, $R^2 = ON^2 + CN^2 = (OA + AN)^2 + CN^2$
$= A^2 + B^2 + 2AB\cos\theta$

$$\therefore \quad R = |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

Special cases: $R = A + B$ when $\theta = 0°$

$R = A - B$ when $\theta = 180° = \sqrt{A^2 + B^2}$ when $\theta = 90°$

Direction

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

Special Cases

Case 1: When the given vectors act in the same direction ($\theta = 0°$).

So $\quad R = \sqrt{A^2 + B^2 + 2AB\cos 0°} = \sqrt{A^2 + B^2 + 2AB}$ $\quad [\because \cos 0° = 1]$

$= \sqrt{(A+B)^2} = A + B$

or $\quad |\vec{R}| = |\vec{A}| + |\vec{B}|$ which represents the magnitude of the resultant.

Case 2: When the given vectors act in opposite directions ($\theta = 180°$).

So $\quad R = \sqrt{A^2 + B^2 + 2AB\cos 180°} = \sqrt{A^2 + B^2 - 2AB}$

$= \sqrt{(A-B)^2}$ $\quad [\because \cos 180° = -1]$

$= \pm(A - B) = A - B$ or $B - A$

or $\quad |\vec{R}| = ||\vec{A}| - |\vec{B}||$ which represents the magnitude of the resultant.

Case 3: When the given vectors $\vec{A}$ and $\vec{B}$ act at right angle to each other ($\theta = 90°$).

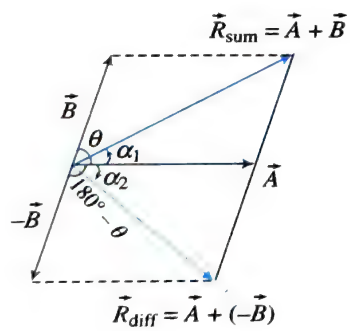
$\therefore \quad R = \sqrt{A^2 + B^2 + 2AB\cos 90°} = \sqrt{A^2 + B^2}$ $[\because \cos 90° = 0]$

or $\quad |\vec{R}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2}$

and $\quad \tan\beta = \dfrac{B\sin 90°}{A + B\cos 90°}$ or $\tan\beta = \dfrac{B}{A}$ $[\because \sin = 90° = 1]$

Subtraction of Vectors

Two vectors $\vec{A}$ and $\vec{B}$ are inclined at as angle θ (figure shown). The summation of both vectors is represented by the bigger diagonal of parallelogram and difference of vectors is represented by the smaller diagonal of parallelogram.



Since, $\quad \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

and $\quad |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos(180° - \theta)}$

Since, $\cos(180° - \theta) = -\cos\theta$

$\Rightarrow \quad |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

$$\tan\alpha_1 = \frac{B\sin\theta}{A + B\cos\theta} \text{ and } \tan\alpha_2 = \frac{B\sin(180° - \theta)}{A + B\cos(180° - \theta)}$$

But $\quad \sin(180° - \theta) = \sin\theta$

and $\quad \cos(180° - \theta) = -\cos\theta$

$\Rightarrow \quad \tan\alpha_2 = \dfrac{B\sin\theta}{A - B\cos\theta}$

ILLUSTRATION 3.2

The greatest and least resultant of two forces acting at a po[int] is 10 N and 6 N, respectively. If each force is increased by [3] N, find the resultant of new forces when acting at a point at an angle of 90° with each other.

Sol.

Let A and B be the two forces.

Greatest resultant $= A + B = 10$

Least resultant $= A - B = 6$

Solving (i) and (ii), we get, $A = 8$ N and $B = 2$ N.

When each force is increased by 3 N then

$A' = A + 3 = 8 + 3 = 11$ N, then

$B' = B + 3 = 2 + 3 = 5$ N

As the new forces are acting at an angle of 90° (i.e., $\theta = 90°$)

So $R' = \sqrt{A'^2 + B'^2} = \sqrt{(11)^2 + (5)^2} = \sqrt{146}$ N

ILLUSTRATION 3.3

Two equal forces have their resultant equal to either. At wh[at] angle are they inclined?

Sol. Here $A = F$; $B = F$; $R = F$; $\theta = ?$

Now $\quad R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$F = \sqrt{F^2 + F^2 + 2F.F.\cos\theta} = F\sqrt{2(1 + \cos\theta)}$

$\Rightarrow \quad 1 = 2(1 + \cos\theta)$

$\cos\theta = -\dfrac{1}{2} = \cos 120° \Rightarrow \theta = 12$

ILLUSTRATION 3.4

Two forces whose magnitudes are in the ratio 3 : 5 give [a] resultant of 28 N. If the angle of their inclination is 60°, fin[d] the magnitude of each force.

Sol. Let A and B be the two forces.

Then $A = 3x$; $B = 5x$; $R = 28$ N and $\theta = 60°$.

Dividing A by B, $A/B = 3/5$

We know that $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\Rightarrow \quad 28 = \sqrt{(3x)^2 + (5x)^2 + 2(3x)(5x)\cos 60°}$

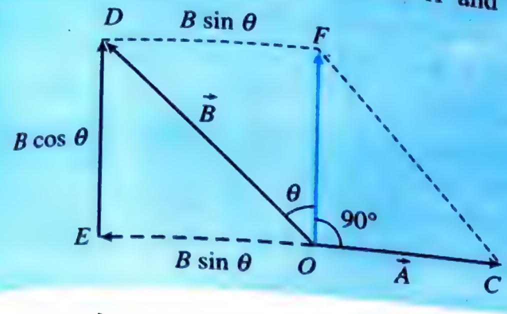
$= \sqrt{9x^2 + 25x^2 + 15x^2} = 7x$

$\Rightarrow \quad x = \dfrac{28}{7} = 4$

Hence, forces are $A = 3 \times 4 = 12\ N$, $B = 5 \times 4 = 20\ N$

ILLUSTRATION 3.5

The resultant of two vectors $\vec{A}$ and $\vec{B}$ is perpendicular to the vector $\vec{A}$ and its magnitude is equal to half of the magnitude of the vector $\vec{B}$. Find out the angle between $\vec{A}$ and $\vec{B}$.



Sol. $R = B\cos\theta$. As per question,

$$R = B/2 = B\cos\theta$$

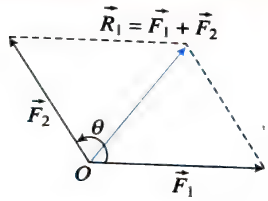
or $\cos\theta = 1/2$ or $\theta = 60°$

Hence, angle between $\vec{A}$ and $\vec{B} = 90° + 60° = 150°$

ILLUSTRATION 3.6

Two forces of unequal magnitude simultaneously act on a particle making an angle $\theta(=120°)$ with each other. If one of them is reversed, the acceleration of the particle is becomes $\sqrt{3}$ times. Calculate the ratio of the magnitude of the forces.

Sol. Let the two forces be $\vec{F}_1$ and $\vec{F}_2$ that are inclined at angle θ.



The resultant of these forces is $\vec{R}_1 = \vec{F}_1 + \vec{F}_2$. Then $|\vec{R}_1| = |\vec{F}_1 + \vec{F}_2|$.

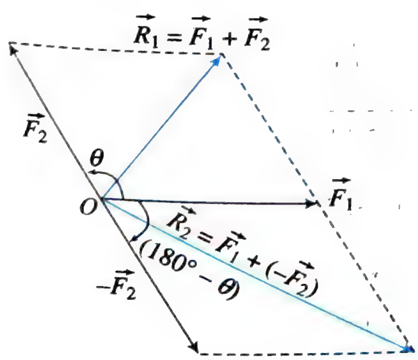
Using parallelogram law of vector addition, we have

$$|\vec{R}_1| = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta} \quad \text{...(i)}$$

If the direction of $\vec{F}_2$ is reversed, new resultant force $\vec{R}_2 = \vec{F}_1 + (-\vec{F}_2)$.

Using parallelogram law of vectors, the magnitude of new resultant is

$$|\vec{R}_2| = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\theta} \quad \text{...(ii)}$$



Since force is directly proportional to acceleration $\vec{F} \propto \vec{a}$.

Hence, $\dfrac{|\vec{R}_2|}{|\vec{R}_1|} = \dfrac{a_2}{a_1} = \dfrac{\sqrt{3}}{1}$

From Eqs. (i) and (ii),

$$\frac{\sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\theta}}{\sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}} = \sqrt{3}$$

$$\Rightarrow \quad F_1^2 + F_2^2 + 4F_1F_2\cos\theta = 0 \quad \text{...(iii)}$$

Here $\theta = 120°$ and dividing Eq. (iii) by F_2^2,

$$\left(\frac{F_1}{F_2}\right)^2 + 1 - 2\left(\frac{F_1}{F_2}\right) = 0 \quad \text{...(iv)}$$

Let $\dfrac{F_1}{F_2} = x$. Equation (iv) becomes $(x)^2 + 1 - 2(x) = 0$

After solving, $x = 1$ or $\dfrac{F_1}{F_2} = 1$.

ILLUSTRATION 3.7

The resultant of $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If $\vec{Q}$ is doubled, $\vec{R}$ is doubled; when $\vec{Q}$ is reversed, $\vec{R}$ is again doubled. Find $P : Q : R$.

Sol. Let θ be the angle between $\vec{P}$ and $\vec{Q}$. Then

$$R^2 = |\vec{P} + \vec{Q}|^2 = P^2 + Q^2 + 2PQ\cos\theta \quad \text{...(i)}$$

If $\vec{Q}$ is doubled, $\vec{R}$ is doubled. That means, the magnitude of resultant of $2\vec{Q}$ and $\vec{P}$ is

$$(2R)^2 = P^2 + (2Q)^2 + 2P(PQ)\cos\theta$$

This yields $4R^2 = P^2 + 4Q^2 + 4PQ\cos\theta$ ...(ii)

When $\vec{Q}$ is reversed, $\vec{R}$ is doubled. Hence, the magnitude of resultant of $\vec{P}$ and $(-\vec{Q})$ is $2R$.

Then $(2R)^2 = P^2 + Q^2 + 2PQ\cos(180° - \theta)$

This yields $4R^2 = P^2 + Q^2 - 2PQ\cos\theta$ ...(iii)

(ii) – (i) yields $\quad 3Q^2 + 2PQ\cos\theta = 3R^2$ ...(iv)

(i) + (iii) yields $\quad P^2 + Q^2 = \dfrac{5R^2}{2}$ ...(v)

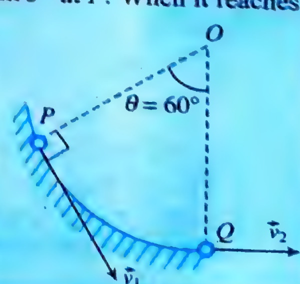
(iii) + (iv) yields $P^2 + 4Q^2 = 7R^2$ ...(vi)

Solving (v) and (vi), we obtain $Q = \sqrt{\dfrac{3}{2}}R$ and $P = R$.

Hence, $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$

ILLUSTRATION 3.8

A particle slides with a speed of 3m s^{-1} at P. When it reaches Q, it acquires a speed of 4 m s^{-1} after describing an angle of 60° at O as shown in the figure. Find the change in the velocity of the particle between P and Q. Assume that the path followed by the particle is circular from P to Q.



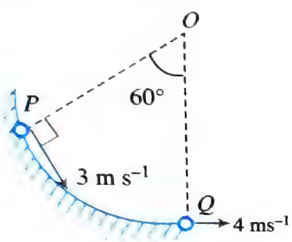
Sol. As the particle moves from P to Q, its speed increases from $v_1 = 3\ \text{m s}^{-1}$ to $v_2 = 4\ \text{m s}^{-1}$ and the angle described by the velocity vector is equal to $\theta = 60°$. Then the change in velocity is given as

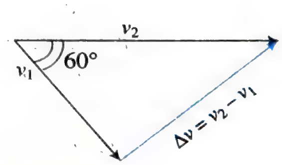
$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\theta}.$$

Substituting $v_1 = 3\ \text{m s}^{-1}$, $v_2 = 4\ \text{m s}^{-1}$ and $\theta = 60°$ in the above expression, we have

$$\Delta v = \sqrt{3^2 + 4^2 - 2.3.4\cos 60°}$$

This yields $\Delta v = \sqrt{13}\ \text{m s}^{-1}$ and the change in velocity is directed as shown in figure.





CONCEPT APPLICATION EXERCISE 3.2

1. Two forces, each of magnitude F, have a resultant of the same magnitude F. The angle between the two forces is
   (1) 45° (2) 120°
   (3) 150° (4) 60°

2. The $|\vec{A} + \vec{B}| = |\vec{A}| + |\vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ will be
   (1) 90° (2) 120°
   (3) 0° (4) 60°

3. The maximum and minimum magnitudes of the resultant of two vectors of magnitudes P and Q are in the ratio 3:1. Which of the following relations is true?
   (1) $P = 2Q$ (2) $P = Q$
   (3) $PQ = 1$ (4) None of these

4. The maximum and minimum magnitude of the resultant of two given vectors are 17 units and 7 unit, respectively. If these two vectors are at right angles to each other, the magnitude of their resultant is
   (1) 14 (2) 16
   (3) 18 (4) 13

5. Which pair of the following forces will never give resultant force of 2 N?
   (1) 2 N and 2 N (2) 1 N and 1 N
   (3) 1 N and 3 N (4) 1 N and 4 N

6. The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their resultant is 12. If the resultant is at 90° with the force of smaller magnitude, what are the magnitudes of forces?

7. At what angle should the two force vectors $2F$ and $\sqrt{2}F$ act so that the resultant force is $\sqrt{10}F$?

8. Two forces, while acting on a particle in opposite directions, have the resultant of 10 N. If they act at right angles to each other, the resultant is found to be 50 N. Find the two forces.

9. Two forces each equal to $F/2$ act at right angle. Their effect may be neutralized by a third force acting along their bisector in the opposite direction. What is the magnitude of that third force?

10. The resultant of two forces has magnitude 20 N. One of the forces is of magnitude $20\sqrt{3}$ N and makes an angle of 30° with the resultant. Then what is the magnitude of the other force?

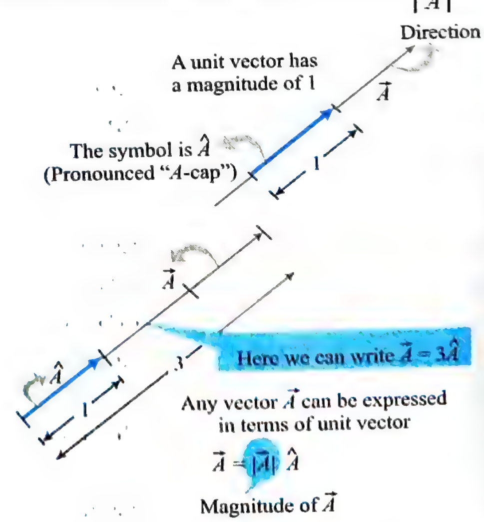
ANSWERS

1. (2)	**2.** (3)	**3.** (1)	**4.** (4)	**5.** (4)
6. 5, 13	**7.** 45°	**8.** 40 N, 30 N		
9. $F/\sqrt{2}$	**10.** 20 N			

UNIT VECTORS

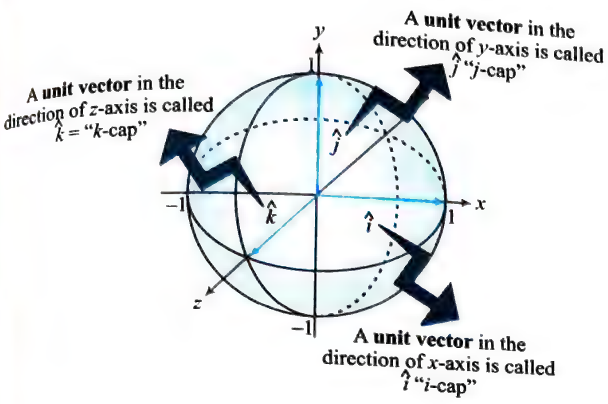
The vector quantities have both direction and magnitude. But sometimes one is interested in only the direction of the vector and not the magnitude. The unit vectors are usually used to specify directions and therefore they do not have any dimension or unit like other vectors. A unit vector is a dimensionless vector having a magnitude of exactly 1.

We can write the unit vector of $\vec{A}$ as $\hat{A} = \dfrac{\vec{A}}{|\vec{A}|}$



The unit vectors are used to specify a given direction and they have no other physical significance. Here we use the symbols $\hat{i}$, $\hat{j}$, and $\hat{k}$ to represent unit vectors pointing in the positive x, y and z directions, respectively. (The "hats," or circumflexes, o

the symbols are a standard notation for unit vectors.) The unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ form a set of mutually perpendicular vectors in a right-handed coordinate system.



$\hat{i}$ is a dimensionless unit vector of length 1 that points in the positive x direction, and $\hat{j}$ is a dimensionless unit vector of length 1 that points in the positive y direction. Similarly $\hat{k}$ is a dimensionless unit vector of length 1 that points in the positive z direction.

ILLUSTRATION 3.9

Represent the following in unit vector notation:

(i) Vector a has magnitude 5 units and is in the direction of the $+x$-axis.

(ii) Vector b has magnitude 3 units in the direction of the $+y$-axis

(iii) Vector c has magnitude 4 units and is in the direction of the $-x$-axis.

(iv) Vector d has magnitude 2 units and is in the direction of the $-y$-axis.

(v) Point A having coordinate (4 m, 2 m)

(vi) Point B having coordinate (–2 m, 3 m)

(vii) Point C having coordinate (–3 m, –2 m)

(viii) Point D having coordinate (4 m, –1 m)

Sol.

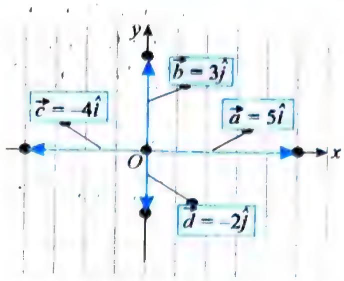
- In the positive x direction, the unit vector is represented by $\hat{i}$
- In negative x direction, the unit vector is represented by $-\hat{i}$
- In the positive y direction, the unit vector is represented by $\hat{j}$
- In negative y direction, the unit vector is represented by $-\hat{j}$

(i) Vector a with magnitude 5 units in the direction of the $+x$-axis : $\vec{a} = 5\hat{i}$

(ii) Vector b with magnitude 3 units in the direction of the $+y$-axis : $\vec{b} = 3\hat{j}$

(iii) Vector c with magnitude 4 units in the direction of the $-x$-axis : $\vec{c} = -4\hat{i}$

(iv) Vector d with magnitude 2 units in the direction of the $-y$-axis : $\vec{d} = -2\hat{j}$

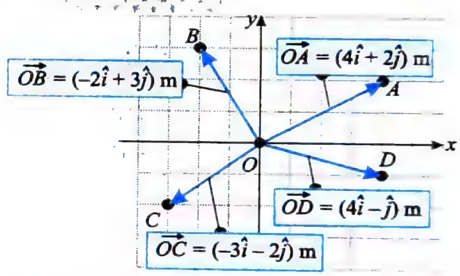


(v) Point A having coordinate (4 m, 2m) has position 4m along positive x-direction and 2 m along y-direction. We can express the position of A as $\overrightarrow{OA} = (4\hat{i} + 2\hat{j})$ m

(vi) Point B having coordinate (4 m, 2 m) has position 4 m along positive x-direction and 2 m along y-direction. We can express the position of B as $\overrightarrow{OB} = (-2\hat{i} + 3\hat{j})$ m

(vii) Point C having coordinate (4 m, 2 m) has position 4 m along positive x-direction and 2 m along y-direction. We can express the position of C as $\overrightarrow{OC} = (-3\hat{i} - 2\hat{j})$ m

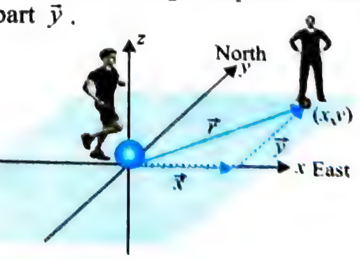
(viii) Point D having coordinate (4 m, 2m) has position 4 m along positive x-direction and 2 m along y-direction. We can express the position of D as $\overrightarrow{OD} = (4\hat{i} - \hat{j})$ m



COMPONENTS OF A VECTOR

In many situations where a vector is directed at a certain angle to the coordinate axes, we can apply a useful mathematical trick to transform the vector into two parts with each part being directed along the coordinate axes.

Let us consider a situation where a ball is displaced from origin to a position having a coordinate (x, y) in this situation the displacement vector of the ball $\vec{r}$ that is directed northeast direction can be thought of as having two parts: an eastward part $\vec{x}$ and a northward part $\vec{y}$.



We can write: $\vec{r} = \vec{x} + \vec{y} = x\hat{i} + y\hat{j}$

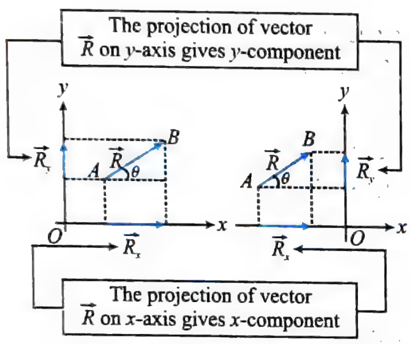
If a vector directed in two dimensions, it can be thought of as having an influence in two different directions. That is, it can be thought of as having two parts. Each part of a two-dimensional vector is known as a **component**.

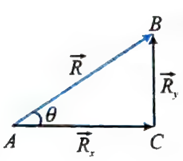
The components of a vector depict the influence of that vector in

a given direction. The combined influence of the two components is equivalent to the influence of a single two-dimensional vector. Therefore, the single two-dimensional vector can be replaced by the two components.

FINDING RECTANGULAR (X AND Y) COMPONENTS OF A VECTOR

A component of a vector is the projection of the vector on an axis. For example, R_x is the component of vector $\vec{R}$ on (or along) the x axis and R_y is the component along the y axis. To find the projection of a vector along an axis, we draw perpendicular lines from the two ends of the vector to the axis, as shown in figure. The projection of a vector on the x axis is its *x component*, and similarly the projection on the y axis is the *y component*. The process of finding the components of a vector is known as resolving the vector.





Hence x-component of a vector $\vec{R}$
$R_x = R\cos\theta$
and y-component of a vector $\vec{R}$
$R_y = R\sin\theta$
and we can write $\vec{R} = \vec{R}_x + \vec{R}_y$

In right angled triangle ABC,

$$\sin\theta = \frac{R_y}{R} \Rightarrow R_y = R\sin\theta \text{ and } \cos\theta = \frac{R_x}{R}$$

$$\Rightarrow \quad R_x = R\cos\theta \text{ and } \tan\theta = \frac{R_y}{R_x}$$

Hence y-component of a vector $\vec{R}: R_y = R\sin\theta$

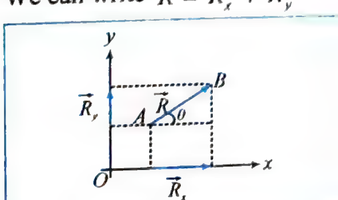
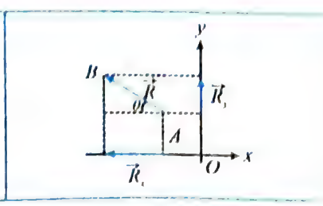
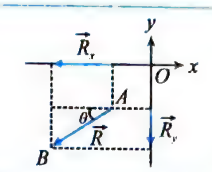
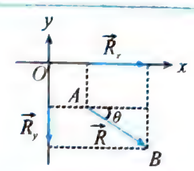
And x-component of a vector $\vec{R}: R_x = R\cos\theta$

And we can write: $\vec{R} = \vec{R}_x + \vec{R}_y = R\cos\theta\hat{i} + R\sin\theta\hat{j}$

WRITING COMPONENTS OF A VECTOR

The signs of the components of a vector $\vec{R}$ depend on the quadrant in which the vector is located.

We can write $\vec{R} = \vec{R}_x + \vec{R}_y$

	
x-component and y-component both are positive $\vec{R} = R\cos\theta\hat{i} + R\sin\theta\hat{j}$ Unit vector in direction of $\vec{R}$: $\hat{R} = \frac{\vec{R}}{\lvert\vec{R}\rvert} = \frac{R\cos\theta\hat{i} + R\sin\theta\hat{j}}{R}$ $= \cos\theta\hat{i} + \sin\theta\hat{j}$	x-component is negative and y-component is positive $\vec{R} = R\cos\theta(-\hat{i}) + R\sin\theta\hat{j}$ $= -R\cos\theta\hat{i} + R\sin\theta\hat{j}$ Unit vector in direction of $\vec{R}$: $\hat{R} = \frac{\vec{R}}{\lvert\vec{R}\rvert} = \frac{-R\cos\theta\hat{i} + R\sin\theta\hat{j}}{R}$ $= -\cos\theta\hat{i} + \sin\theta\hat{j}$
	
Both x-component and and y-component are negative $\vec{R} = R\cos\theta(-\hat{i}) + R\sin\theta(-\hat{j})$ $= -R\cos\theta\hat{i} - R\sin\theta\hat{j}$ Unit vector in direction of $\vec{R}$: $\hat{R} = \frac{\vec{R}}{\lvert\vec{R}\rvert} = \frac{-R\cos\theta\hat{i} - R\sin\theta\hat{j}}{R}$ $= -\cos\theta\hat{i} - \sin\theta\hat{j}$	x-component is positive and y-component is negative $\vec{R} = R\cos\theta\hat{i} + R\sin\theta(-\hat{j})$ $= R\cos\theta\hat{i} - R\sin\theta\hat{j}$ Unit vector in direction of $\vec{R}$: $\hat{R} = \frac{\vec{R}}{\lvert\vec{R}\rvert} = \frac{R\cos\theta\hat{i} - R\sin\theta\hat{j}}{R}$ $= \cos\theta\hat{i} - \sin\theta\hat{j}$

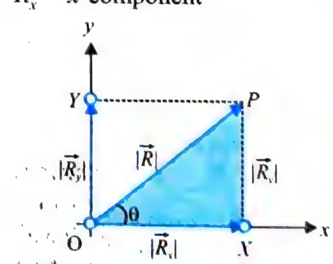
FINDING A VECTOR IF X AND Y COMPONENTS OF THE VECTOR ARE GIVEN

Let the magnitude of the x-component and the y-component of a vector $\vec{R}$ be R_x and R_y, respectively.

$$\vec{R} = \vec{R}_x + \vec{R}_y \Rightarrow \vec{R} = R_x\cos\theta\hat{i} + R_y\sin\theta\hat{j}$$

In right-angled triangle OPX, we can write $R^2 = R_x^2 + R_y^2$ and inclination of vector with x-axis:

$$\tan\theta = \frac{R_y}{R_x} = \frac{y\text{-component}}{x\text{-component}} \Rightarrow \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$



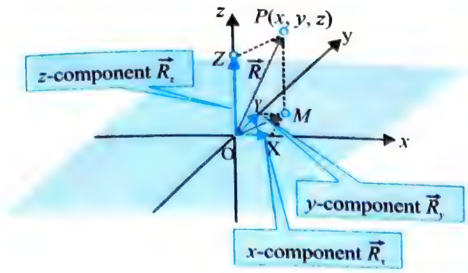
FINDING RECTANGULAR COMPONENTS OF A VECTOR IN THREE DIMENSIONS

We can write the position vector of point P with (coordinate x, y, z)

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z$$

in unit vector notation $\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$

From the figure, we can observe the projection of point P on xy plane is M and OX and OY represents x and y components of vector $\vec{R}$. Similarly projection of P on z-axis gives the z-component of $\vec{R}$. In figure, OZ represents z-component of the vector $\vec{R}$.

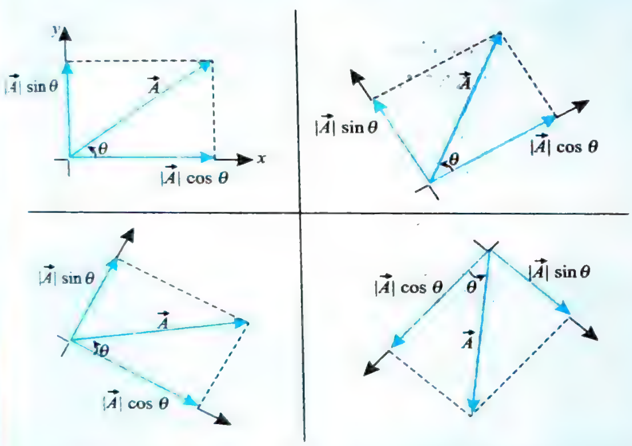


We can find the magnitude of $\vec{R}$ in same way as we have calculated in two dimensions.

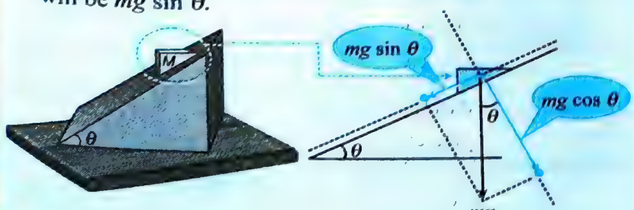
$$R^2 = OM^2 + PM^2 = (R_x^2 + R_y^2) + R_z^2 \Rightarrow R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Important Points:

- We can have two mutually perpendicular components of a vector other than x-direction or y-direction. If a vector makes an angle θ with a given line, the component of the vector along the given is called **"cos" component** and the component perpendicular to given line as **"sin" component**. In the figure given, we can observe that the component of vector $\vec{A}$ in two mutually perpendicular directions are $|\vec{A}|\cos\theta$ and $|\vec{A}|\sin\theta$.



- **Finding the components of weight parallel to and perpendicular to an inclined plane:**
  The gravitational pull on the block always acts in vertical downward direction which is equal to mg. The line of action of force mg makes an angle θ with the line which is perpendicular to inclined surface. The projection of mg on this perpendicular line will be $mg\cos\theta$. Similarly the projection of mg on the line parallel to inclined surface will be $mg\sin\theta$.



- **Acceleration of the block sliding on smooth inclined plane:**
  If a block slides on a smooth inclined plane, the only force responsible is the 'component of force parallel to incline' Which is $mg\sin\theta$. Hence, the acceleration (a) of the block

$$a = \frac{\text{Force on the block}}{\text{Mass of the block}} = \frac{mg\sin\theta}{m} = g\sin\theta$$

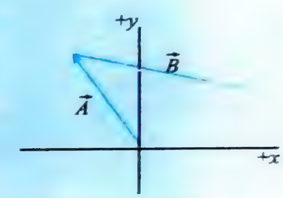
ILLUSTRATION 3.10

Two vectors $\vec{A}$ and $\vec{B}$ are shown in the drawing.

(a) What are the signs (+ or –) of the scalar components, A_x and A_y, of the vector $\vec{A}$?

(b) What are the signs of the scalar components, B_x and B_y, of the vector $\vec{B}$?

(c) What are the signs of the scalar components, R_x and R_y, of the vector $\vec{R}$, where $\vec{R} = \vec{A} + \vec{B}$?



Sol.

(a) If we make components of $\vec{A}$ in x and y direction, the x-component of $\vec{A}$ points in left direction or in $-x$ direction. Hence, A_x is negative. The y-component of $\vec{A}$ points in upwards direction or in $+y$ direction, hence A_y is positive.

(b) The components of $\vec{B}$: the x-component of $\vec{B}$ points in right direction or in $+x$ direction, hence B_x is positive and, the y-component of $\vec{B}$ points in downwards direction or in $-y$ direction, hence A_y is negative.

(c) The components of $\vec{R}$: the x-component of $\vec{R}$ points in right direction or in $+x$ direction, hence R_x is positive and, the y-component of $\vec{R}$ points in upwards direction or in $+y$ direction, hence R_y is positive.

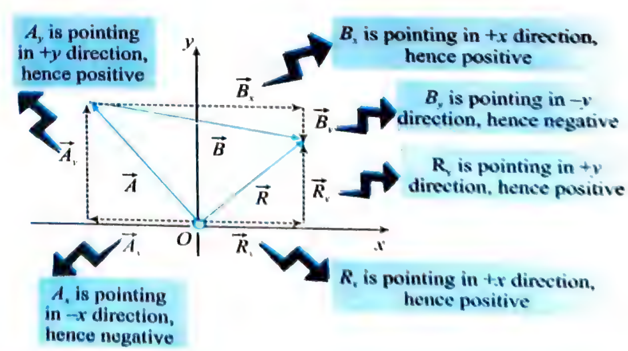


ILLUSTRATION 3.11

Find the magnitude and direction of the following vectors

(a) $\vec{a} = 3\hat{i} + 4\hat{j}$ (b) $\vec{b} = -2\hat{i} + 2\sqrt{3}\hat{j}$

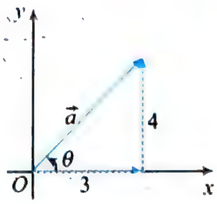
(c) $\vec{c} = -2\hat{i} - 2\hat{j}$ (d) $\vec{d} = \sqrt{3}\hat{i} - \hat{j}$

Sol.

(a) $\vec{a} = 3\hat{i} + 4\hat{j}$

$|\vec{a}| = \sqrt{3^2 + 4^2} = 5$ units

$\tan\theta = \frac{4}{3} \Rightarrow \theta = 53°$

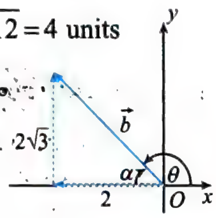


(b) $\vec{b} = -2\hat{i} + 2\sqrt{3}\hat{j}$

$|\vec{b}| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$ units

$\tan\alpha = \frac{|2\sqrt{3}|}{|2|} = \sqrt{3}$ $\therefore \alpha = 60°$

where $\theta = 180° - \alpha = 120°$ with +ve x-axis.

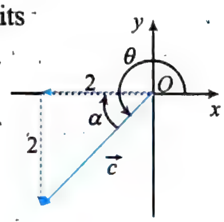


(c) $\vec{c} = -2\hat{i} - 2\hat{j}$

$|\vec{c}| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ units

$\tan\alpha = \frac{2}{2} = 1$ $\therefore \alpha = 45°$

where $\theta = 180° + \alpha = 225°$ with +ve x-axis.



(d) $\vec{d} = \sqrt{3}\hat{i} - 3\hat{j}$

$|\vec{d}| = \sqrt{(\sqrt{3})^2 + (-3)^2} = 2\sqrt{3}$ units

$\tan\alpha = \frac{3}{\sqrt{3}} = \sqrt{3}$ $\therefore \alpha = 60°$

where $\theta = 360° - \alpha = 300°$ with +ve x-axis.

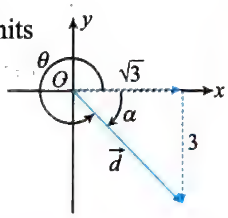
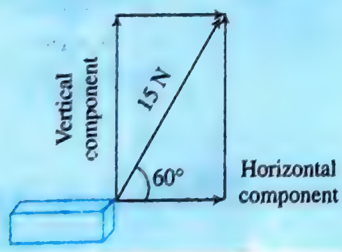


ILLUSTRATION 3.12

A force of 15 N acts on a box as shown in Figure. What are the horizontal component and vertical components of the force?

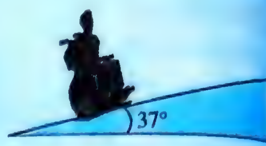


Sol. Horizontal component $F_x = 15\cos 60° = 7.5$ N

Vertical component $F_y = 15\sin 60° = 12.99$ N

ILLUSTRATION 3.13

A person in a wheelchair is moving up a ramp at constant speed. Their total weight is 900 N. The ramp makes an angle of 37° with the horizontal. Calculate the component of its weight parallel and perpendicular to the ramp.



Sol. Component of weight parallel to ramp

$W_{\parallel} = 900\sin 37° = 540$ N

Component of weight perpendicular to ramp

$W_{\perp} = 900\cos 37° = 720$ N

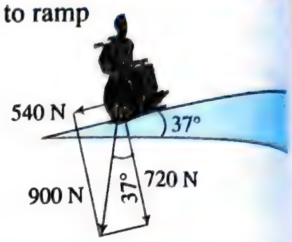


ILLUSTRATION 3.14

A particle is moving with velocity $v = 100$ m s^{-1}. If one of the rectangular components of a velocity is 50 m s^{-1}. Find the other component of velocity and its angle with the given component of velocity.

Sol. Here we are given net velocity $v = 100$ m s^{-1}.

Let $v_x = 50$ m s^{-1} $\Rightarrow v_x = v\cos\theta$

$\Rightarrow$ $50 = 100\cos\theta$

$\Rightarrow$ $\cos\theta = \frac{1}{2}$ or $\theta = 60°$

$\therefore$ $v_y = v\sin\theta = 100\sin 60°$

$= 100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3}$ m s^{-1}

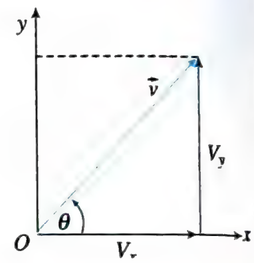
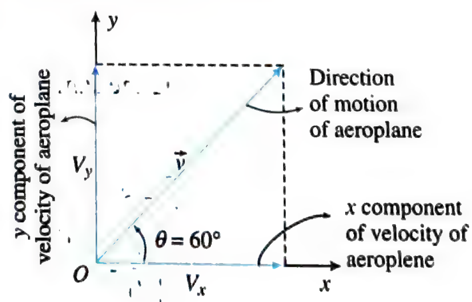


ILLUSTRATION 3.15

An aeroplane takes off at an angle of 60° to the horizontal. If the velocity of the plane is 200 km h^{-1}, calculate its horizontal and vertical component of its velocity.

Sol. Here $v = 200$ km h^{-1}, $\theta = 60°$



$\therefore$ Horizontal component

$v_x = v\cos\theta = 200\cos 60°$

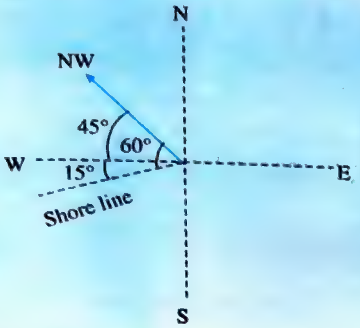
$= 200 \times \frac{1}{2} = 100$ km h^{-1}

Vertical component

$v_y = v\sin\theta = 200\sin 60° = 200 \times \frac{\sqrt{3}}{2} = 100\sqrt{3}$ km h^{-1}

ILLUSTRATION 3.16

A man rows a boat with a speed of 18 km h^{-1} in the north-west direction. The shoreline makes an angle of 15° south of west. Obtain the components of the velocity of the boat along the shoreline and perpendicular to the shoreline.



Sol. The north-west direction of the boat makes an angle of 60° with the shoreline.

Component of the velocity of boat along the shoreline

$= 18 \cos 60° \text{ km h}^{-1} = 9 \text{ km h}^{-1}$

Component of the boat velocity along a line normal to the shoreline

$$= 18 \sin 60° \text{ km h}^{-1} = 18 \times \frac{\sqrt{3}}{2} \text{ km h}^{-1} = 9\sqrt{3} \text{ km h}^{-1}$$

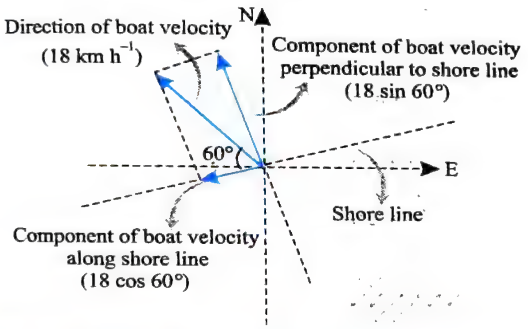


ILLUSTRATION 3.17

If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 7\hat{i} + 24\hat{j}$. Find a vector having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$.

Sol. We can write a vector ($\vec{C}$) having the same magnitude as $\vec{B}$ and parallel to $\vec{A}$, $\vec{C} = |\vec{B}|\hat{A}$

The magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{3^2 + 4^2} = 5$

A unit vector parallel to $\vec{A}$ will be $\hat{A} = \frac{\vec{A}}{|A|} = \frac{(3\hat{i} + 4\hat{j})}{5}$

Now magnitude of $\vec{B}$ is $|\vec{B}| = \sqrt{7^2 + (24)^2} = 25$

Therefore a vector parallel to $\vec{A}$ and having magnitude of $\vec{B}$ will be $\vec{C} = |\vec{B}|\hat{A} = 25\left[\frac{(3\hat{i} + 4\hat{j})}{5}\right] = (15\hat{i} + 20\hat{j})$

ILLUSTRATION 3.18

A boat is moving in direction $\vec{a} = -4\hat{i} + 3\hat{j}$ with a speed of 10 m/s. Write velocity vector of boat in unit vector notation.

Sol. We can write velocity vector of boat $\vec{v} = |\vec{v}|\hat{a}$

$\hat{a}$ is the unit vector in the direction of motion of boat.

Hence $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-4\hat{i} + 3\hat{j}}{\sqrt{4^2 + 3^2}} = -\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

Thus velocity vector of boat

$$\vec{v} = |\vec{v}|\hat{a} \Rightarrow \vec{v} = (10 \text{ ms}^{-1})\left(-\frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}\right)$$

$$= (-8\hat{i} + 6\hat{j}) \text{ ms}^{-1}$$

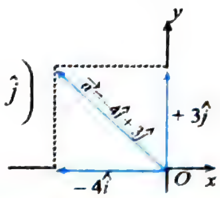
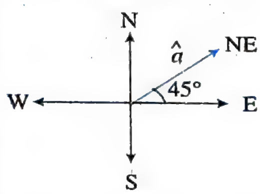
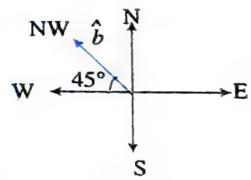
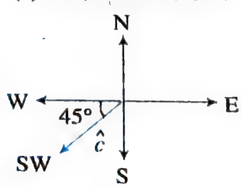
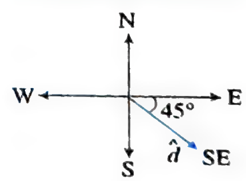


ILLUSTRATION 3.19

A car is moving with a speed of 10 m s^{-1}. If the east direction taken as x-axis and the north direction as y-axis. Write the velocity vector of car in unit vector notation. If it is moving (a) in the direction of N-E, (b) in the direction of N-W, (c) in the direction of S-W, and (d) in the direction of S-E.

Sol. Let us write the unit vectors along N-E direction, N-W direction, S-W direction, and S-E direction.

(a) in direction of N-E  $\hat{a} = (\cos 45°)\hat{i} + (\sin 45°)\hat{j} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$	(b) in direction of N-W  $\hat{b} = -(\cos 45°)\hat{i} + (\sin 45°)\hat{j} = \frac{-\hat{i} + \hat{j}}{\sqrt{2}}$
(c) in direction of S-W  $\hat{c} = -(\cos 45°)\hat{i} - (\sin 45°)\hat{j} = \frac{-\hat{i} - \hat{j}}{\sqrt{2}}$	(d) in direction of S-E  $\hat{d} = (\cos 45°)\hat{i} - (\sin 45°)\hat{j} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$

(a) Writing the velocity vector of car in the direction of N-E

$$\vec{v}_1 = |\vec{v}_1|\hat{a} = 10\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right) = 5\sqrt{2}(\hat{i} + \hat{j}) \text{ m s}^{-1}$$

(b) Writing the velocity vector of car in the direction of N-W

$$\vec{v}_2 = |\vec{v}_2|\hat{a} = 10\left(\frac{-\hat{i}+\hat{j}}{\sqrt{2}}\right) = 5\sqrt{2}(-\hat{i}+\hat{j})\ \text{m s}^{-1}$$

(c) Writing the velocity vector of car in the direction of S-W

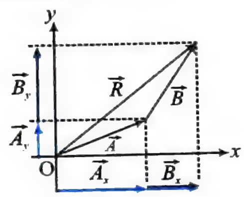
$$\vec{v}_3 = |\vec{v}_3|\hat{a} = 10\left(\frac{-\hat{i}-\hat{j}}{\sqrt{2}.}\right) = -5\sqrt{2}(\hat{i}+\hat{j})\ \text{m s}^{-1}$$

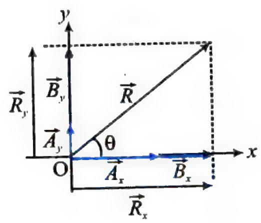
(d) Writing the velocity vector of car in the direction of S-E

$$\vec{v}_4 = |\vec{v}_4|\hat{a} = 10\left(\frac{\hat{i}-\hat{j}}{\sqrt{2}}\right) = 5\sqrt{2}(\hat{i}-\hat{j})\ \text{m s}^{-1}$$

Addition of Vectors by Means of Components

The components of a vector provide us with the most convenient and accurate way of adding (or subtracting) any given number of vectors. For example, if vector $\vec{A}$ is added to vector $\vec{B}$, the resultant vector is $\vec{R}$, where $\vec{R} = \vec{A} + \vec{B}$. The figure below illustrates this vector addition, along with the x and y vector components of $\vec{A}$ and $\vec{B}$.





The x components are colinear and add together to give the x component of the resultant vector $\vec{R}$. Similarly, the y components are colinear and add together to give the y component of $\vec{R}$. In terms of scalar components, we can write:

$R_x = A_x + B_x$ and $R_y = A_y + B_y$

$\vec{R} = (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j})$ or $\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$

Therefore, we see that in the component method of adding vectors, we add all the x components together to find the x component of the resultant vector and use the same process for the y components.

The magnitude of $\vec{R}$ and the angle it makes with the x axis are obtained from its components using the relationships

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

$$\tan\theta = \frac{R_y}{R_x} = \frac{A_y + B_y}{A_x + B_x}$$

In some situations, we need to consider the situations involving motion in three component directions. We can extend our methods to three-dimensional vectors is straight-forward. If $\vec{A}$ and $\vec{B}$ both have x, y, and z components, they can be expressed in the form

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \text{ and } \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

The sum of $\vec{A}$ & $\vec{B}$ is $\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$

If a vector $\vec{R}$ has x, y, and z components, the magnitude of the vector is $R = \sqrt{R_x^2 + R_y^2 + R_z^2}$.

The extension of our method to adding more than two vectors is also straightforward. For example,

$$\vec{A} + \vec{B} + \vec{C} = (A_x + B_x + C_x)\hat{i} + (A_y + B_y + C_y)\hat{j} + (A_z + B_z + C_z)\hat{k}.$$

We have described adding displacement vectors in this section because these types of vectors are easy to visualize. We can also add other types of vectors, such as displacements, velocity and acceleration vectors, etc., which we will do in later chapters.

ILLUSTRATION 3.20

Given: $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = 3\hat{j} + 4\hat{k}$

Sol. $(\vec{a}+\vec{b}) = (3\hat{i}+4\hat{j}) + (3\hat{j}+4\hat{k}) = 3\hat{i}+7\hat{j}+4\hat{k}$

Magnitude of resultant

$$|\vec{a}+\vec{b}| = \sqrt{3^2+7^2+4^2} = \sqrt{9+49+16} = \sqrt{74} \text{ units}$$

ILLUSTRATION 3.21

Given: $\vec{A} = (2\hat{i} - \hat{j} + 3\hat{k})$ and $\vec{B} = (3\hat{i} - 2\hat{j} - 2\hat{k})$.

Find the unit vector of (i) $(\vec{A}+\vec{B})$ and (ii) $(\vec{A}-\vec{B})$

Sol.

(i) $(\vec{A}+\vec{B}) = (2\hat{j}-\hat{j}+3\hat{k}) + (3\hat{i}-2\hat{j}-2\hat{k}) = 5\hat{i}-3\hat{j}+\hat{k} = \vec{C}$(say)

Hence $|C| = \sqrt{5^2+3^2+1^2} = \sqrt{35}$

Unit vector, $\hat{C} = \dfrac{\vec{C}}{|C|} = \dfrac{(5\hat{i}-3\hat{j}+\hat{k})}{\sqrt{35}}$

(ii) $(\vec{A}-\vec{B}) = (2\hat{j}-\hat{j}+3k) - (3\hat{i}-2\hat{j}-2\hat{k})$

$= -\hat{i}+\hat{j}+5\hat{k} = \vec{D}$(say)

$|D| = \sqrt{1^2+1^2+5^2} = \sqrt{27}$

Unit vector, $\hat{D} = \dfrac{\vec{D}}{|D|} = \dfrac{(-\hat{i}+\hat{j}+5\hat{k})}{\sqrt{27}}$

ILLUSTRATION 3.22

Determine that vector which when added to the resultant of $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$ gives unit vector along y-direction.

Sol. Here $\vec{A} = 3\hat{i} - 5\hat{j} + 7\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 3\hat{k}$

Resultant, $\vec{R} = \vec{A} + \vec{B}$

$= (3\hat{i} - 5\hat{j} + 7\hat{k}) + (2\hat{i} + 4\hat{j} - 3\hat{k}) = (5\hat{i} - \hat{j} + 4\hat{k})$

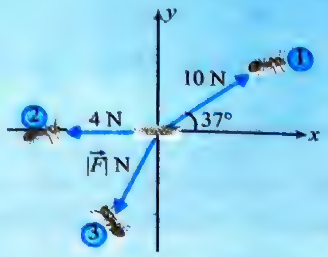
The unit vector along y direction $= \hat{j}$. Let required vector is $\vec{C}$.

$\hat{j} = (5\hat{i} - \hat{j} + 4\hat{k}) + \vec{C} \Rightarrow \vec{C} = \hat{j} - (5\hat{i} - \hat{j} + 4\hat{k})$

Required vector $= (-5\hat{i} + 2\hat{j} - 4\hat{k})$

ILLUSTRATION 3.23

Three ants 1, 2, and 3 are pulling a grain with forces of magnitudes 10 N, 4 N, and $|\vec{F}|$ N as shown in the figure. Find the force $\vec{F}$ if the grain remains in equilibrium under the action of the above forces.



Sol. The forces exerted by the ants are

$$\vec{F}_1 = 10\cos 37^\circ\,\hat{i} + 10\sin 37^\circ\,\hat{j} = 10 \times \frac{4}{5}\hat{i} + 10\times\frac{3}{5}\hat{j} = (8\hat{i}+6\hat{j})\text{N}$$

$\vec{F}_2 = 4(-\hat{i})\text{N}$ and $\vec{F}_3 (=\vec{F}) = x\hat{i} + y\hat{j}$ (say)

Since the grain is in equilibrium, it experiences no net force,

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

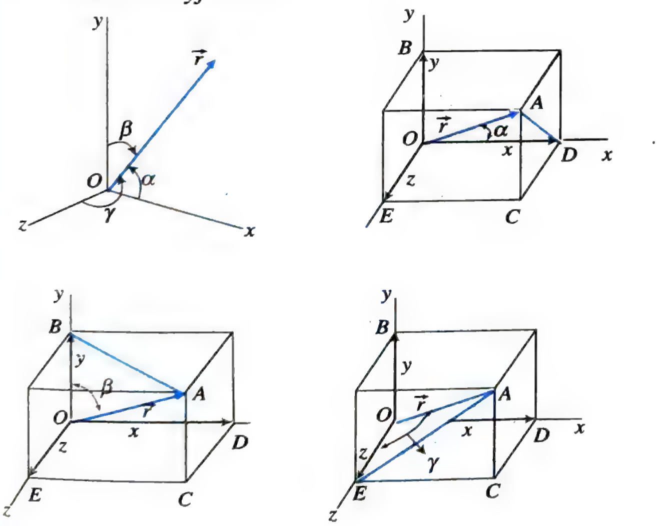
$$\Rightarrow \quad (8\hat{i}+6\hat{j}) + 4(-\hat{i}) + \vec{F} = 0$$

$$\Rightarrow \quad \vec{F} = -(4\hat{i}+6\hat{j})\text{N}$$

Direction Cosines

Let A be a point in space whose coordinates are (x, y, z). We can represent the position of this point with respect to origin as

$$\vec{r} = \overline{OA} = x\hat{i} + y\hat{j} + z\hat{k}.$$



and $r = OA = \sqrt{x^2 + y^2 + z^2}$

Angles with x-, y-, and z-axes, respectively, are given by:

$$\cos\alpha = \frac{x}{r} = l,\ \cos\beta = \frac{y}{r} = m,\ \cos\gamma = \frac{z}{r} = n$$

The direction cosines l, m, and n of a vector are the cosines of the angles α, β, and γ, which a given vector makes with x-axis, y-axis, and z-axis, respectively.

Now squaring and adding l, m, and n,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2+y^2+z^2}{r^2}$$

or $\quad l^2 + m^2 + n^2 = \dfrac{r^2}{r^2} = 1$

It means the sum of the squares of the direction cosines of a vector is always unity.

ILLUSTRATION 3.24

Given $\vec{A} = 5\hat{i} + 2\hat{j} + 4\hat{k}$. Find (a) $|\vec{A}|$ and (b) the direction cosines of vector $\vec{A}$.

Sol.

(a) As $\vec{A} = 5\hat{i} + 2\hat{j} + 4\hat{k} \Rightarrow |\vec{A}| = \sqrt{25+4+16} = \sqrt{45}$

(b) $\cos\alpha = l = \dfrac{x}{r} = \dfrac{5}{\sqrt{45}}$; $\cos\beta = m = \dfrac{y}{r} = \dfrac{2}{\sqrt{45}}$;

$$\cos\gamma = n = \frac{z}{r} = \frac{4}{\sqrt{45}}$$

ILLUSTRATION 3.25

A bird moves with velocity 20 ms^{-1} in a direction making an angle of 60° with the eastern line and 60° with vertical upward. If the eastern direction is taken as x-axis, northen direction as y-axis and upward direction as z-axis, then represent the velocity vector in rectangular form.

Sol.

Let the velocity v makes angles α, β, and γ with x-, y-, and z-axes, respectively. Then $\alpha = 60^\circ$ and $\gamma = 60^\circ$.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We have $\cos^2 60^\circ + \cos^2\beta + \cos^2 60^\circ = 1$

or $\quad \cos\beta = \dfrac{1}{\sqrt{2}}$

$$\vec{v} = v\cos\alpha\,\hat{i} + v\cos\beta\,\hat{j} + v\cos\gamma\,\hat{k}$$

$$= 20\left[\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right] = 10\hat{i} + 10\sqrt{2}\hat{j} + 10\hat{k}$$

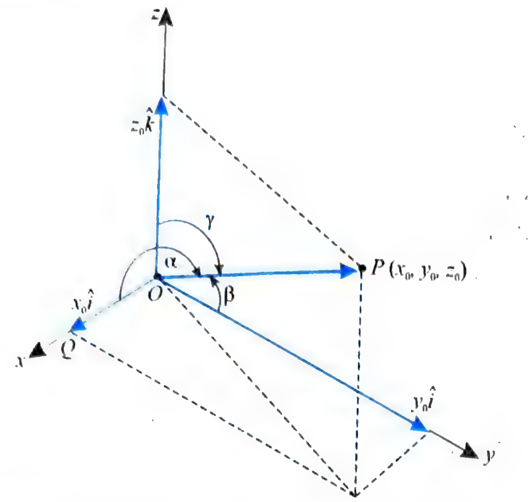
POSITION VECTOR

Vectors find numerous applications in the study of physics. Various physical quantities follow the rules of vector algebra. Using vector notation, physical quantities can be concisely expressed and conveniently calculated.

The position of a particle can be expressed in terms of its **position vector**. The position vector is a physical quantity that specifies the position of a particle in space with respect to an arbitrarily chosen origin. It is a vector quantity whose magnitude is given by the distance between the point and the origin; and it is directed towards the point from the origin. It is denoted as

$$\vec{r} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k} \qquad \text{...(i)}$$

where (x_0, y_0, z_0) are the coordinates of the point P in the Cartesian coordinate system.



For example, let us consider a particle is moving in a three dimensional space and is located at a distance r from the origin at angle α with x-axis, β from y-axis and γ from z-axis. We can express its position vector by calculating its x, y and z components and writing the expression in unit vector notation.

x-component of position vector $x_0 = r\cos\alpha$

y-component of position vector $y_0 = r\cos\beta$

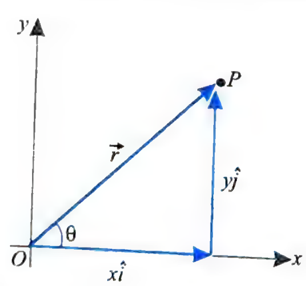
z-component of position vector $z_0 = r\cos\gamma$

Hence position vector of the particle,

$$\vec{r} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$$

$$\Rightarrow \quad \vec{r} = r\cos\alpha\,\hat{i} + r\cos\beta\,\hat{j} + r\cos\gamma\,\hat{k}$$

If a particle is moving in a two-dimensional plane and located at a distance r from the origin at angle θ with x-axis, we can express its position vector in unit vector notation as



x-component of position vector $x = r\cos\theta$,

y-component of position vector $y = r\sin\theta$

Hence position vector of the particle, $\vec{r} = x\hat{i} + y\hat{j}$

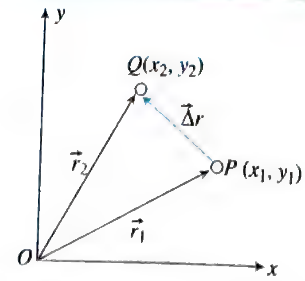
$$\Rightarrow \quad \vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$$

Important Points:

Position vector is defined with respect to an arbitrarily chosen origin, and therefore, its value depends on the reference point.

DISPLACEMENT VECTOR

Displacement vector is a vector which gives the position of a point with reference to a point other than the origin of the co-ordinate system.



It gives the position with reference to a point other than the origin.

Applying the triangle law of vectors, we get

$$\vec{r}_1 + \overrightarrow{\Delta r} = \vec{r}_2 \quad \text{or} \quad \overrightarrow{\Delta r} = \vec{r}_2 - \vec{r}_1$$

Thus, displacement vector is merely the difference of two position vectors.

If (x_1, y_1) and (x_2, y_2) are the co-ordinates of P and Q respectively, then $\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j}$.

$$\therefore \quad \overrightarrow{\Delta r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$|\overrightarrow{\Delta r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Generalizing this result for three dimensions, we get

$$|\overrightarrow{\Delta r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

ILLUSTRATION 3.26

A particle is initially at point $A(2, 4, 6)$ m moves finally to the point $B(3, 2, -3)$ m. Write the initial position vector, final position vector, and displacement vector of the particle.

Sol.

Initial position vector: $\vec{r}_1 = 2\hat{i} + 4\hat{j} + 6\hat{k}$

Final position vector: $\vec{r}_2 = 3\hat{i} + 2\hat{j} - 3\hat{k}$

Displacement: $\vec{d} = \vec{r}_2 - \vec{r}_1 = (3-2)\hat{i} + (2-4)\hat{j} + (-3-6)\hat{k}$

$$= \hat{i} - 2\hat{j} - 9\hat{k}$$

ILLUSTRATION 3.27

A particle starts moving from origin, moves 10 m in negative x-direction then turns 90° and moves further 10 m along y-direction and finely again turns 90° and after moving 20 m, stops. Find the position vector of the particle at the end of his journey.

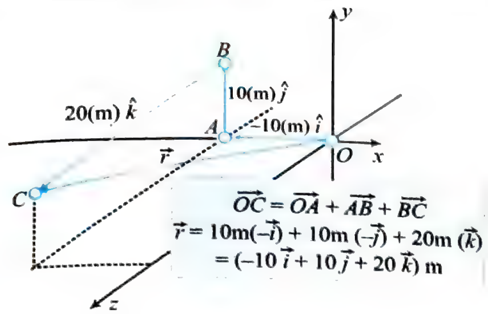
Sol.

Ist part of journey: 10 m in negative x-direction

In negative x direction the unit vector is represented by $-\hat{i}$. The magnitude of displacement is 10 m. Hence vector with magnitude 10m in the direction of the $-x$-axis: $\overrightarrow{OC} = 10\,\text{m}(-\hat{i}) = -10\hat{i}$ m

IInd part of journey: 10 m in y-direction

In y direction the unit vector is represented by $\hat{j}$. The magnitude of displacement is 10 m. Hence vector with magnitude 10m in the direction of y-axis: $\overrightarrow{AB} = 10\text{ m}(\hat{j}) = 10\hat{j}$ m



IIIrd part of journey: 20 m in z-direction

In z direction the unit vector is represented by $\hat{k}$. The magnitude of displacement is 20 m. Hence vector with magnitude 20 m in the direction of z-axis: $\overrightarrow{BC} = 20\,\text{m}(\hat{k}) = 20\hat{k}\,\text{m}$

Net displacement of the particle: $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

Final position vector of the particle:

$$\vec{r} = 10\,\text{m}(-\vec{i}) + 10\,\text{m}(\vec{j}) + 20\,\text{m}(\vec{k}) = (-10\vec{i} + 10\vec{j} + 20\vec{k})\,\text{m}$$

ILLUSTRATION 3.28

A bird flies due east through a distance of 100 m, then heading due north by a distance of 50 m, it flies vertically up through a distance of 20 m. Find the position of the bird relative to its initial position.

Sol. The final position $\vec{r}$ of the bird is followed by three successive displacements $\vec{d}_1$, $\vec{d}_2$ and $\vec{d}_3$.

$$\vec{r} = \vec{d}_1 + \vec{d}_2 + \vec{d}_3 = 100\hat{i} + 50\hat{j} + 20\hat{k}$$

Then, $|\vec{r}| = \sqrt{100^2 + 50^2 + 20^2} = 10\sqrt{129}\,\text{m}$

Hence, the bird is at a distance $10\sqrt{129}\,\text{m}$ from initial position.

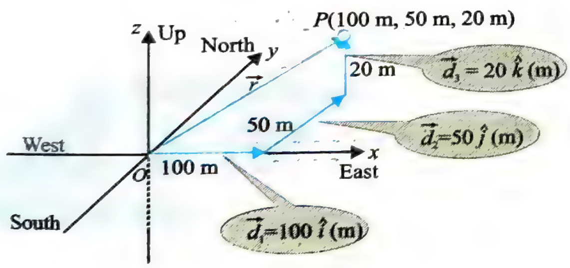
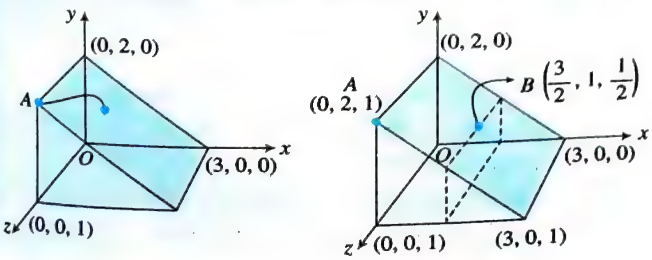


ILLUSTRATION 3.29

An insect crawls from A to B where B is the center of the rectangular slant face. Find the (a) initial and final position vector of the insect and (b) displacement vector of the insect.



Sol.

Initial position vector of insect

$$\overrightarrow{OA} = 2\hat{j} + \hat{k}\ (\text{m})$$

$$\overrightarrow{OB} = \frac{3}{2}\hat{i} + \hat{j} + \frac{1}{2}\hat{k}\ (\text{m})$$

Displacement of insect $\overrightarrow{AB}$ = Position vector of $\vec{B}$ − Position vector of $\vec{A}$.

$$\overrightarrow{AB} = \left(\frac{3}{2} - 0\right)\hat{i} + (1-2)\hat{j} + \left(\frac{1}{2} - 1\right)\hat{k} = \frac{3}{2}\hat{i} - \hat{j} - \frac{1}{2}\hat{k}\ (\text{m})$$

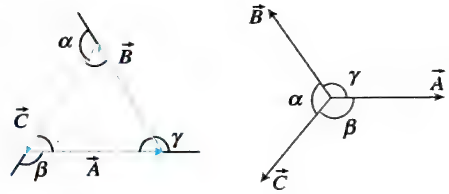
Hence, magnitude of displacement

$$|\overrightarrow{AB}| = \sqrt{\left(\frac{3}{2}\right)^2 + (1)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{9}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{7}{2}}\ \text{m}$$

Important Points:

Lami's Theorem

It states that if the resultant of three vectors is zero, then the magnitude of a vector is directly proportional to the sine of angle between other two vectors. Or it can be stated as if the resultant of three vectors is zero, then the ratio of magnitude of a vector to the sine of angle between other two vectors is constant, i.e.,

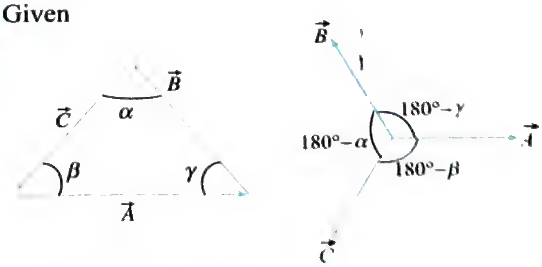


$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$

ILLUSTRATION 3.30

Given that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. Out of three vectors, two are equal in magnitude and the magnitude of the third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. Find the angles between the vectors.

Sol. Given



$$A = B,\ C = \sqrt{2}A = \sqrt{2}B.$$

From figure, $\alpha = \beta$ and $\alpha + \beta + \gamma = 180°$

$\Rightarrow \quad \gamma = 180° - 2\alpha$

Apply Lami's theorem: $\dfrac{A}{\sin\alpha} = \dfrac{C}{\sin\gamma}$

$$\Rightarrow \quad \frac{A}{\sin\alpha} = \frac{\sqrt{2}A}{\sin(180^\circ - 2\alpha)}$$

$$\Rightarrow \quad \frac{1}{\sin\alpha} = \frac{\sqrt{2}}{\sin 2\alpha}$$

$$\frac{1}{\sin\alpha} = \frac{\sqrt{2}}{2\sin\alpha\cos\alpha} \Rightarrow \cos\alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = 45^\circ$$

$$\Rightarrow \quad \beta = 45^\circ \quad \text{and} \quad \gamma = 180^\circ - 2\alpha = 90^\circ$$

Angle between $\vec{A}$ and $\vec{B}$ $= 180^\circ - \gamma = 90^\circ$,

Angle between $\vec{B}$ and $\vec{C}$ $= 180 - \alpha = 135^\circ$,

Angle between $\vec{C}$ and $\vec{A}$ $= 180^\circ - \beta = 135^\circ$.

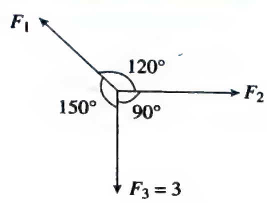
ILLUSTRATION 3.31

A bob of weight 3 N is in equilibrium under the action of two strings 1 and 2. Find the tension forces in the strings.

Sol. $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$

According to Lami's theorem,

$$\frac{F_1}{\sin 90^\circ} = \frac{F_2}{\sin 150^\circ} = \frac{F_3}{\sin 120^\circ}$$

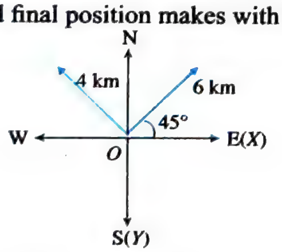


or $\quad F_1 = \dfrac{F_2}{1/2} = \dfrac{3}{\sqrt{3}/2} = 2\sqrt{3}$ N

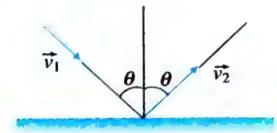
and $\quad F_2 = \sqrt{3}$ N

CONCEPT APPLICATION EXERCISE 3.3

1. A car travels 6 km towards north at an angle of 45° to the east and then travels distance of 4 km towards north at an angle of 135° to the east. How far is the point from the starting point? What angle does the straight line joining its initial and final position makes with the east?



2. If $\vec{A} = 4\hat{i} - 3\hat{j}$ and $\vec{B} = 6\hat{i} + 8\hat{j}$, then find the magnitude and direction of $\vec{A} + \vec{B}$.
3. A truck travelling due north at 20 m s^{-1} turns west and travels at the same speed. Find the change in its velocity.
4. If the sum of two unit vectors is a unit vector, then find the magnitude of difference.
5. Two forces F_1 = 1 N and F_2 = 2 N act along the lines $x = 0$ and $y = 0$, respectively. Then find the resultant of forces.
6. Let $\vec{A} = 2\hat{i} + \hat{j}$, $B = 3\hat{j} - \hat{k}$ and $\vec{C} = 6\hat{i} - 2\hat{k}$. Find the value of $\vec{A} - 2\vec{B} + 3\vec{C}$.
7. An object of m kg with speed of v m s^{-1} strikes a wall at an angle θ and rebounds at the same speed and same angle. Find the magnitude of change in the momentum of object.



8. If $A = 3\hat{i} + 4\hat{j}$ and $B = 7\hat{i} + 24\hat{j}$, find the vector having the same magnitude as B and parallel to A.
9. Vector $\vec{A}$ makes equal angles with x-, y-, and z-axes. Find the value of its components (in terms of magnitude of $\vec{A}$)
10. Find the vector that must be added to the vector $\hat{i} - 3\hat{j} + 2\hat{k}$ and $3\hat{i} + 6\hat{j} - 7\hat{k}$ so that the resultant vector is a unit vector along the y-axis.

ANSWERS

1. $\tan^{-1}(5)$
2. $5\sqrt{5}$, $\tan^{-1}\left(\dfrac{1}{2}\right)$
3. $20\sqrt{2}$
4. $\sqrt{3}$
5. $2\hat{i} + \hat{j}$
6. $20\hat{i} - 5\hat{j} - 4\hat{k}$
7. $2m\,v\cos\theta$
8. $15\hat{i} + 20\hat{j}$
9. $\dfrac{A}{\sqrt{3}}$
10. $-4\hat{i} - 2\hat{j} + 5\hat{k}$

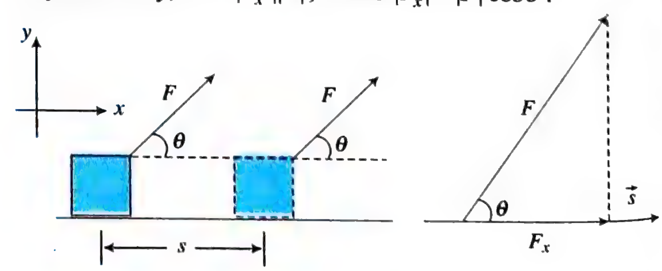
PRODUCT OF TWO VECTORS

SCALAR PRODUCT OF TWO VECTORS

If we multiple a vector by a scalar produces another vector. The scalar product is different from this multiplication. The scalar product is a way to multiply two vectors to yield a scalar result.

In physics, we defined some scalar quantities such as work, power, flux, etc., as the product of two vectors. For example, work done W by a force $\vec{F}$ is defined as the product of the displacement of a point at which the force is acting and the component of the force along the displacement (direction of motion of the point).

Symbolically, $W = |\vec{F}_x||\vec{s}|$, where $|\vec{F}_x| = |\vec{F}|\cos\theta$.



This gives $W = |\vec{F}||\vec{s}|\cos\theta$. Since the scalar quantity W is defined as a product of two vectors $\vec{F}$ and $\vec{s}$, we can call this product "scalar product."

In short, substituting cos θ by a dot (.), we write $W = \vec{F}\cdot\vec{s}$. Hence the scalar product can also be termed as a "dot product."

There are many instants in physics where we need to express some scalar quantities such as flux, pressure, power as products of two vectors. For this, we need to develop the idea of scalar (dot) product of two vectors.

Note: If, in general, a scalar quantity C is defined as the scalar (or dot) product of any two vectors $\vec{A}$ and $\vec{B}$, C is given as $C = |\vec{A}|\,|\vec{B}|\cos\theta = \vec{A}\cdot\vec{B}$, where θ is the smaller angle between the vectors $\vec{A}$ and $\vec{B}$.

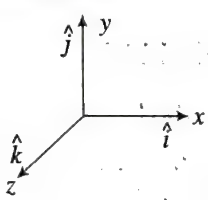
Special Cases

1. If $\theta = 0°$, $\vec{A}\cdot\vec{B} = AB$ (maximum value) $[\because \cos 0° = 1]$
2. If $\theta = 180°$, $\vec{A}\cdot\vec{B} = -AB$ (negative maximum value) $[\because \cos 180° = -1]$
3. If $\theta = 90°$, $\vec{A}\cdot\vec{B} = 0$ (minimum value) $[\because \cos 90° = 0]$
   So if two vectors are perpendicular, then their dot product is zero.
4. If θ is acute then, $\vec{A}\cdot\vec{B}$ is +ve.
   $[\because \cos\theta$ is +ve when θ is acute.$]$
5. If θ is obtuse, then $\vec{A}\cdot\vec{B}$ is –ve.
   $[\because \cos\theta$ is –ve when θ is obtuse.$]$

Note: The dot product of two vectors is always a scalar quantity.

Dot Product of Unit Vectors

The dot product of a unit vector with itself is unity and with other perpendicular unit vector is zero.



$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1 \text{ and}$$
$$\hat{i}.\hat{j} = \hat{j}.\hat{i} = \hat{j}.\hat{k} = \hat{k}.\hat{j} = \hat{k}.\hat{i} = \hat{i}.\hat{k} = 0$$

In component form, the product is expressed as:

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}, \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Then $\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$$= A_x\hat{i}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$
$$+ A_z\hat{k}(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}).$$

So $\vec{A}\cdot\vec{B} = A_xB_x(\hat{i}.\hat{i}) + A_xB_y(\hat{i}.\hat{j}) + A_xB_z(\hat{i}.\hat{k})$

$$+ A_yB_x(\hat{j}.\hat{i}) + A_yB_y(\hat{j}.\hat{j}) + A_yB_z(\hat{j}.\hat{k})$$
$$+ A_zB_x(\hat{k}.\hat{i}) + A_zB_y(\hat{k}.\hat{j}) + A_yB_z(\hat{k}.\hat{k})$$
$$= A_xB_x + A_yB_y + A_zB_z$$

Properties of Dot Product

Dot product is commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Applying the commutative property to unit vectors, we get

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i}, \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j}, \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k}$$

Dot product is distributive: $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

This may be extended to any number of vectors.

$$\vec{A}\cdot(\vec{B}+\vec{C}+\vec{D}+\cdots) = \vec{A}\cdot\vec{B} + \vec{A}.\vec{C} + \vec{A}\cdot\vec{D} + \cdots$$

Dot product of perpendicular vectors is zero

Proof. If $\vec{A}$ is perpendicular to $\vec{B}$, then $\theta = 90°$.

$$\vec{A}\cdot\vec{B} = AB\cos 90° = 0$$

This leads us to the following condition of perpendicularity of two vectors.

"Two given non-zero vectors will be perpendicular to each other if and only if their dot product is zero."

Applying the result to unit vectors, we get

$$\hat{i}\cdot\hat{j} = 0, \hat{j}\cdot\hat{k} = 0, \text{ and } \hat{k}\cdot\hat{i} = 0.$$

Dot product of a vector with itself: A vector is parallel to itself. So, angle of a vector with itself is zero.

$\therefore \quad \vec{A}\cdot\vec{A} = A\,A\cos 0° = A^2 \qquad [\because \cos 0° = 1]$

Hence, the dot product of a vector with itself is the square of its magnitude.

We can also write: $\vec{A}\cdot\vec{A} = |\vec{A}|^2 \Rightarrow |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

For example: $|\vec{P}+\vec{Q}| = \sqrt{(\vec{P}+\vec{Q})\cdot(\vec{P}+\vec{Q})}$ (taking $\vec{A} = \vec{P}+\vec{Q}$)

$$\Rightarrow \quad |\vec{P}+\vec{Q}|^2 = (\vec{P}+\vec{Q})\cdot(\vec{P}+\vec{Q})$$
$$= P^2 + Q^2 + 2\vec{P}\cdot\vec{Q} = P^2 + Q^2 + 2PQ\cos\theta$$

Similarly, $|\vec{P}-\vec{Q}|^2 = P^2 + Q^2 - 2\vec{P}\cdot\vec{Q}$

$$= P^2 + Q^2 - 2PQ\cos\theta$$

Important Points:

1. Angle between two vectors can be calculated from:

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

2. If $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ is 90°

   Proof: Given $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$

   Squaring both sides, we get $|\vec{A}+\vec{B}|^2 = |\vec{A}-\vec{B}|^2$

   $\Rightarrow \quad A^2 + B^2 + 2\vec{A}\cdot\vec{B} = A^2 + B^2 - 2\vec{A}\cdot\vec{B} \Rightarrow 4\vec{A}\cdot\vec{B} = 0$

   $\Rightarrow \quad \vec{A}\cdot\vec{B} = 0$

   Hence, $\vec{A}$ is perpendicular to $\vec{B}$.

3. If $(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = 0$, then $\vec{A}$ and $\vec{B}$ are equal in magnitude, i.e., $A = B$.

   Proof: Given $(\vec{A}+\vec{B})\cdot(\vec{A}-\vec{B}) = 0$

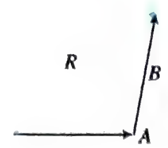
   $\Rightarrow \quad \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} - \vec{B}\cdot\vec{B} = 0$

   $\Rightarrow \quad A^2 - B^2 = 0$

   $\Rightarrow \quad A^2 = B^2$

   $\Rightarrow \quad A = B$ (Hence proved)

4. We can find the addition of two vectors using dot product.



In Figure $\vec{R}=\vec{A}+\vec{B}$

$\Rightarrow \quad |\vec{R}|=|\vec{A}+\vec{B}|$

or $\quad |\vec{R}|^2=|\vec{A}+\vec{B}|^2$

or $\quad R^2=(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = A^2+B^2+2\vec{A}\cdot\vec{B}$

or $\quad R=\sqrt{A^2+B^2+2\vec{A}\cdot\vec{B}}=\sqrt{A^2+B^2+2AB\cos\theta}$

Similarly, we can find subtraction of two vectors also.

ILLUSTRATION 3.32

Find the dot product of two vectors $\vec{A}=3\hat{i}+2\hat{j}-4\hat{k}$ and $\vec{B}=2\hat{i}-3\hat{j}-6\hat{k}$.

Sol. $\vec{A}\cdot\vec{B}=3\times2+2\times(-3)+(-4)\times(-6)=24$

ILLUSTRATION 3.33

Find the value of m so that the vector $3\hat{i}-2\hat{j}+\hat{k}$ may be perpendicular to the vector $2\hat{i}+6\hat{j}+m\hat{k}$.

Sol. The given vectors will be perpendicular if their dot product is zero.

$$(3\hat{i}-2\hat{j}+\hat{k})\cdot(2\hat{i}+6\hat{j}+m\hat{k})=0$$

$$6(\hat{i}\cdot\hat{i})-12(\hat{j}\cdot\hat{j})+m(\hat{k}\cdot\hat{k})=0$$

or $\quad 6-12+m=0$

or $\quad m-6=0$ or $m=6$

ILLUSTRATION 3.34

What is the angle between the following pair of vectors ?

$\vec{A}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{B}=-2\hat{i}-2\hat{j}-2\hat{k}$.

Sol. $\vec{A}\cdot\vec{B}=AB\cos\theta$ or $\cos\theta=\dfrac{\vec{A}.\vec{B}}{AB}$ ...(i)

But, $\vec{A}\cdot\vec{B}=(\hat{i}+\hat{j}+\hat{k})\cdot(-2\hat{i}-2\hat{j}-2\hat{k})$

$\vec{A}\cdot\vec{B}=-2-2-2=-6$

Again $A=|\vec{A}|=\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$;

$B=|\vec{B}|=\sqrt{(-2)^2+(-2)^2+(-2)^2}=\sqrt{12}=2\sqrt{3}$

Now, $\cos\theta=\dfrac{-6}{\sqrt{3}\times2\sqrt{3}}=-1 \quad\Rightarrow\quad \theta=180^\circ$

ILLUSTRATION 3.35

If the sum of two unit vectors is a unit vector, then find the magnitude of their difference.

Sol. Let $\hat{n}_1$ and $\hat{n}_2$ are the two unit vectors, then their sum is $\vec{n}_S=\hat{n}_1+\hat{n}_2$.

$\Rightarrow \quad n_S^2=n_1^2+n_2^2+2n_1n_2\cos\theta=1+1+2\cos\theta$

Since it is given that $\vec{n}_s$ is a unit vector, so $n_s=1$.

$\therefore \quad 1=1+1+2\cos\theta \Rightarrow \cos\theta=-\dfrac{1}{2} \Rightarrow \theta=120^\circ$

Now the difference vector is $\vec{n}_d=\vec{n}_1-\vec{n}_2$

$\Rightarrow \quad n_d^2=n_1^2+n_2^2-2n_1n_2\cos\theta=1+1-2\cos(120^\circ)=3$

$\Rightarrow \quad n_d=\sqrt{3}$

ILLUSTRATION 3.36

Prove that $(\vec{A}+2\vec{B})\cdot(2\vec{A}-3\vec{B})=2A^2+AB\cos\theta-6B^2$.

Sol. $(\vec{A}+2\vec{B})\cdot(2\vec{A}-3\vec{B})$

$=2\vec{A}\cdot\vec{A}-3\vec{A}\cdot\vec{B}+4\vec{B}\cdot\vec{A}-6(\vec{B}\cdot\vec{B})$

$=2(\vec{A}\cdot\vec{A})-3\vec{A}\,\vec{B}\cos\theta+4\vec{A}\,\vec{B}\cos\theta-6(\vec{B}\cdot\vec{B})$

$=2A^2+AB\cos\theta-6B^2$

ILLUSTRATION 3.37

A body constrained to move along the z-axis of a co-ordinate system is subjected to a constant force $\vec{F}$ given by $\vec{F}=-\hat{i}+2\hat{j}+3\hat{k}$ newton where $\hat{i}$, $\hat{j}$, and $\hat{k}$ represent unit vectors along x-, y-, and z-axes of the system, respectively. Calculate the work done by this force in displacing the body through a distance of 4 m along the z-axis.

Sol. Displacement $=4\hat{k}$, Force: $\vec{F}=-\hat{i}+2\hat{j}+3\hat{k}$

Since work W is the scalar product of force and displacement,

$\therefore \quad W=(-\hat{i}+2\hat{j}+3\hat{k})\cdot4\hat{k}$

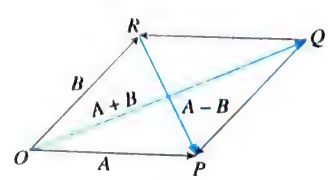
$=-4(\hat{i}\cdot\hat{k})+8(\hat{j}\cdot\hat{k})+12(\hat{k}\cdot\hat{k})=12$ J

[Because $\hat{i}\cdot\hat{k}=0=\hat{j}\cdot\hat{k}$ and $\hat{k}\cdot\hat{k}=1$]

ILLUSTRATION 3.38

By vector method, prove that if the diagonals of a parallelogram intersect perpendicularly, then the parallelogram is a rhombus.

Sol. Assuming the sides of a parallelogram as $\overrightarrow{OP}=\vec{A}$, and $\overrightarrow{OR}=\vec{B}$, its diagonals can be given as $\overrightarrow{OQ}=\vec{A}+\vec{B}$ and $\overrightarrow{RP}=\vec{A}-\vec{B}$, respectively.



Since $\overline{OQ} \perp \overline{RP}$, $\overline{OQ} \cdot \overline{RP} = 0$; substituting $OQ = \vec{A} + \vec{B}$ and $RP = \vec{A} - \vec{B}$, we have $(\vec{A} + \vec{B}) \cdot (\vec{A} - \vec{B}) = 0$.

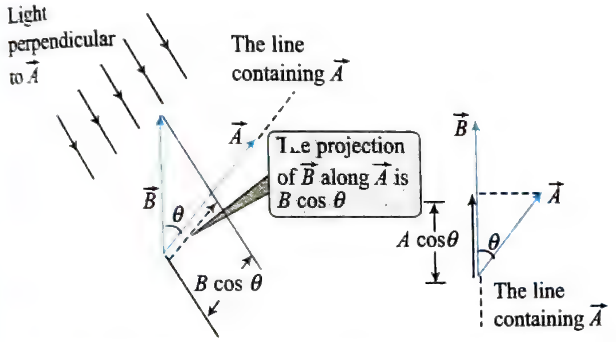
This gives $\vec{A} \cdot \vec{A} - \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} - \vec{B} \cdot \vec{B} = 0$

Since $\vec{A} \cdot \vec{A} = |\vec{A}|^2$, $\vec{B}.\vec{B} = |\vec{B}|^2$, and $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$, we have

$|\vec{A}|^2 - |\vec{B}|^2 = 0$. Hence, $|\vec{A}| = |\vec{B}|$. It means that $OP = OR$. This tells us that the given parallelogram is a rhombus.

Geometric Interpretation of Dot Product

In order to geometrically interpret the scalar product, we draw $\vec{A}$ and $\vec{B}$ drawn with their tails together. we drop a perpendicular from the tip of $\vec{B}$ to line containing $\vec{A}$. The quantity $B \cos \theta$ is called projection of $\vec{B}$ or component of $\vec{B}$ on the line containing $\vec{A}$. Imagine light shinning perpendicular to $\vec{A}$ then the shadow of vector $\vec{B}$ on the line containing $\vec{A}$, has length equal to the projection of $\vec{B}$ or component of $\vec{B}$ on line of $\vec{A}$.



We can also take projection the other way around.

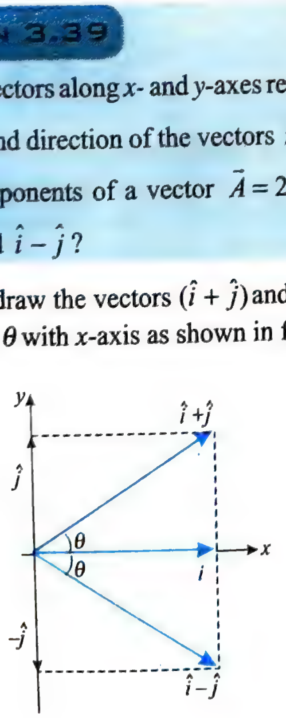
$$\vec{A} \cdot \vec{B} = A(B\cos\theta) = (A\cos\theta)B$$

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

ILLUSTRATION 3.39

$\hat{i}$ and $\hat{j}$ are unit vectors along x- and y-axes respectively. What is the magnitude and direction of the vectors $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$? What are the components of a vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the direction $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$?

Sol. Let us first draw the vectors $(\hat{i} + \hat{j})$ and $(\hat{i} - \hat{j})$. Both the vectors make angle θ with x-axis as shown in figure.



(a) Magnitude of $\hat{i} + \hat{j}$

$$= \sqrt{1^2 + 1^2 + 2 \times 1 \times 1 \times \cos 90^\circ} = \sqrt{2} = 1.414 \text{ units}$$

$$\tan\theta = \frac{1}{1} = 1$$

$\therefore \quad \theta = 45^\circ$

So, the vector $\hat{i} + \hat{j}$ makes an angle of 45° with x-axis.

(b) Magnitude of $\hat{i} - \hat{j} = |\hat{i} + (-\hat{j})|$

$$= \sqrt{1^2 + (-1)^2 + 2(1)(-1)\cos 90^\circ} = \sqrt{2}$$

$$\tan\theta = \frac{1}{-1} = -1 \quad \therefore \quad \theta = -45^\circ$$

The vector $\hat{i} - \hat{j}$ makes an angle of –45° with x-axis.

Let us now determine the component of $\vec{A} = 2\hat{i} + 3\hat{j}$ in the direction of $\hat{i} + \hat{j}$.

Then, $\vec{B} = \hat{i} + \hat{j}$

$$\vec{A}.\vec{B} = AB\cos\theta = (A\cos\theta)B$$

So component of $\vec{A}$ in the direction of $\vec{B}$

$$= \frac{\vec{A}.\vec{B}}{B} = \frac{(2\hat{i} + 3\hat{j}).(\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}} = \frac{2.1 + 3.1}{\sqrt{2}} = \frac{5}{\sqrt{2}} \text{ units}$$

Component of $\vec{A}$ in the direction of $\hat{i} - \hat{j}$

$$= \frac{(2\hat{i} + 3\hat{j}).(\hat{i} - \hat{j})}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \text{ units}$$

Here, PM is the component of $\vec{A}$ in the direction of $\vec{B}$. So, dot product of two vectors $\vec{A}$ and $\vec{B}$ may be defined as the product of the magnitude of $\vec{B}$ and the component of $\vec{A}$ in the direction of $\vec{B}$.

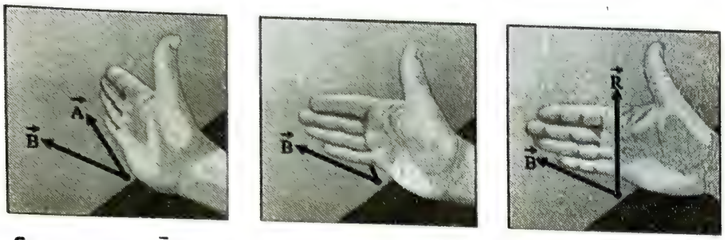
Vector Product (Cross Product)

If the product of two vector quantities is a vector quantity, then the product is called vector product or cross product. We express the vector product of $\vec{A}$ and $\vec{B}$ as $\vec{A} \times \vec{B}$, produces a third vector $\vec{R}$ whose magnitude is $|\vec{A} \times \vec{B}| = |A||B| \sin\theta$ where θ is the smaller of the two angles between $\vec{A}$ and $\vec{B}$. (You must use the smaller of the two angles between the vectors because $\sin \theta$ and $\sin (360° - \theta)$ differ in algebraic sign.) Because of the notation, $\vec{A} \times \vec{B}$ is also known as the cross product, and in speech it is "A cross B." We can write $\vec{R} = \vec{A} \times \vec{B}$.

If two vectors $\vec{A}$ and $\vec{B}$ are parallel or antiparallel, then $\vec{A} \times \vec{B} = 0$. Hence $\hat{i} \times \hat{i} = 0, \hat{j} \times \hat{j} = 0, \hat{k} \times \hat{k} = 0$. The magnitude of $\vec{A} \times \vec{B}$, which can be written as $|\vec{A} \times \vec{B}|$, is maximum when $\vec{A}$ and $\vec{B}$ are perpendicular to each other.

The direction of resultant of cross product $\vec{R}$ is perpendicular to the plane that contains $\vec{A}$ and $\vec{B}$. The first figure shows how to determine the direction of $\vec{A} \times \vec{B}$ with what is known as a

right-hand rule. Place the vectors $\vec{A}$ and $\vec{B}$ tail to tail without altering their orientations, and imagine a line that is perpendicular to their plane where they meet. Pretend to place your right hand around that line in such a way that your fingers would sweep $\vec{A}$ into $\vec{B}$ through the smaller angle between them. Your outstretched thumb points in the direction of $\vec{R}$.



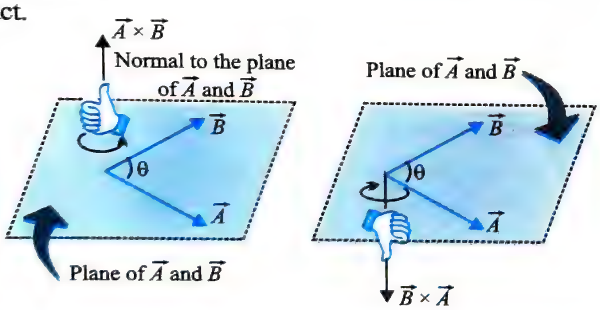
Sweep vector $\vec{A}$ into vector $\vec{B}$ with the fingers of your right hand. Your outstretched thumb shows the direction of vector $\vec{R} = \vec{A} \quad \vec{B}$

We can completely write vector product (or cross product) as $\vec{A} \times \vec{B} = (|A||B|\sin\theta)\,\hat{n}$, $\hat{n}$ is the direction of unit vector normal to plane containing vectors $\vec{A}$ and $\vec{B}$.

Properties of Cross Product of Vectors

The order of the vector multiplication is important. In the figure below, we are determining the direction of $\vec{B} \times \vec{A}$, so the fingers are placed to sweep $\vec{A}$ into $\vec{B}$ through the smaller angle. The thumb ends up in the opposite direction from previously, that is, $\vec{B} \times \vec{A} = -(\vec{A} \times \vec{B})$

In other words, the commutative law does not apply to a vector product.

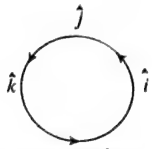


Right Hand Thumb Rule for Finding the Direction of Cross Product of Unit Vectors

To check whether any *xyz* coordinate system is a right-handed coordinate system, use the right-hand rule for the cross product $\hat{i} \times \hat{j} = \hat{k}$ with that system. If your fingers sweep (positive direction of *x*) into $\hat{j}$ (positive direction of *y*) with the outstretched thumb pointing in the positive direction of *z* (not the negative direction), then the system is right-handed.

Unit Vectors and their Cross Product

$\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along *x*, *y*, and *z* axis, respectively. The magnitude of each vector is 1 and the angle between any of two unit vectors is 90°.



So $\hat{i} \times \hat{j} = (1)\,(1)\sin 90°\ \hat{n} = \hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane containing vector, $\hat{i}$ and $\hat{j}$.

To find out the resultant of any unit vector in cross product, use the following rules.

Rule 1. Multiplication of any two unit vectors in anticlockwise direction gives the third unit vector with positive sign.

Rule 2. Multiplication of any two unit vectors in clockwise direction gives the third unit vector with negative sign.

From these rules, we obtain the following results:

From Rule 1

1. $\hat{i} \times \hat{j} = \hat{k}$ **2.** $\hat{j} \times \hat{k} = \hat{i}$

3. $\hat{k} \times \hat{i} = \hat{j}$

From Rule 2

1. $\hat{j} \times \hat{i} = -\hat{k}$ **2.** $\hat{k} \times \hat{j} = -\hat{i}$

3. $\hat{i} \times \hat{k} = -\hat{j}$

Cross product method 1: Using component form

$$\vec{A} \times \vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \times (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k})$$
$$= A_xB_x(\hat{i}\times\hat{i}) + A_yB_x(\hat{j}\times\hat{i}) + A_zB_x(\hat{k}\times\hat{i})$$
$$+A_xB_y(\hat{i}\times\hat{j}) + A_yB_y(\hat{j}\times\hat{j}) + A_zB_y(\hat{k}\times\hat{j})$$
$$+A_xB_z(\hat{i}\times\hat{k}) + A_yB_z(\hat{j}\times\hat{k}) + A_zB_z(\hat{k}\times\hat{k})$$

[As $\hat{i}\times\hat{i} = 0,\ \hat{j}\times\hat{j} = 0,\ \hat{k}\times\hat{k} = 0$ and $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{i} = \hat{j}, \hat{i}\times\hat{k} = -\hat{j}, \hat{j}\times\hat{k} = \hat{i}\ \hat{k}\times\hat{j} = -\hat{i}$]

So, we have $\vec{A} \times \vec{B}$

$$= A_yB_x(-\hat{k}) + A_zB_x\hat{j} + A_xB_y\hat{k}$$
$$+ A_zB_y(-\hat{i}) + A_xB_z(-\hat{j}) + A_yB_z(\hat{i})$$
$$= (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

Cross product method 2: Determinant method

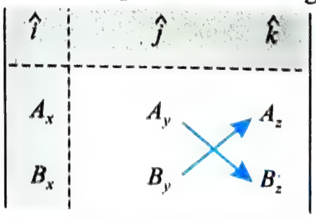
We have $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

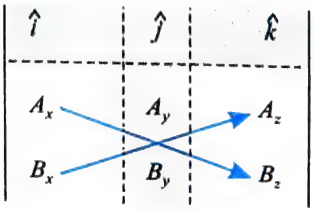
The cross product of $\vec{A}$ and $\vec{B}$ can be written as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

For solving above determinant, we pick $\hat{i}$, $\hat{j}$ and $\hat{k}$ one by one. When $\hat{i}$ is chosen, its corresponding row and column becomes bond remaining elements are subtracted after cross multiplication. Process is given in following figures.



$$= \hat{i}\,(A_yB_z - B_yA_z) \qquad \ldots(i)$$



$$= -\hat{j}\,(A_xB_z - B_xA_z) \qquad \ldots(ii)$$

$$\begin{array}{ccc|c} \hat{i} & \hat{j} & \hat{k} \\ \hline A_x & A_y & A_z \\ B_x & B_y & B_z \end{array} = \hat{k}\,(A_xB_y - B_xA_y) \qquad \ldots\text{(iii)}$$

Now adding right hand side of equations (i), (ii) and (iii), we get

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\left(A_yB_z - B_yA_z\right) - \hat{j}\left(A_xB_z - B_xA_z\right) + \hat{k}\left(A_xB_y - B_xA_y\right)$$

Properties of Vector Cross Product

Anti-commutative property: The vector product of two vectors is anti commutative

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n} \text{ and } \vec{B}\times\vec{A} = BA\sin\theta(-\hat{n})$$

$$= -AB\sin\theta\,\hat{n} = -(\vec{A}\times\vec{B})$$

So $\vec{B}\times\vec{A} = -(\vec{A}\times\vec{B})$

It means $\vec{B}\times\vec{A} \neq \vec{A}\times\vec{B}$

Distributive property: Vector product is distributive, i.e.,

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

Associative Property:

$$(\vec{A}+\vec{B})\times(\vec{C}+\vec{D}) = \vec{A}\times\vec{C} + \vec{A}\times\vec{D} + \vec{B}\times\vec{C} + \vec{B}\times\vec{D}$$

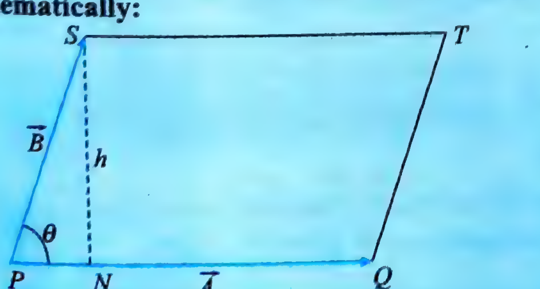
Cross product of two parallel vectors: Cross product of the parallel vectors is zero.

As $\theta = 0°$ (for parallel vectors)

So $(\vec{A}\times\vec{B}) = AB\sin 0°\,\hat{n} = 0$

Important Points:

1. **If two vectors represent the two adjacent sides of a parallelogram, then the magnitude of the cross product will give the area of the parallelogram. Mathematically:**



Two vectors $\vec{A}$ and $\vec{B}$ are represented by the two adjacent sides PQ and PS, respectively, of the parallelogram as shown in figure.

Now, from magnitude of cross product:

$$|\vec{A}\times\vec{B}| = AB\sin\theta = Ah$$

ILLUSTRATION 3.40

Calculate the area of the triangle determined by the two vectors $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = -3\hat{i} + 7\hat{j}$.

Sol. We know that the half of magnitude of the cross product of two vectors gives the area of the triangle.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -3 & 7 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0-0) - \hat{j}(0-0) + \hat{k}(21+12) = 33\hat{k}$$

Taking magnitude $|\vec{A}\times\vec{B}| = \sqrt{33^2} = 33$.

So, area of triangle $= \dfrac{1}{2}|\vec{A}\times\vec{B}| = \dfrac{33}{2}$ sq. unit

ILLUSTRATION 3.41

Calculate the area of the parallelogram when adjacent sides are given by the vectors $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} - 3\hat{j} + \hat{k}$.

Sol. We know that area of the parallelogram is equal to magnitude of the cross product of given vectors. Now,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= \hat{i}\,(2+9) + \hat{j}\,(6-1) + \hat{k}\,(-3-4) = 11\,\hat{i} + 5\,\hat{j} - 7\,\hat{k}$$

So area of parallelogram: $|\vec{A}\times\vec{B}| = \sqrt{11^2 + 5^2 + (-7)^2}$

$$= \sqrt{195} \text{ sq·unit}$$

ILLUSTRATION 3.42

(a) Prove that the vectors $\vec{A} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 5\hat{k}$, and $\vec{C} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right-angled triangle.

(b) Determine the unit vector parallel to the cross product of the vectors $\vec{A} = 3\hat{i} - 5\hat{j} + 10\hat{k}$ & $\vec{B} = 6\hat{i} + 5\hat{j} + 2\hat{k}$.

Sol.

(a) The given vectors will constitute a triangle only if one of the given vectors is equal to vector sum of the remaining two vectors. In the given problem, $\vec{B} + \vec{C} = \vec{A}$

So, the given vectors do form a triangle. This triangle will be right-angled only if the dot product of two vectors (out of the given three) is zero.

$$\vec{A}\cdot\vec{B} = (3\hat{i} - 2\hat{j} + \hat{k})\cdot(\hat{i} - 3\hat{j} + 5\hat{k})$$

$$= 3(\hat{i}\cdot\hat{i}) + 6(\hat{j}\cdot\hat{j}) + 5(\hat{k}\cdot\hat{k}) = 3+6+5+ = 14$$

$$\vec{B}\cdot\vec{C} = (\hat{i} - 3\hat{j} + 5\hat{k})\cdot(2\hat{i} + \hat{j} - 4\hat{k})$$

$$= 2(\hat{i}\cdot\hat{i}) - 3(\hat{j}\cdot\hat{j}) - 20(\hat{k}\cdot\hat{k}) = 2 - 3 - 20 = -21$$

$$\vec{C}\cdot\vec{A} = (2\hat{i} + \hat{j} - 4\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k})$$
$$= 6(\hat{i}\cdot\hat{i}) - 2(\hat{j}\cdot\hat{j}) - 4(\hat{k}\cdot\hat{k}) = 6 - 2 - 4 = 0$$

Since the dot product of $\vec{C}$ and $\vec{A}$ is zero, therefore, it implies that $\vec{C}$ is perpendicular to $\vec{A}$.

(b) The unit vector parallel to $(\vec{A}\times\vec{B})$ is given by $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$ So, let us first determine $\vec{A}\times\vec{B}$.

Now $\vec{A}\times\vec{B} = (3\hat{i} - 5\hat{j} + 10\hat{k}) \times (6\hat{i} + 5\hat{j} + 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & 10 \\ 6 & 5 & 2 \end{vmatrix} = \hat{i}(-10-50) + \hat{j}(60-6) + \hat{k}(15+30)$$

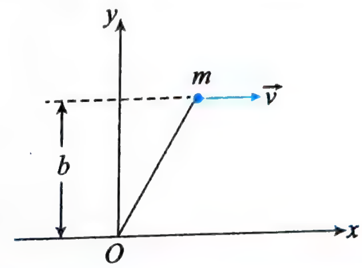
$$= -60\hat{i} + 54\hat{j} + 45\hat{k}$$

Magnitude: $|\vec{A}\times\vec{B}| = \sqrt{(-60)^2 + (54)^2 + (45)^2} = \sqrt{8541}$

So, required unit vector: $\hat{n} = \dfrac{-60\hat{i} + 54\hat{j} + 45\hat{k}}{\sqrt{8541}}$

Concept Application Exercise 3.4

1. $\vec{A} = 2\hat{i} + 4\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 2\hat{j} - 4\hat{k}$ are two vectors. Find the angle between them.
2. If two vectors $2\hat{i} + 3\hat{j} - \hat{k}$ and $-4\hat{i} - 6\hat{j} + \lambda\hat{k}$ are parallel to each other, then find the value of λ.
3. In Q. 2, if vectors are perpendicular to each other then find the value of λ.
4. If $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = -\hat{i} + 3\hat{j} + 4\hat{k}$, then find the projection of $\vec{A}$ on $\vec{B}$.
5. A body, acted upon by a force of 50 N, is displaced through a distance 10 m in a direction making an angle of 60° with the force. Find the work done by the force.
6. A particle moves from position $3\hat{i} + 2\hat{j} - 6\hat{k}$ to $14\hat{i} + 13\hat{j} + 9\hat{k}$ due to a uniform force of $4\hat{i} + \hat{j} + 3\hat{k}$ N. If the displacement is in meters, then find the work done.
7. If for two vectors $\vec{A}$ and $\vec{B}$, sum $(\vec{A} + \vec{B})$ is perpendicular to the difference $(\vec{A} - \vec{B})$. Find the ratio of their magnitude.
8. A force $\vec{F} = -K(y\hat{i} + x\hat{j})$ (where K is a positive constant) acts on a particle moving in the x-y plane. Starting from the origin, the particle is taken along the positive x-axis to the point $(a, 0)$ and then parallel to the y-axis to the point (a, a). Find the total work done by the forces $\vec{F}$ on the particle.
9. If $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$, then find the value of $|\vec{A}\times\vec{B}|$.
10. The vectors from origin to the points A and B are $\vec{A} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} + \hat{j} - 2\hat{k}$, respectively. Find the area of the triangle OAB.
11. The angle between the vectors $\vec{A}$ and $\vec{B}$ is θ. Find the value of the triple product $\vec{A}\cdot(\vec{B}\times\vec{A})$.
12. Find the torque of the force $\vec{F} = (2\hat{i} - 3\hat{j} + 4\hat{k})$ N acting at the point $\vec{r} = (3\hat{i} + 2\hat{j} + 3\hat{k})$ m about the origin.
13. If a particle of mass m is moving with constant velocity v parallel to x-axis in xy plane as shown in Figure, Find its angular momentum with respect to origin at any time t.



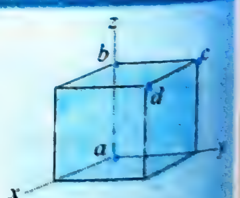
Answers

1. 90°
2. 2
3. −26
4. $\dfrac{3}{\sqrt{26}}$
5. 250 J
6. 100 J
7. 1
8. $-Ka^2$
9. $8\sqrt{3}$
10. $\dfrac{5\sqrt{17}}{2}$ sq. unit
11. 0
12. $17\hat{i} - 6\hat{j} - 13\hat{k}$
13. $-mvb\,\hat{k}$

Solved Examples

Example 3.1

A cube is placed so that one corner is at the origin and three edges are along the x-, y-, and z-axes of a coordinate system. Use vectors to compute



(a) The angle between the edge along the z-axis (line ab) and the diagonal from the origin to the opposite corner (line ad).

(b) The angle between line ac (the diagonal of a face) and line ad.

Sol.

(a) Let side of the cube is d, then $\vec{ab} = d\hat{k}$ and $\vec{ad} = d\hat{i} + d\hat{j} + d\hat{k}$.

Let angle between $\vec{ab}$ and $\vec{ad}$ is θ.

then, $\cos\theta = \dfrac{\vec{ab}\cdot\vec{ad}}{|\vec{ab}||\vec{ad}|} = \dfrac{d^2}{d\sqrt{3}d} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

(b) We can represent $\vec{ac}$ and $\vec{ad}$ as $\vec{ac} = d\hat{j} + d\hat{k}$,

$\vec{ad} = d\hat{i} + d\hat{j} + d\hat{k}$

Let angle between $\vec{ac}$ and $\vec{ad}$ is θ'.

$$\cos\theta' = \frac{\overline{ac}\cdot\overline{ad}}{|\overline{ac}||\overline{ad}|} = \frac{2d^2}{\sqrt{2}d\sqrt{3}d} = \frac{\sqrt{2}}{\sqrt{3}}$$

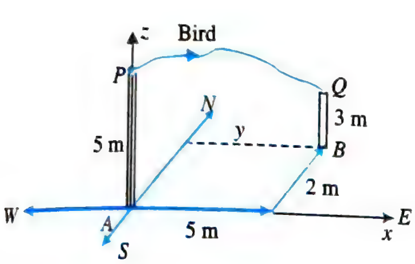
$$\Rightarrow \quad \theta' = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$$

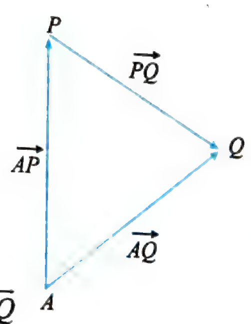
EXAMPLE 3.2

On a horizontal flat ground, a person is standing at a point A. At this point, he installs a 5 m long pole vertically. Now, he moves 5 m towards east and then 2 m towards north and reaches at a point B. There he installs another 3 m long vertical pole. A bird flies from the top of the first pole to the top of the second pole. Find the displacement and magnitude of the displacement of the bird.

Sol. Let us represent the situation digramatically as shown in figure.

$$\overline{AP} = 5\hat{k},\ \overline{AQ} = 5\hat{i} + 2\hat{j} + 3\hat{k}$$





From vector triangle, $\overline{AP} + \overline{PQ} = \overline{AQ}$

Hence, the displacement of the bird

$$= \overline{PQ} = \overline{AQ} - \overline{AP} = 5\hat{i} + 2\hat{j} - 2\hat{k}$$

$$PQ = \sqrt{5^2 + 2^2 + 2^2} = \sqrt{33}\text{ m}$$

EXAMPLE 3.3

Unit vectors $\hat{P}$ and $\hat{Q}$ are inclined at an angle θ. Prove that $|\hat{P} - \hat{Q}| = 2\sin(\theta/2)$.

Sol.

$|\hat{P}| = |\hat{Q}| = 1$, because they are unit vectors.

$$\text{LHS} = |\hat{P} - \hat{Q}|$$

$$= \sqrt{(\hat{P} - \hat{Q})\cdot(\hat{P} - \hat{Q})} = \sqrt{P^2 + Q^2 - 2\hat{P}\cdot\hat{Q}}$$

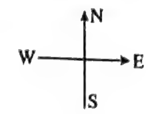
$$= \sqrt{1^2 + 1^2 - 2PQ\cos\theta} = \sqrt{2 - 2\times 1\times 1\cos\theta}$$

$$= \sqrt{2(1-\cos\theta)} = \sqrt{2\left[2\sin^2(\theta/2)\right]} = 2\sin(\theta/2) = \text{RHS}$$

EXAMPLE 3.4

A sail boat sails 2 km due east, 5 km 37° south of east, and finally an unknown displacement. If the final displacement of the boat from the starting point is 6 km due east, determine the third displacement.

Sol. Given that the resultant displacement is 6 km due east which may be expressed as $6\hat{i}$.



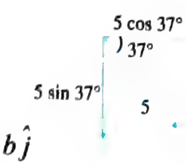
Given displacements are:

(a) 2 km due east or $2\hat{i}$

(b) 5 km 37° south of east its components

Along x-axis $= 5\cos 37° = 5\times\frac{4}{5} = 4\hat{i}$

Along y-axis $= 5\sin 37° = 5\times\frac{3}{5} = -3\hat{j}$



(c) Let the unknown displacement be $a\hat{i} + b\hat{j}$

$\therefore \quad 2\hat{i} + 4\hat{i} - 3\hat{j} + a\hat{i} + b\hat{j} = 6\hat{i}$

$\therefore \quad 6\hat{i} + a\hat{i} - 3\hat{j} + b\hat{j} = 6\hat{i}$

Comparing the two sides, we get $a = 0$ and $b = 3$.

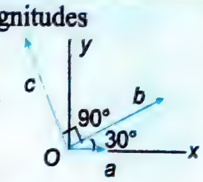
Therefore, unknown displacement $= 3\hat{j}$

EXAMPLE 3.5

Three vectors as shown in figure have magnitudes $|\vec{a}| = 3, |\vec{b}| = 4$, and $|\vec{c}| = 10$.

(a) Find the x and y components of these vectors.

(b) Find the numbers p and q such that $\vec{c} = p\vec{a} + q\vec{b}$.



Sol.

(a) $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 10$

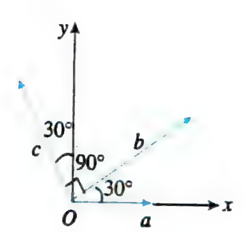
$a_x = 3,\ a_y = 0;$

$b_x = b\cos 30° = 4\times\frac{\sqrt{3}}{2} = 2\sqrt{3}$

$b_y = 4\sin 30° = \frac{4}{2} = 2;$

$c_x = c\cos 120° = -c\times\frac{1}{2} = \frac{-10}{2} = -5,$

$c_y = c\sin 120° = 10\times\frac{\sqrt{3}}{2} = 5\sqrt{3}$



(b) Given $c = p\vec{a} + q\vec{b}$

$$c_x = p\vec{a}_x + q\vec{b}_x \Rightarrow -5 = 3p + 2\sqrt{3}q$$

$$\vec{c}_y = p\vec{c}_y + q\vec{b}_y \Rightarrow 5\sqrt{3} = 0 + 2q \Rightarrow q = \frac{5\sqrt{3}}{2}$$

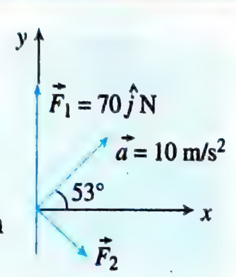
Replacing value of q,

$$-5 = 3p + 2\sqrt{3}\times\frac{5\sqrt{3}}{2}$$

or $\quad -5 = 3p + 15 \quad$ or $\quad p = \frac{-20}{3}$

EXAMPLE 3.6

A particle of $m = 5$ kg is momentarily at rest at $x = 0$ at $t = 0$. It is acted upon by two forces $\vec{F}_1$ and $\vec{F}_2$. $\vec{F}_1 = 70\hat{j}$ N The direction and magnitude of $\vec{F}_2$ are unknown. The particle experiences a constant acceleration, $\vec{a}$, in the direction as shown in figure. Neglect gravity.



(a) Find the missing force $\vec{F}_2$.
(b) What is the velocity vector of the particle at $t = 10$ s?
(c) What third force, $\vec{F}_3$, is required to make the acceleration of the particle zero? Either give magnitude and direction of $\vec{F}_3$ or its components.

Sol. Acceleration of the particle

$$\vec{a} = 10\cos 53^\circ\,\hat{i} + 10\sin 53^\circ\,\hat{j}$$
$$= 10\times\frac{3}{5}\hat{i} + 10\times\frac{4}{5}\hat{j} = (6\hat{i}+8\hat{j})\,\text{m/s}$$

As $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 = m\vec{a}$

$$70\hat{j} + F_2\hat{i} + F_2\hat{j} = 5(6\hat{i}+8\hat{j})$$
$$= 30\hat{i} + 40\hat{j}$$

By comparing two sides, we get

$$F_2\hat{i} = 30\hat{i} \quad\Rightarrow\quad F_2\hat{j} = -30\hat{j}$$
$$\therefore\quad \vec{F}_2 = 30\hat{i} - 30\hat{j}$$
$$|F_2| = \sqrt{(30)^2+(30)^2} = 30\sqrt{2}\ \text{N}$$

At $t = 10$ s

$$v = \vec{u} + at = 0 + (6\hat{i}+8\hat{j})\times 10 = (60\hat{i}+80\hat{j})\,\text{m/s}$$

Acceleration will be zero if a third force makes the resultant force = 0

$$\therefore\quad \vec{F} + \vec{F}_2 + \vec{F}_3 = 0$$
$$\text{or}\quad \vec{F}_3 = -(\vec{F}_1 + \vec{F}_2)$$
$$= -(70\hat{j} + 30\hat{i} - 30\hat{j}) = (-30\hat{i} - 40\hat{j})\,\text{N}$$

EXAMPLE 3.7

A spy plane is being tracked by a radar. At $t = 0$, its position is reported as (100 m, 200 m, 1000 m). 130 s later, its position is reported to be (2500 m, 1200 m, 1000 m). Find a unit vector in the direction of plane velocity and the magnitude of its average velocity.

Sol. In 130 s, the displacement is

$$(2500-100)\hat{i} + (1200-200)\hat{j} + (100-100)\hat{k} = (2400\hat{i} + 1000\hat{j})\,\text{m}$$

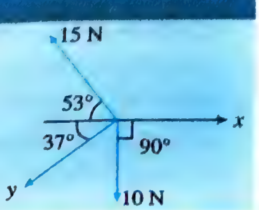
$$\text{Magnitude of displacement} = \sqrt{(2400)^2 + (1000)^2} = 2600\ \text{m}$$

$$\therefore\quad \text{Average velocity} = \frac{2600}{130} = 20\ \text{m s}^{-1}$$

$$\text{And unit vector along velocity} = \frac{1}{2600}(2400\hat{i} + 1000\hat{j})$$
$$= \frac{12\hat{i} + 5\hat{j}}{13}$$

EXAMPLE 3.8

Find the magnitude of the unknown forces if the sum of all forces is zero figure.

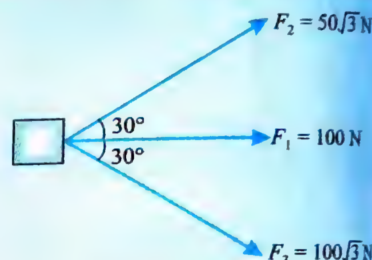


Sol. Sum of all given forces = 0

$$\therefore\quad x\hat{i} - 10\hat{j} - 15\cos 53^\circ\hat{i} + 15\sin 53^\circ\hat{j} - y\cos 37^\circ\hat{i} - y\sin 37^\circ\hat{j} = 0$$
$$x\hat{i} - 15\times\frac{3}{5}\hat{i} - y\cdot\frac{4}{5}\hat{i} = 0$$
$$-10\hat{j} + 15\times\frac{4}{5}\hat{j} - y\cdot\frac{3}{5}\hat{j} = 0$$
$$x - \frac{4}{5}y - 9 = 0 \quad\text{or}\quad x - \frac{4}{5}y = 9$$
$$-10 + 12 - \frac{3}{5}y = 0 \quad\text{or}\quad \frac{3}{5}y = 2 \quad\text{or}\quad y = \frac{10}{3}$$
$$\text{and}\quad x = 9 + \frac{4}{5}\times\frac{10}{3} = \frac{25}{3}\text{N}$$

EXAMPLE 3.9

Three boys are pulling a heavy trolley by means of three ropes. The boy in the middle is exerting a pull of 100 N. The other two boys, whose ropes both make an angle of 30° with the central rope, are pulling with forces of $50\sqrt{3}$ N and $100\sqrt{3}$ N. Find the magnitude of the resultant pull on the trolley.

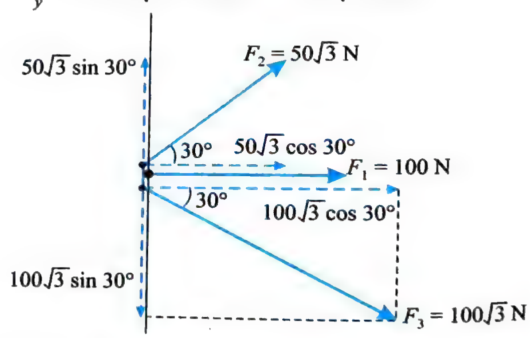


Sol. x-component

$$F_x = 100 = 50\sqrt{3}\cos 30^\circ + 100\sqrt{3}\cos 30^\circ$$
$$= 100 + 50\sqrt{3}\times\frac{\sqrt{3}}{2} + 100\sqrt{3}\times\frac{\sqrt{3}}{2} = 325\,\text{N}$$

y-component

$$F_y = -100\sqrt{3}\times\sin 30^\circ + 50\sqrt{3}\sin 30^\circ = -25\sqrt{3}\,\text{N}$$



In unit vector notation, we can write

$$\vec{F} = (325\hat{i} - 25\sqrt{3}\hat{j})\,\text{N}$$

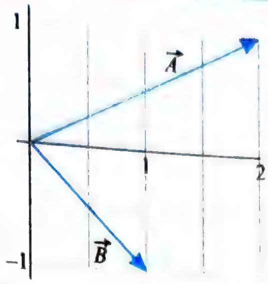
Resultant

$$= \sqrt{(325)^2 + (-25\sqrt{3})^2} = \sqrt{107500} = 327.9\text{N}$$

EXAMPLE 3.10

If $\vec{A} = 2\hat{i} + \hat{j}$ and $\vec{B} = \hat{i} - \hat{j}$, sketch vectors graphically and find the component of $\vec{A}$ along $\vec{B}$ and perpendicular to $\vec{B}$.

Sol. $\vec{A} = 2\hat{i} + \hat{j}$, $\vec{B} = \hat{i} - \hat{j}$



Component of A along B

$$= \frac{A \cdot B}{B} \cdot \hat{B} = A\cos\theta \cdot \hat{B}$$

$$= \frac{2-1}{\sqrt{2}} \cdot \frac{(\hat{i} - \hat{j})}{\sqrt{2}} = \frac{1}{2}(\hat{i} - \hat{j})$$

Component of A perpendicular to B

$$= \vec{A} - (\text{component of } \vec{A} \text{ along } \vec{B})$$

$$= 2\hat{i} + \hat{j} - \frac{1}{2}(\hat{i} - \hat{j}) = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} = \frac{3}{2}(\hat{i} + \hat{j})$$

Exercises

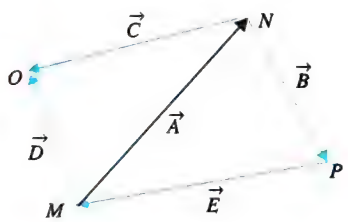
Single Correct Answer Type

1. The sum and difference of two perpendicular vectors of equal length are
 (1) Perpendicular to each other and of equal length
 (2) Perpendicular to each other and of different lengths
 (3) Of equal length and have an obtuse angle between them
 (4) Of equal length and have an acute angle between them

2. The minimum number of vectors having different planes which can be added to give zero resultant is
 (1) 2 (2) 3
 (3) 4 (4) 5

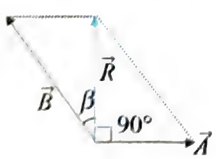
3. A vector perpendicular to $\hat{i}+\hat{j}+\hat{k}$ is
 (1) $\hat{i}-\hat{j}+\hat{k}$ (2) $\hat{i}-\hat{j}-\hat{k}$
 (3) $-\hat{i}-\hat{j}-\hat{k}$ (4) $3\hat{i}+2\hat{j}-5\hat{k}$

4. From figure, the correct relation is
 (1) $\vec{A}+\vec{B}+\vec{E}=\vec{0}$ (2) $\vec{C}-\vec{D}=-\vec{A}$
 (3) $\vec{B}+\vec{E}-\vec{C}=-\vec{D}$ (4) All of the above



5. Out of the following set of forces, the resultant of which cannot be zero?
 (1) 10, 10, 10 (2) 10, 10, 20
 (3) 10, 20, 20 (4) 10, 20, 40

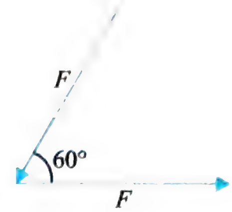
6. The resultant of two vectors $\vec{A}$ and $\vec{B}$ is perpendicular to the vector $\vec{A}$ and its magnitude is equal to half of the magnitude of vector $\vec{B}$ figure. The angle between $\vec{A}$ and $\vec{B}$ is



 (1) 120° (2) 150°
 (3) 135° (4) None of these

7. The ratio of maximum and minimum magnitudes of the resultant of two vectors $\vec{a}$ and $\vec{b}$ is 3:1. Now, $|\vec{a}|$ is equal to
 (1) $|\vec{b}|$ (2) $2|\vec{b}|$
 (3) $3|\vec{b}|$ (4) $4|\vec{b}|$

8. Two forces, each equal to F, act as shown in figure. Their resultant is



 (1) $F/2$ (2) F
 (3) $\sqrt{3}F$ (4) $\sqrt{5}F$

9. Vector $\vec{A}$ is 2 cm long and is 60° above the x-axis in the first quadrant. Vector $\vec{B}$ is 2 cm long and is 60° below the x-axis in the fourth quadrant. The sum $\vec{A}+\vec{B}$ is a vector of magnitude
 (1) 2 cm along positive y-axis
 (2) 2 cm along positive x-axis
 (3) 2 cm along negative y-axis
 (4) 2 cm along negative x-axis

10. What is the angle between two vector forces of equal magnitude such that their resultant is one-third of either of the original forces?
 (1) $\cos^{-1}\left(-\frac{17}{18}\right)$ (2) $\cos^{-1}\left(-\frac{1}{3}\right)$
 (3) 45° (4) 120°

11. The angle between $\vec{A}+\vec{B}$ and $\vec{A}\times\vec{B}$ is
 (1) 0 (2) $\pi/4$
 (3) $\pi/2$ (4) π

12. The projection of a vector $\vec{r}=3\hat{i}+\hat{j}+2\hat{k}$ on the x-y plane has magnitude
 (1) 3 (2) 4
 (3) $\sqrt{14}$ (4) $\sqrt{10}$

13. If $|\vec{A}+\vec{B}|=|\vec{A}|=|\vec{B}|$, then the angle between $\vec{A}$ and $\vec{B}$ is
 (1) 120° (2) 60°
 (3) 90° (4) 0°

14. If vectors $\vec{A}=\hat{i}+2\hat{j}+4\hat{k}$ and $\vec{B}=5\hat{i}$ represent the two sides of a triangle, then the third side of the triangle can have length equal to
 (1) 6 (2) $\sqrt{56}$
 (3) Both of the above (4) None of the above

15. Given $|\vec{A}_1|=2, |\vec{A}_2|=3$ and $|\vec{A}_1+\vec{A}_2|=3$. Find the value of $(\vec{A}_1+2\vec{A}_2)\cdot(3\vec{A}_1-4\vec{A}_2)$.
 (1) –64 (2) 60
 (3) –60 (4) 64

16. Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfy the relation $\vec{A}\cdot\vec{B}=0$ and $\vec{A}\cdot\vec{C}=0$. The vector $\vec{A}$ is parallel to

(1) $\vec{B}$
(2) $\vec{C}$
(3) $\vec{B}\cdot\vec{C}$
(4) $\vec{B}\times\vec{C}$

17. If $\vec{A}=\vec{B}+\vec{C}$, and the magnitudes of $\vec{A}$, $\vec{B}$, $\vec{C}$ are 5, 4, and 3 units, then the angle between $\vec{A}$ and $\vec{C}$ is
(1) $\cos^{-1}\left(\frac{3}{5}\right)$
(2) $\cos^{-1}\left(\frac{4}{5}\right)$
(3) $\sin^{-1}\left(\frac{3}{4}\right)$
(4) $\frac{\pi}{2}$

18. Given: $\vec{A} = A\cos\theta\hat{i} + A\sin\theta\hat{j}$. A vector $\vec{B}$, which is perpendicular to $\vec{A}$, is given by
(1) $B\cos\theta\hat{i} - B\sin\theta\hat{j}$
(2) $B\sin\theta\hat{i} - B\cos\theta\hat{j}$
(3) $B\cos\theta\hat{i} + B\sin\theta\hat{j}$
(4) $B\sin\theta\hat{i} + B\cos\theta\hat{j}$

19. The angle which the vector $\vec{A} = 2\hat{i}+3\hat{j}$ makes with the y-axis, where $\hat{i}$ and $\hat{j}$ are unit vectors along x- and y-axes, respectively, is
(1) $\cos^{-1}(3/5)$
(2) $\cos^{-1}(2/3)$
(3) $\tan^{-1}(2/3)$
(4) $\sin^{-1}(2/3)$

20. Given $\vec{P}=3\hat{i}-4\hat{j}$. Which of the following is perpendicular to $\vec{P}$?
(1) $3\hat{i}$
(2) $4\hat{j}$
(3) $4\hat{i}+3\hat{j}$
(4) $4\hat{i}-3\hat{j}$

21. In going from one city to another, a car travels 75 km north, 60 km north-west and 20 km east. The magnitude of displacement between the two cities is (take $1/\sqrt{2}=0.7$)
(1) 170 km
(2) 137 km
(3) 119 km
(4) 140 km

22. What is the angle between $\vec{A}$ and $\vec{B}$, if $\vec{A}$ and $\vec{B}$ are the adjacent sides of a parallelogram drawn from a common point and the area of the parallelogram is $AB/2$?
(1) 15°
(2) 30°
(3) 45°
(4) 60°

23. Two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$. What is the angle between $\vec{a}$ and $\vec{b}$?
(1) 0°
(2) 90°
(3) 60°
(4) 180°

24. Given $\vec{A}=4\hat{i}+6\hat{j}$ and $\vec{B}=2\hat{i}+3\hat{j}$. Which of the following is correct?
(1) $\vec{A}\times\vec{B}=\vec{0}$
(2) $\vec{A}\cdot\vec{B}=24$
(3) $\frac{|\vec{A}|}{|\vec{B}|}=\frac{1}{2}$
(4) $\vec{A}$ and $\vec{B}$ are antiparallel

25. Given $\vec{A}=2\hat{i}+p\hat{j}+q\hat{k}$ and $\vec{B}=5\hat{i}+7\hat{j}+3\hat{k}$. If $\vec{A}\parallel\vec{B}$, then the values of p and q are, respectively,
(1) $\frac{14}{5}$ and $\frac{6}{5}$
(2) $\frac{14}{3}$ and $\frac{6}{5}$
(3) $\frac{6}{5}$ and $\frac{1}{3}$
(4) $\frac{3}{4}$ and $\frac{1}{4}$

26. If $\vec{A}$ is perpendicular to $\vec{B}$, then
(1) $\vec{A}\times\vec{B}=0$
(2) $\vec{A}\cdot\left[\vec{A}+\vec{B}\right]=A^2$
(3) $\vec{A}\cdot\vec{B}=AB$
(4) $\vec{A}\cdot\left[\vec{A}+\vec{B}\right]=A^2+AB$

27. If the angle between the vectors $\vec{a}$ and $\vec{b}$ is an acute angle, then the difference $\vec{a}-\vec{b}$ is
(1) The major diagonal of the parallelogram
(2) The minor diagonal of the parallelogram
(3) Any of the above
(4) None of the above

28. Given that $\vec{A}+\vec{B}=\vec{C}$. If $|\vec{A}|=4, |\vec{B}|=5$ and $|\vec{C}|=\sqrt{61}$, the angle between $\vec{A}$ and $\vec{B}$ is
(1) 30°
(2) 60°
(3) 90°
(4) 120°

29. If $\vec{b}=3\hat{i}+4\hat{j}$ and $\vec{a}=\hat{i}-\hat{j}$, the vector having the same magnitude as that of $\vec{b}$ and parallel to $\vec{a}$ is
(1) $\frac{5}{\sqrt{2}}(\hat{i}-\hat{j})$
(2) $\frac{5}{\sqrt{2}}(\hat{i}+\hat{j})$
(3) $5(\hat{i}-\hat{j})$
(4) $5(\hat{i}+\hat{j})$

30. Choose the wrong statement.
(1) Three vectors of different magnitudes may be combined to give zero resultant.
(2) Two vectors of different magnitudes can be combined to give a zero resultant.
(3) The product of a scalar and a vector is a vector quantity.
(4) All of the above are wrong statements.

31. What displacement at an angle 60° to the x-axis has an x-component of 5 m? $\hat{i}$ and $\hat{j}$ are unit vectors in x and y directions, respectively.
(1) $5\hat{i}$
(2) $5\hat{i}+5\hat{j}$
(3) $5\hat{i}+5\sqrt{3}\hat{j}$
(4) All of the above

32. Mark the correct statement.
(1) $|\vec{a}+\vec{b}|\ge|\vec{a}|+|\vec{b}|$
(2) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$
(3) $|\vec{a}-\vec{b}|\ge|\vec{a}|+|\vec{b}|$
(4) All of the above

33. Out of the following forces, the resultant of which cannot be 10 N?
(1) 15 N and 20 N
(2) 10 N and 10 N
(3) 5 N and 12 N
(4) 12 N and 1 N

34. Which of the following pairs of forces cannot be added to give a resultant force of 4 N?
(1) 2 N and 8 N
(2) 2 N and 2 N
(3) 2 N and 6 N
(4) 2 N and 4 N

35. In an equilateral triangle ABC, AL, BM, and CN are medians. Forces along BC and BA represented by them will have a resultant represented by
(1) $2AL$
(2) $2BM$
(3) $2CN$
(4) AC

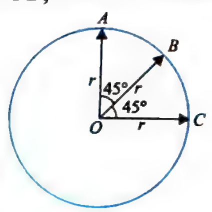
36. The vector sum of two forces is perpendicular to their vector difference. The forces are
(1) Equal to each other
(2) Equal to each other in magnitude
(3) Not equal to each other in magnitude
(4) Cannot be predicted

37. If a parallelogram is formed with two sides represented by vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}+\vec{b}$ represents the
(1) Major diagonal when the angle between vectors is acute
(2) Minor diagonal when the angle between vectors is obtuse
(3) Both of the above
(4) None of the above

38. The resultant $\vec{C}$ of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Also, $|\vec{A}|=|\vec{C}|$. The angle between $\vec{A}$ and $\vec{B}$ is
(1) $\frac{\pi}{4}$ rad (2) $\frac{3\pi}{4}$ rad
(3) $\frac{5\pi}{4}$ rad (4) $\frac{7\pi}{4}$ rad

39. Two forces $\vec{F}_1 = 500$ N due east and $\vec{F}_2 = 250$ N due north have their common initial point. $\vec{F}_2 - \vec{F}_1$ is
(1) $250\sqrt{5}$ N, $\tan^{-1}(2)$ W of N
(2) 250 N, $\tan^{-1}(2)$ W of N
(3) Zero
(4) 750 N, $\tan^{-1}(3/4)$ N of W

40. The resultant of the three vectors $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ shown in figure is



(1) r
(2) $2r$
(3) $r(1+\sqrt{2})$
(4) $r(\sqrt{2}-1)$

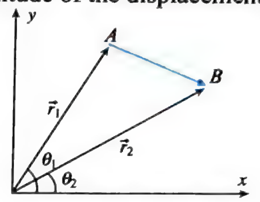
41. Two vectors $\vec{a}$ and $\vec{b}$ are at an angle of 60° with each other. Their resultant makes an angle of 45° with $\vec{a}$. If $|\vec{b}| = 2$ units, then $|\vec{a}|$ is
(1) $\sqrt{3}$ (2) $\sqrt{3}-1$
(3) $\sqrt{3}+1$ (4) $\sqrt{3}/2$

42. The resultant of two vectors $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. If the magnitude of $\vec{Q}$ is doubled, the new resultant vector becomes perpendicular to $\vec{P}$. Then, the magnitude of $\vec{R}$ is equal to
(1) $P+Q$ (2) P
(3) $P-Q$ (4) Q

43. A vector $\vec{A}$ when added to the vector $\vec{B} = 3\hat{i}+4\hat{j}$ yields a resultant vector that is in the positive y-direction and has a magnitude equal to that of $\vec{B}$. Find the magnitude of $\vec{A}$.
(1) $\sqrt{10}$ (2) 10
(3) 5 (4) $\sqrt{15}$

44. $ABCDEF$ is a regular hexagon with point O as center. The value of $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}$ is
(1) $2\overrightarrow{AO}$ (2) $4\overrightarrow{AO}$
(3) $6\overrightarrow{AO}$ (4) 0

45. In a two-dimensional motion of a particle, the particle moves from point A, with position vector $\vec{r}_1$, to point B, with position vector $\vec{r}_2$. If the magnitudes of these vectors are, respectively, $r_1 = 3$ and $r_2 = 4$ and the angles they make with the x-axis are $\theta_1 = 75°$ and $\theta_2 = 15°$, respectively, then find the magnitude of the displacement vector.



(1) 15 (2) $\sqrt{13}$
(3) 17 (4) $\sqrt{15}$

46. The sum of the magnitudes of two forces acting at a point is 16 N. The resultant of these forces is perpendicular to the smaller force and has a magnitude of 8 N. If the smaller force is of magnitude x, then the value of x is
(1) 2 N (2) 4 N
(3) 6 N (4) 7 N

47. The angle between two vectors $\vec{A}$ and $\vec{B}$ is θ. The resultant of these vectors $\vec{R}$ makes an angle of $\theta/2$ with A. Which of the following is true?
(1) $A = 2B$ (2) $A = B/2$
(3) $A = B$ (4) $AB = 1$

48. The resultant of three vectors 1, 2, and 3 units whose directions are those of the sides of an equilateral triangle is at an angle of
(1) 30° with the first vector
(2) 15° with the first vector
(3) 100° with the first vector
(4) 150° with the first vector

49. A unit vector along the incident ray of light is $\hat{i}$. The unit vector for the corresponding refracted ray of light is $\hat{r}\cdot\hat{n}$, a unit vector normal to the boundary of the medium and directed towards the incident medium. If μ is the refractive index of the medium, then Snell's law (second law) of refraction is
(1) $\hat{i}\times\hat{n} = \mu(\hat{n}+\hat{r})$
(2) $\hat{i}\cdot\hat{n} = \mu(\hat{r}\cdot\hat{n})$
(3) $\hat{i}\times\hat{n} = \mu(\hat{r}\times\hat{n})$
(4) $\hat{i}\times\hat{n} = \mu(\hat{r}\times\hat{n})$

50. The components of a vector along the x- and y-directions are $(n+1)$ and 1, respectively. If the coordinate system is rotated by an angle $\theta = 60°$, then the components change to n and 3. The value of n is
(1) 2 (2) cos 60°
(3) sin 60° (4) 3.5

51. Two point masses 1 and 2 move with uniform velocities $\vec{v}_1$ and $\vec{v}_2$, respectively. Their initial position vectors are $\vec{r}_1$ and $\vec{r}_2$, respectively. Which of the following should be satisfied for the collision of the point masses?

(1) $\dfrac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_2 - \vec{r}_1|} = \dfrac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|}$ (2) $\dfrac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} = \dfrac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|}$

(3) $\dfrac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 + \vec{r}_1|} = \dfrac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 + \vec{v}_1|}$ (4) $\dfrac{\vec{r}_2 + \vec{r}_1}{|\vec{r}_2 + \vec{r}_1|} = \dfrac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 + \vec{v}_1|}$

52. In a methane (CH_4) molecule each hydrogen atom is at a corner of a regular tetrahedron with the carbon atom at the centre. In coordinates where one of the C–H bonds is in the direction of $\hat{i} + \hat{j} + \hat{k}$, an adjacent C–H bond in the $\hat{i} - \hat{j} - \hat{k}$ direction. Then angle between these two bonds.

(1) $\cos^{-1}\left(-\dfrac{2}{3}\right)$ (2) $\cos^{-1}\left(\dfrac{2}{3}\right)$

(3) $\cos^{-1}\left(-\dfrac{1}{3}\right)$ (4) $\cos^{-1}\left(\dfrac{1}{3}\right)$

53. Consider east as positive x-axis, north as positive y-axis. A girl walks 10 m east first time then 10 m in a direction 30° west of north for the second time and then third time in unknown direction and magnitude so as to return to her initial position. What is her third displacement in unit vector notation?

(1) $-5\hat{i} - 5\sqrt{3}\,\hat{j}$ (2) $5\hat{i} - 5\sqrt{3}\,\hat{j}$

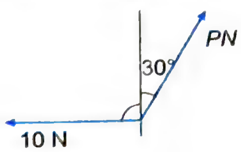
(3) $-5\hat{i} + 5\sqrt{3}\,\hat{j}$ (4) She cannot return

54. A sail boat sails 2 km due East, 5 km 37° South of East and finally an unknown displacement. If the final displacement of the boat from the starting point is 6 km due East, determine the third displacement.

(1) 3 km, North (2) 4 km, South

(3) 5 km, East (4) 3 km, West

55. Two horizontal forces of magnitudes 10 N and P N act on a particle. The force of magnitude 10 N acts due west and the force of magnitude P N acts on a bearing of 30° east of north as shown in figure. The resultant of these two force acts due north. Find the magnitude of this resultant.



(1) $10\sqrt{2}$ N (2) $15\sqrt{3}$ N

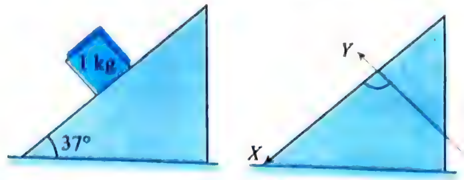
(3) $12\sqrt{5}$ N (4) $10\sqrt{3}$ N

56. A particle whose speed is 50 m/s moves along the line from $A(2, 1)$ to $B(9, 25)$. Find its velocity vector in the form of $a\hat{i} + b\hat{j}$.

(1) $(7\hat{i} + 24\hat{j})$ m/s (2) $2(7\hat{i} + 24\hat{j})$ m/s

(3) $4(7\hat{i} + 24\hat{j})$ m/s (4) $5(7\hat{i} + 24\hat{j})$ m/s

57. Weight mg of a block is a force acting downward towards centre of the earth. A block of mass 1 kg is placed on an inclined plane as shown in the figure. Find the x-component and y-component of weight of the block are

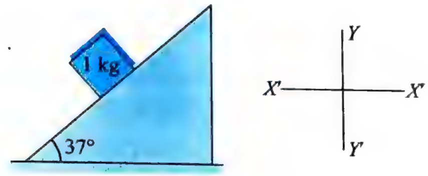


(1) 6 N, – 8 N (2) 6 N, 8 N

(3) 8 N, 6 N (4) 8 N, –6 N

58. Normal reaction N is a force exerted by the surface on the block perpendicular to the surface of contact. A block of mass 1 kg is placed on inclined plane of inclination 37° as shown in the figure.

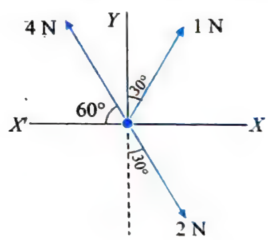
Find the component of normal reaction N = 8 N on the block X-axis and Y-axis.



(1) –4.8 N, 6.4 N (2) 6.4 N, 4.8 N

(3) 10 N,0 (4) 4.8 N, 6.4 N

59. Three forces are acting on a particle as shown in the figure. To have the resultant force only along the Y-direction, the magnitude of the minimum additional force needed is



(1) 0.866 N (2) 1.732 N

(2) 0.5 N (4) 4 N

60. A car going due North at $20\sqrt{2}$ ms^{-1} turns right through an angle of 90° without changing speed. The change in velocity of car is

(1) 20 ms^{-1} in South-East direction

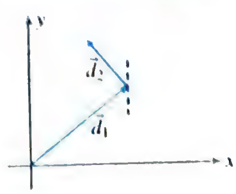
(2) $20\sqrt{2}$ ms^{-1} in South-East direction

(3) 20 ms^{-1} North-East direction

(4) 20 ms^{-1} in North-West direction

Multiple Correct Answers Type

1. Which of the following statements is/are correct?



(1) The sign of the x-component of $\vec{d}_1$ is positive and that of $\vec{d}_2$ is negative.

(2) The signs of the y-components of $\vec{d}_1$ and $\vec{d}_2$ are positive and negative, respectively.

(3) The signs of the x- and y-components of $\vec{d}_1 + \vec{d}_2$ are positive.

(4) None of these.

2. Given two vectors $\vec{A} = = 3\hat{i} + 4\hat{j}$ and $\vec{B} = = \hat{i} + \hat{j}$. θ is the angle between $\vec{A}$ and $\vec{B}$. Which of the following statements is/are correct?

(1) $|\vec{A}|\cos\theta\left(\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ along $\vec{B}$.

(2) $|\vec{A}|\sin\theta\left(\dfrac{\hat{i}-\hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ perpendicular to $\vec{B}$.

(3) $|\vec{A}|\cos\theta\left(\dfrac{\hat{i}-\hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ along $\vec{B}$.

(4) $|\vec{A}|\sin\theta\left(\dfrac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$ is the component of $\vec{A}$ perpendicular to $\vec{B}$.

3. If $\vec{A} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ are two vectors, then the unit vector is

(1) Perpendicular to $\vec{A}$ is $\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}$

(2) Parallel to $\vec{A}$ is $\dfrac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

(3) Perpendicular to $\vec{B}$ is $\left(\dfrac{-\hat{j}+\hat{k}}{\sqrt{2}}\right)$

(4) Parallel to $\vec{A}$ is $\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

4. If $\vec{v}_1 + \vec{v}_2$ is perpendicular to $\vec{v}_1 - \vec{v}_2$, then

(1) $\vec{v}_1$ is perpendicular to $\vec{v}_2$ c

(2) $|\vec{v}_1| = |\vec{v}_2|$

(3) $\vec{v}_1$ is a null vector

(4) The angle between $\vec{v}_1$ and $\vec{v}_2$ can have any value

5. Two vectors $\vec{A}$ and $\vec{B}$ lie in one plane. Vector $\vec{C}$ lies in a different plane. Then, $\vec{A} + \vec{B} + \vec{C}$

(1) Cannot be zero

(2) Can be zero

(3) Lies in the plane of $\vec{A}$ or $\vec{B}$

(4) Lies in a plane different from that of any of the three vectors

6. Two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point and

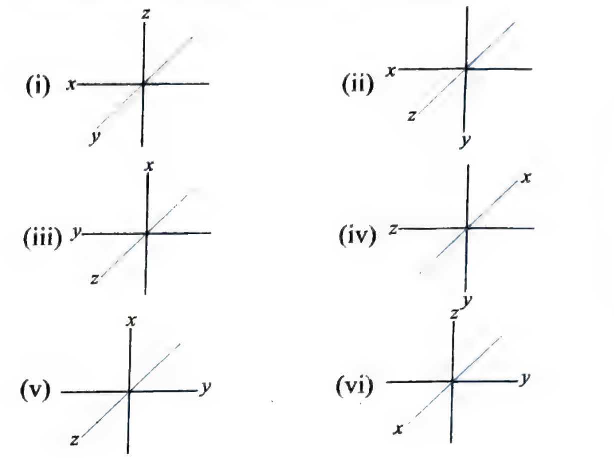
(1) If $C^2 = A^2 + B^2$, the angle between vectors $\vec{A}$ and $\vec{B}$ is $90°$

(2) If $C^2 < A^2 + B^2$, the angle between $\vec{A}$ and $\vec{B}$ is greater than $90°$

(3) If $C^2 > A^2 + B^2$ then angle between the vectors $\vec{A}$ and $\vec{B}$ is between $0°$ and $90°$

(4) If $C = A - B$, angle between $\vec{A}$ and $\vec{B}$ is $180°$

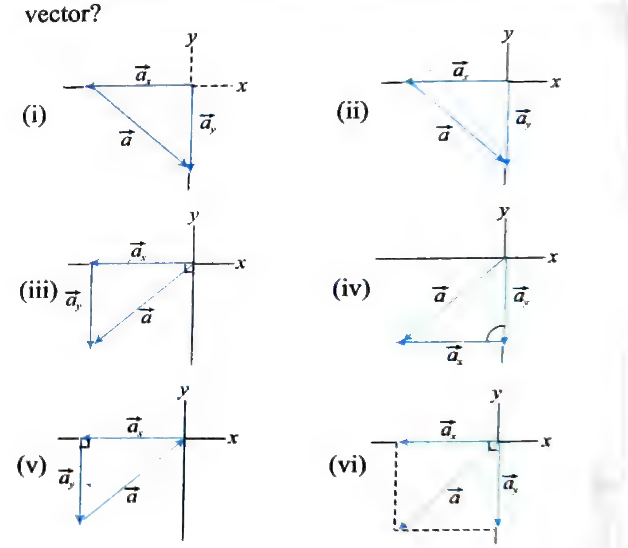
7. Which of the arrangement of axes in figure can be labelled "right-handed coordinate system"? As usual, each axis label indicates the positive side of the axis.



(1) (i), (ii) (2) (iii), (iv)

(3) (vi) (4) (v)

8. In the figure which of the ways indicated for combining the x and y components of a vector are proper to determine that vector?



(1) (iii) (2) (iv)

(3) (vi) (4) (i), (ii) and (v)

9. If the resultant of several vectors is zero. Then,

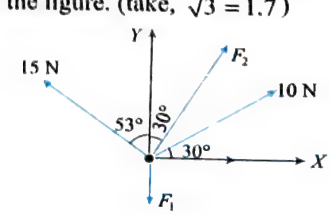
(1) vectors must obey polygon law of vector

(2) the x-component of resultant vector must be zero

(3) the y-component of resultant vector must be zero

(4) the z-component of the resultant vector must be zero

10. A particle is in equilibrium in the presence of four forces as shown in the figure. (take, $\sqrt{3} = 1.7$)



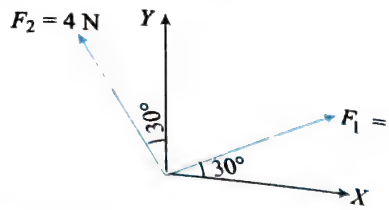
(1) $F_1 = 19.4$ N

(2) $F_2 = 7$ N

(3) $F_1 = 7$ N

(4) $F_2 = 19.4$ N

Mark the correct option(s).



(1) The x-component of F_1 is $3\sqrt{3}$ N

(2) The x-component of F_2 is -2 N

(3) The magnitude of resultant force is $\sqrt{52}$ N

(4) None of the above

If unit vector $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. The possible values of a_1, a_2 and a_3 are

(1) $a_1 = \frac{2}{3}, a_2 = \frac{2}{3}$ and $a_3 = \frac{1}{3}$

(2) $a_1 = 0.3, a_2 = 0.4$ and $a_3 = \frac{\sqrt{3}}{2}$

(3) $a_1 = 1, a_2 = 1$ and $a_3 = 1$

(4) $a_1 = -\frac{2}{3}, a_2 = \frac{2}{3}$ and $a_3 = -\frac{1}{3}$

A force of $\sqrt{3}$ N makes equal angles with X-axis and Z-axis. The possible value of force are

(1) $(\hat{i} + \hat{j} + \hat{k})$N

(2) $(-\hat{i} + \hat{j} - \hat{k})$N

(3) $(-\hat{i} - \hat{j} - \hat{k})$N

(4) $(-\hat{i} + \hat{j} + \hat{k})$N

Find the component of a 50 N force which makes an angle of 45° with Z-axis and whose projection in XY-plane makes 45° with the X-axis

(1) $F_x = 25$ N

(2) $F_y = 25\sqrt{2}$ N

(3) $F_y = 25$ N

(4) $F_z = 25\sqrt{2}$ N

If $a = 3\hat{i} + 4\hat{j}$ and $b = 4\hat{i} - 3\hat{j}$, then

(1) the magnitude of resultant of a and b is $5\sqrt{2}$ units

(2) the angle between a and b is 90°

(3) the magnitude of resultant of a and b is 5 units

(4) the angle between a and b is zero

5. A ray of light is incident along vector $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} + \hat{k}$ on plane mirror placed in XY-plane normal on incidence point is along Z-axis

(1) The normal on incidence point is along Z-axis

(2) The angle of incidence is 30°

(3) The angle of reflection is 30°

(4) The angle of incidence is 45°

7. If C is perpendicular to $r = x\hat{i} + y\hat{j}$. The possible values of C are

(1) $x\hat{i} - y\hat{j}$

(2) $y\hat{i} - x\hat{j}$

(3) $+x\hat{j} - y\hat{i}$

(4) $\hat{k}$

18. Which of the following is perpendicular to $\hat{i} + \hat{j} - \hat{k}$?

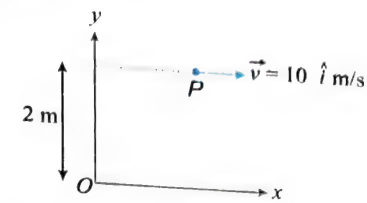
(1) $2\hat{i} - \hat{j} + \hat{k}$

(2) $\hat{i} + \hat{j} + 2\hat{k}$

(3) $-\hat{i} + 2\hat{j} + \hat{k}$

(4) $-\hat{i} - \hat{j} - \hat{k}$

19. Angular momentum of a particle P about point O is $\vec{L} = \vec{OP} \times m\vec{v}$,



where m is the mass of particle and v is its velocity. A particle of mass 2 kg is moving with speed 10 m/s as shown in the figure. Which of the following statements is/are correct.

(1) The magnitude of angular momentum of particle is 40 kg–m^{2}s^{-1}

(2) The angular momentum is directed along negative Z-axis

(3) The angular momentum is directed along positive Z-axis

(4) The angular momentum is time dependent

20. The magnetic force on a moving charge is $\vec{F} = q\vec{v} \times \vec{B}$. Here, q = electric charge on particle, v = velocity of the particle, B = magnetic field.

A charged particle of charge 1 C moving with a velocity $(\hat{i} + \hat{j} - \hat{k})$ms^{-1} in a magnetic field $B = (2\hat{i} - \hat{j} + 4\hat{k})$T. Choose the correct option(s).

(1) The component of magnetic force in the direction of velocity is zero

(2) The component of magnetic force in the direction of magnetic field is zero

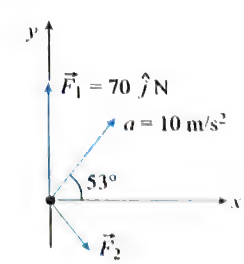
(3) The magnitude of magnetic force is $\sqrt{54}$ N

(4) None of the above

Linked Comprehension Type

For Problems 1–3

A particle of $m = 5$ kg is momentarily at rest at $x = 0$ at $t = 0$. It is acted upon by two forces $\vec{F}_1$ and $\vec{F}_2$. $\vec{F}_1 = 70\hat{j}$ N The direction and magnitude of $\vec{F}_2$ are unknown. The particle experiences a constant acceleration $\vec{a}$, in the direction as shown. Neglect gravity.



1. Find the missing force $\vec{F}_2$.

(1) $(20\hat{i} - 30\hat{j})$N

(2) $(25\hat{i} + 40\hat{j})$N

(3) $(30\hat{i} - 30\hat{j})$N

(4) $(30\hat{i} - 20\hat{j})$N

2. What is the velocity vector of the particle at $t = 10$ sec?

(1) $(30\hat i + 50\hat j)$ m/s (2) $(50\hat i + 75\hat j)$ m/s

(3) $(30\hat i - 45\hat j)$ m/s (4) $(60\hat i + 80\hat j)$ m/s

3. What third force, $\vec F_3$ is required to make the acceleration of the particle zero?

(1) $(20\hat i - 30\hat j)$N (2) $(-30\hat i - 40\hat j)$N

(3) $(30\hat i - 30\hat j)$N (4) $(30\hat i - 20\hat j)$N

For Problems 4 and 5

The motion of a particle is defined by the position vector

$$\vec r = A(\cos t + t\sin t)\hat i + A(\sin t - t\cos t)\hat j$$

where t is expressed in seconds. Determine the value of t for which:

4. The positions vectors and acceleration vectors are perpendicular

(1) at $t = 1$ sec (2) at $t = 0$

(3) at $t = \sqrt{2}$ sec (4) at $t = 1.5$ sec

5. The positions vectors and acceleration vectors are parallel

(1) at $t = 1$ sec (2) at $t = 0$

(3) at $t = \sqrt{2}$ sec (4) at $t = 1.5$ sec

For Problems 6 and 7

Three forces $\vec F_1$, $\vec F_2$ and $\vec F_3$ are acting on a particle of mass 10 kg. The particle moves in x-y plane such that its coordinates are given by $(5t^2\text{m}, 15t^2\text{ m})$. Find:

6. Acceleration vector of the particle

(1) $(5\hat i + 15\hat j)$ m/s^2 (2) $(10\hat i + 30\hat j)$ m/s^2

(3) $(10\hat i + 25\hat j)$ m/s^2 (4) $(5\hat i + 30\hat j)$ m/s^2

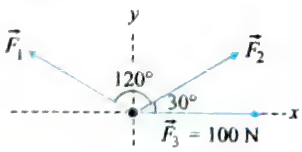
7. If the acceleration which is produced by these three forces. They are acting in the direction shown in the figure. The magnitude of forces $\vec F_1$ and $\vec F_2$ are

(1) 250 N, 450 N

(2) 250 N, 500 N

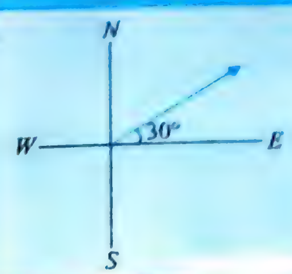
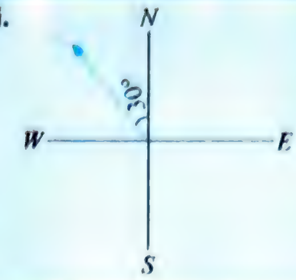
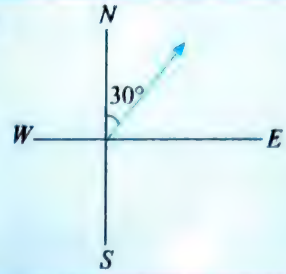
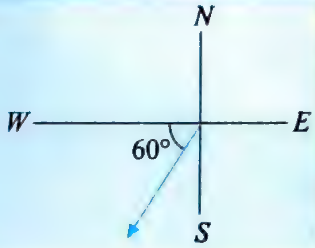
(3) 300 N, 500 N

(4) 300 N, 300 N

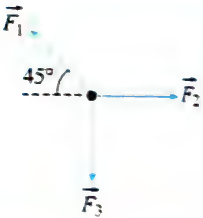


Matrix Match Type

1. In column I some straight lines graphs are given and in column II corresponding signs of slopes and intercepts are given. Match the type of graphs of column I corresponding to signs of slopes and intercepts in column II

Column I	Column II
i. 	a. 30° East of North
ii. 	b. 30° North of East
iii. 	c. 30° west of North
iv. 	d. 60° South of West

2. Three forces $\vec F_1$, $\vec F_2$ and $\vec F_3$ are represented as shown. Each of them is of equal magnitude.



Column I (Combination)	Column II (Approximate direction)
i. $\vec F_1 + \vec F_2 + \vec F_3$	a. 
ii. $\vec F_1 - \vec F_2 + \vec F_3$	b. 
iii. $\vec F_1 - \vec F_2 - \vec F_3$	c. 
iv. $\vec F_2 - \vec F_1 - \vec F_3$	d. 

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	a	b	c	d
(2)	d	c	b	a
(3)	b	c	a	d
(4)	c	d	a	b

3. For problem of column I,

$$a = 3\hat i + 4\hat j,\ b = 2\hat i + 2\hat j - \hat k \text{ and } c = 3\hat i - 4\hat k.$$

Match the column I with column II and select the correct option of code.

Column I		Column II	
i.	$\lvert a\rvert$	a.	5
ii.	$\lvert a+b\rvert$	b.	1/3
iii.	$\lvert a-b\rvert$	c.	7.87
iv.	$\lvert sb\rvert = 1$, then s is equal to	d.	$4\sqrt{2}$

Codes:

	i	ii	iii	iv
(1)	a	b	c	d
(2)	d	c	b	a
(3)	a, d	b, c	b	d
(4)	a	c	d	b

4. Match the information given in Column I with Column II. Select the correct option from the codes given below.

Column I		Column II	
i.	The component of vector $6\hat{i}-3\hat{j}-6\hat{k}$ parallel to the vector $\hat{i}+\hat{j}+\hat{k}$.	a.	0
ii.	The component of vector $6\hat{i}-3\hat{j}-6\hat{k}$ perpendicular to vector $\hat{i}+\hat{j}+\hat{k}$.	b.	$-\sqrt{3}$
iii.	The component of vector $6\hat{i}-3\hat{j}-6\hat{k}$ perpendicular to vector $2\hat{i}-\hat{j}-2\hat{k}$.	c.	$7\hat{i}-2\hat{j}-5\hat{k}$
iv.	The component of vector $6\hat{i}-3\hat{j}-6\hat{k}$ parallel to vector $2\hat{j}-\hat{k}$.	d.	$-\hat{i}-\hat{j}-\hat{k}$

Codes:

	i.	ii.	iii.	iv.
(1)	d	c	b	a
(2)	d	c	a	a
(3)	a	b	c	d
(4)	b, d	a	c	a

5. If $a=\hat{i}-2\hat{j}-3\hat{k}, b=2\hat{i}+\hat{j}-\hat{k}$ and $c=\hat{i}+3\hat{j}-2\hat{k}$. Match the Column I with Column II and select the correct option from the given code below.

Column I		Column II	
i.	$\lvert(a\times b)\times c\rvert$	a.	$35\sqrt{3}$
ii.	$\lvert a\times(b\times c)\rvert$	b.	20
iii.	$\lvert a\cdot(b\times c)\rvert$	c.	$3\sqrt{10}$
iv.	$\lvert(a\times b)(b\cdot c)\rvert$	d.	$5\sqrt{26}$

Codes:

	i.	ii.	iii.	iv.
(1)	a	b	c	d
(2)	b	a	d	c
(3)	d	c	b	a
(4)	a, b	b, c	c	a, d

6. Match the Column I with Column II and select the option from the given code below.

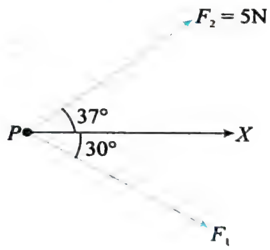
Column I		Column II	
i.	$2\hat{i}\times(\hat{i}+\hat{j})$	a.	$2\hat{k}$
ii.	$4\hat{i}\times(2\hat{i}-\hat{j})$	b.	0
iii.	$3\hat{i}\times(3\hat{i}-4\hat{j})$	c.	$-4\hat{k}$
iv.	$2(\hat{i}+\hat{j})\times 2\hat{i}$	d.	$-12\hat{k}$

Codes:

	i.	ii.	iii.	iv.
(1)	a	b	c	d
(2)	b	a	d	c
(3)	a	c	d	c
(4)	d	c	b	a

Numerical Value Type

1. According to Newton's second law of motion, resultant force on a particle is in the direction of acceleration of the particle.



Two forces F_1 and F_2 are acting on a particle as shown in the figure. The acceleration of the particle is along X-axis. Find the value of F_1 (in newton).

2. In vectors A and B be respectively equal to $3\hat{i}-4\hat{j}+5\hat{k}$ and $2\hat{i}+3\hat{j}-4\hat{k}$. The unit vector parallel to $A+B$ is $\frac{1}{\sqrt{27}}(5\hat{i}-a\hat{j}+\hat{k})$. Find the value of a.

3. How many unit vectors are there for which $\cos\alpha=\frac{1}{2}$ and $\cos\beta=\frac{1}{2}$, where α and β are angles made with X-axis and Y-axis, respectively.

4. If a vector $r=x\hat{i}+y\hat{j}+z\hat{k}$, makes angle $\frac{\pi}{3},\frac{\pi}{3}$ and $\frac{\pi}{n}$ with X-axis, Y-axis and Z-axis respectively. Find the value of n.

5. Two point charges q_1 and q_2 are placed at (0,0,0) and (1,2,2) m respectively. They repel each other with force of 3 N. The force on q_2 due to q_1 is $F_{21} = (x\hat{i} + y\hat{j} + z\hat{k})$N. Find the value of $x + y + z$.

6. Four forces are acting on a particle. One force of magnitude 3N is directed upward, another is directed 37° East of North having magnitude 5N, third is directed in South-West direction is of magnitude $4\sqrt{2}$ N and fourth force is $\sqrt{5n}$ N. If the particle is in equilibrium. Find the value on n.

7. If $a = 2\hat{i} - 3\hat{j} + \hat{k}$ and $b = x\hat{i} + \hat{j} + \hat{k}$ are mutually perpendicular, Find the value of x.

8. Find the volume of parallelepiped (in m³) whose edges are represented by
$a = (2\hat{i} - 3\hat{j} + 4\hat{k})$m, $b = (\hat{i} + 2\hat{j} - \hat{k})$m
and $c = (3\hat{i} - \hat{j} + 2\hat{k})$m.

9. If $a = 2\hat{i} - \hat{j} + \hat{k}, b = \hat{i} + 2\hat{j} - 3\hat{k}$ and $c = 3\hat{i} - y\hat{j} + 5\hat{k}$ are coplanar. Find the value of y.

10. A force of 1000 N in a particular direction must be applied to a block. For same reason. It is not possible to apply the force in that direction but two forces can be applied to 30° and 45° on either side of it in the same plane containing the given force. If the ratio of magnitude of forces as $\sqrt{n}$. Find the value of n.

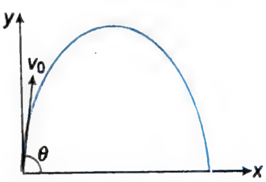
Archives

JEE Main

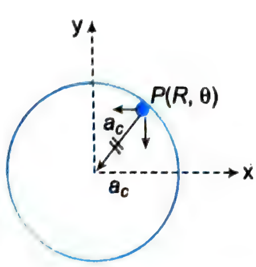
Single Correct Answer Type

1. A particle has an initial velocity $3\hat{i} + 4\hat{j}$ and an acceleration of $0.4\hat{i} + 0.3\hat{j}$. Its speed after 10 s **(AIEEE 2009)**
(1) 10 units (2) $7\sqrt{2}$ units
(3) 7 units (4) 8.5 units

2. A small particle of mass m is projected at an angle q with the x-axis with an initial velocity v_0 in the x-y plane as shown in the figure. At a time $t < \frac{v_0 \sin\theta}{g}$, the angular momentum of the particle is **(AIEEE 2010)**
(1) $-mgv_0 t^2 \cos\theta\,\hat{j}$
(2) $mgv_0 t \cos\theta\,\hat{k}$
(3) $-\frac{1}{2}mgv_0 t^2 \cos\theta\,\hat{k}$
(4) $\frac{1}{2}mgv_0 t^2 \cos\theta\,\hat{i}$



3. For a particle in uniform circular motion the acceleration $\vec{a}$ at a point $P(R, \theta)$ on the circle of radius R is (here θ is measured from the x-axis)
(1) $-\frac{v^2}{R}\cos\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$
(2) $-\frac{v^2}{R}\sin\theta\,\hat{i} + \frac{v^2}{R}\cos\theta\,\hat{j}$
(3) $-\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$
(4) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$ **(AIEEE 2010)**



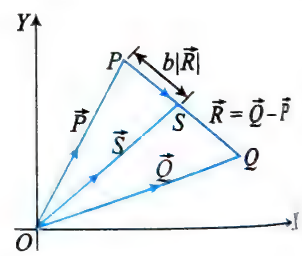
4. A particle is moving with velocity $\vec{v} = K(y\hat{i} + x\hat{j})$, where K is a constant. The general equation for its path is
(1) $y = x^2$ + constant (2) $y^2 = x$ + constant
(3) xy = constant (4) $y^2 = x^2$ + constant
(AIEEE 2010)

5. A projectile is given an initial velocity of $(\hat{i} + 2\hat{j})$ m/s where $\hat{i}$ is along the ground and $\hat{j}$ is along the vertical. If $g = 10$ m/s², the equation of its trajectory is
(1) $y = x - 5x^2$ (2) $y = 2x - 5x^2$
(3) $4y = 2x - 5x^2$ (4) $4y = 2x - 25x^2$
(JEE Main 2013)

JEE Advanced

Single Correct Answer Type

1. Three vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are shown in the figure. Let S be any point on the vector $\vec{R}$. The distance between the points P and S is $b\,|\vec{R}|$. The general relation among vectors $\vec{P}, \vec{Q}$ and $\vec{S}$ is:
(1) $\vec{S} = (1-b)\vec{P} + b^2\vec{Q}$
(2) $\vec{S} = (b-1)\vec{P} + b\vec{Q}$
(3) $\vec{S} = (1-b)\vec{P} + b\vec{Q}$
(4) $\vec{S} = (1-b^2)\vec{P} + b\vec{Q}$



(JEE Advanced 2017)

Numerical Value Type

1. Two vectors $\vec{A}$ and $\vec{B}$ are defined as $\vec{A} = a\hat{i}$ and $\vec{B} = a(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$ where a is a constant and $\omega = \frac{\pi}{6}$ rad s⁻¹. If $|\vec{A} + \vec{B}| = \sqrt{3}\,|\vec{A} - \vec{B}|$ at time $t = \tau$ for the first time, the value of τ, in seconds, is ______.
(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (3)	3. (4)	4. (4)	5. (4)
6. (2)	7. (2)	8. (2)	9. (2)	10. (1)
11. (3)	12. (4)	13. (1)	14. (3)	15. (1)
16. (4)	17. (1)	18. (2)	19. (3)	20. (3)
21. (3)	22. (2)	23. (2)	24. (1)	25. (1)
26. (2)	27. (2)	28. (2)	29. (1)	30. (2)
31. (3)	32. (2)	33. (4)	34. (1)	35. (2)
36. (2)	37. (3)	38. (2)	39. (1)	40. (3)
41. (2)	42. (4)	43. (1)	44. (3)	45. (2)
46. (3)	47. (3)	48. (4)	49. (3)	50. (4)
51. (1)	52. (3)	53. (1)	54. (1)	55. (4)
56. (2)	57. (1)	58. (1)	59. (3)	60. (1)

Multiple Correct Answers Type

1. (1),(3)	2. (1),(2)	3. (1),(2),(3)
4. (2),(4)	5. (1),(4)	6. (1),(2),(3),(4)
7. (1),(2),(3)	8. (1),(2),(3)	9. (1),(3),(4)
10. (1),(2)	11. (1),(2),(3)	12. (1),(2),(4)
13. (1),(3)	14. (1),(3),(4)	15. (1),(2)
16. (1),(4)	17. (2),(3),(4)	18. (1),(2),(3)
19. (1),(2)	20. (1),(2),(3)	

Linked Comprehension Type

1. (3)	2. (4)	3. (2)	4. (1)	5. (2)
6. (2)	7. (4)			

Matrix Match Type

1. i $\rightarrow$ b; ii $\rightarrow$ c; iii $\rightarrow$ a; iv $\rightarrow$ d
2. i $\rightarrow$ b; ii $\rightarrow$ c; iii $\rightarrow$ a; iv $\rightarrow$ d
3. i $\rightarrow$ a; ii $\rightarrow$ c; iii $\rightarrow$ d; iv $\rightarrow$ b
4. i $\rightarrow$ b,d; ii $\rightarrow$ c; iii $\rightarrow$ a; iv $\rightarrow$ a
5. i $\rightarrow$ d; ii $\rightarrow$ c; iii $\rightarrow$ b; iv $\rightarrow$ a
6. i $\rightarrow$ a; ii $\rightarrow$ c; iii $\rightarrow$ d; iv $\rightarrow$ c

Numerical Value Type

1. (6)	2. (1)	3. (2)	4. (4)	5. (5)
6. (2)	7. (1)	8. (7)	9. (4)	10. (2)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	2. (3)	3. (3)	4. (4)	5. (1)

JEE Advanced

Single Correct Answer Type

1. (3)

Numerical Value Type

1. (2)

4 Kinematics I

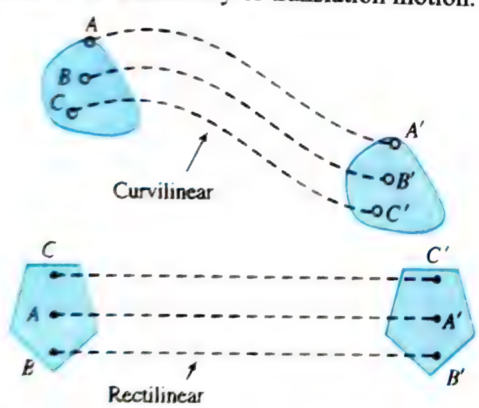
INTRODUCTION

Kinematics is the study of motion of an object ignoring the agents causing the motion. Usually, the nature of motion of bodies in general is highly complicated, being composed of translation, rotation, vibration, etc. To simplify the complex nature of motion, a body can be treated as a particle, if the dimensions of motion are too large compared to its size. Thus, a particle in the strictest sense means an object without dimensions (i.e., having no length, breadth, and thickness). Even though, no such thing actually exists in nature, still, the smaller the dimensions of an object, the more closer it approaches to the concept of a particle.

Usually, a "body" is modeled as a "particle," if its size is negligible and irrelevant to the description of motion.

TRANSLATORY MOTION

When a body moves in such a way that the linear distance covered by each particle of the body is same during the motion, then the motion is said to be translatory or translation motion.



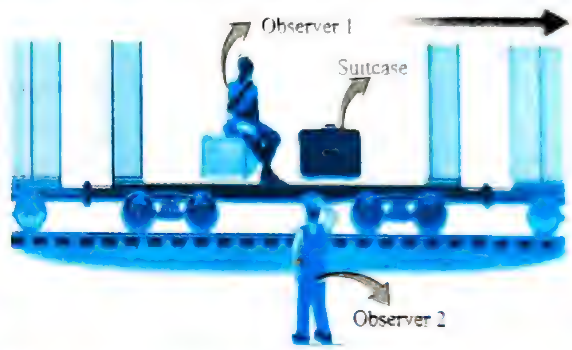
Translatory motion can be again of two types, viz., curvilinear or rectilinear, accordingly as the path of every constituent particle is similarly curved or straight line path. Here it is important that the body does not change its orientation.

FRAME OF REFERENCE

In particle kinematics, we assess the position, displacement, velocity and acceleration of a particle. For this observer is taken as reference point. Generally, the observer is considered to be at the origin of a coordinate system while the place where the observer (reference point) is placed (fixed) is known as "frame of reference".

Description of the state of a body requires a location of an observer, with respect to which the state must be specified, otherwise it would have no sense. In the figure below, two observers are observing a suitcase which is placed on a moving trolley. Observer (1) is observing the suitcase from the same moving trolley but observer (2) is observing it from ground.

Observer (1) who is observing from the frame of reference attached on trolley says "the suitcase is at rest" while Observer (2) who is observing from the ground frame of reference says "the suitcase is moving". Both observers are looking at same object but they have different observations about the status of the suitcase and no one will be wrong. Because they are observing the object from different frames of reference.



REST AND MOTION

We say a body is said to be at rest when it does not change its position with time, while it is said to be in motion if it changes its position continuously with time. In the previous figure, the position of the suitcase kept on the floor of the trolley does not change with time. Thus, the suitcase is at rest with respect to the trolley frame of reference. But the position of the suitcase is changing with time with respect to the observer from ground frame of reference. So rest and motion are relative terms and they depend upon the frame of references.

TRAJECTORY

The path followed by a point object during its motion is called its "trajectory."

When we write with a pen (or pencil) on a paper, the tip of the pen leaves a clear mark on the paper. The mark appearing as a line (curve or straight) represents the path followed by the tip of the pen. This is what we call the trajectory of the moving point (tip of the pen).

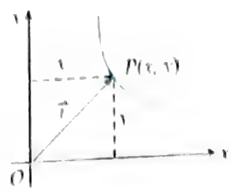
Sometimes when we look at the sky we find a long thin line of smoke. If we curiously try to trace the object causing the smoke track, we can see a point-like object, that is, a jet plane at the end of the track of smoke. It means the points of space through which the jet plane passes are traced by the smoke of condensed water vapor. This is called "trajectory" of the jet.

Even though neither the jet plane nor the tip of the pen is ideally a point, following the assumption as discussed in the beginning of

this chapter, we can practically treat them as points.

Now, we can define trajectory as the path traced by a moving point.

TRAJECTORY EQUATION



Let us consider a particle moving in x-y plane. If x coordinates of all points of the path traced by the particle are released with the corresponding y-coordinates by a definite relation (mathematically known as function) such as $y = x^2$, $y = \sin x$, etc. We call it "equation of trajectory." In general, the function $y = f(x)$ represents the path of a particle moving in the x-y plane.

In other words, we can state that the equation that relates the position vector of displacement (but not distance) of a particle along x and y axes by a simple relation, that is, "function," is known as "trajectory equation".

When the displacement of a particle along y-axis is expressed as the function of displacement of the particle along x-axis, we call is "locus equation" or "equation of trajectory."

FINDING TRAJECTORY EQUATION

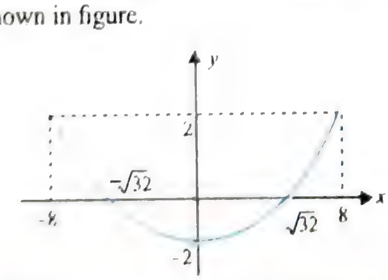
Let us find the equation of trajectory of a particle whose position vector is given as $\vec{r} = x\hat{i} + y\hat{j}$, where x and y are the function of time, $x = g(t)$ and $y = h(t)$. First of all, choose a simpler equation, say $x = g(t)$ and solve it for time t. Then substitute the obtained value of t (as the function of x) in the other equation $y = h(t)$ to express $y = f(x)$ as the locus equation of the particle.

A particle moves in the x-y plane according to the scheme $x = -8 \sin \pi t$ and $y = -2 \cos^2 2\pi t$, where t is time. Find the equation of the path of the particle. Show the path on a graph.

$y = -2 \cos^2 2\pi t = -2(1 - 2 \sin^2 \pi t)$

$$= -2\left(1 - 2\left(\frac{-x}{8}\right)^2\right) = -2 + \frac{x^2}{16} \qquad (\because \sin \pi t = -x/8)$$

This is the equation of path, which is a parabola. The path of the particle is shown in figure.



A particle moves in x-y plane such that its position vector varies with time as $\vec{r} = (2\sin 3t)\hat{i} + 2(1 - \cos 3t)\hat{j}$. Find the equation of the trajectory of the particle.

Comparing $\vec{r} = (2\sin 3t)\hat{i} + 2(1 - \cos 3t)\hat{j}$

with $\vec{r} = x\hat{i} + y\hat{j}$, we have $x = 2 \sin 3t$ and $y = 2(1 - \cos 3t)$.

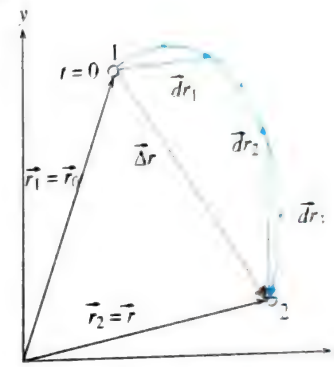
This gives $\sin 3t = \frac{x}{2}$ and $\cos 3t = 1 - \frac{y}{2}$

Eliminating t by squaring and adding the above terms, we have

$$\frac{x^2}{4} + \left(1 - \frac{y}{2}\right)^2 = 1$$

DISPLACEMENT AND DISTANCE

When we speak of distance, we should not mistake it for displacement. These two are entirely different quantities. As explained earlier, displacement is a vector quantity. Its magnitude is equal to the "shortest distance" between the initial position 1 and final position 2 of the particle (as the particle moves from position 1 to position 2 following an arbitrary curved path) as shown in figure.



If we split the curve path 1 to 2 traced by a particle into elementary segments, each elementary segment is represented by an elementary displacement vector $d\vec{r}$ as a tiny arrow as shown in figure. If we add all elementary displacements vectorially between any two points 1 and 2, we will obtain the total (net) displacement $\vec{s}$ between these points.

$$\vec{s} = \Delta\vec{r} = \lim_{\Delta\vec{r}_i \to 0} \sum_{i=1}^{i=n} \Delta\vec{r}_i = \int d\vec{r}$$

On the other hand, the distance D (length of the path followed covered by the particle between the points 1 and 2 can be measured by adding all elementary distance dD. Since $dD = |d\vec{r}|$, the total distance D is given as

$$D = \lim_{\Delta\vec{r}_i \to 0} \sum_{i=1}^{i=n} |\Delta\vec{r}_i| = \int |d\vec{r}|$$

Note: Be careful while using the equation $\Delta\vec{r} = \int d\vec{r}$ and $D = \int |d\vec{r}|$. The former is the integration (summation) of $d\vec{r}$, and the latter is the integration of the magnitude of $d\vec{r}$. $\int |d\vec{r}|$ and $\left|\int d\vec{r}\right|$ are different from each other.

$\left|\int d\vec{r}\right|$ = length of line joining 1 to 2 = $|\Delta\vec{r}|$.

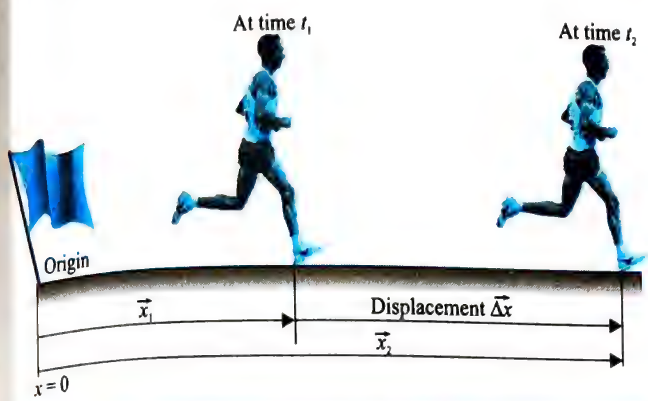
$\int |d\vec{r}|$ = length of curve path traced by the particle.

The simplest motion we investigate is that of an object moving in a straight line. The orientation of line may be vertical, horizontal, or slanted, but must be straight.

Expressing Displacement in Case of Motion in One Dimension

In the figure shown, the initial position of a man is indicated by the vector labeled $\vec{x}_1$. The magnitude $|\vec{x}_1|$ is the distance of the man from an arbitrarily chosen origin. After some time the man has moved to a new position, which is indicated by the vector $\vec{x}_2$. The displacement of the man, $\Delta\vec{x}$ is a vector drawn from the initial position to the final position. We write the displacement as

$$\Delta\vec{x} = \vec{x}_2 - \vec{x}_1 \quad \text{...(i)}$$



Important Points:

Here we use the notation $\Delta\vec{x}$ for the displacement vector to indicate that it is a difference between two position vectors. It should be noted that the displacement vector is independent of the location of the origin of the coordinate system. Why? Any shift of the coordinate system will add to the position vector x_2 the same amount that it adds to the position vector x_1; hence the difference between the position vectors, or Δx, will not change.

In motion in one dimension, we will discuss about the motion along a straight line. In such a case, the displacement in one direction along the line is assigned a positive value, while the displacement in the opposite direction is assigned a negative value.

For instance, let us consider that a man is moving along an east/west direction and that a positive (+) sign is used to denote a direction due east. If the mam moves from $x_1 = -100$ m to $x_2 = 400$ m, then the displacement is $\Delta x = (400 \text{ m}) - (-100 \text{ m}) = +500$ m. The positive result indicates that the motion is in the positive direction. If, instead, the man moves from $x_1 = 200$ m to $x_2 = -300$ m, then $\Delta x = (-300 \text{ m}) - (200 \text{ m}) = -500$m. The negative result represents that the motion is in the negative direction.

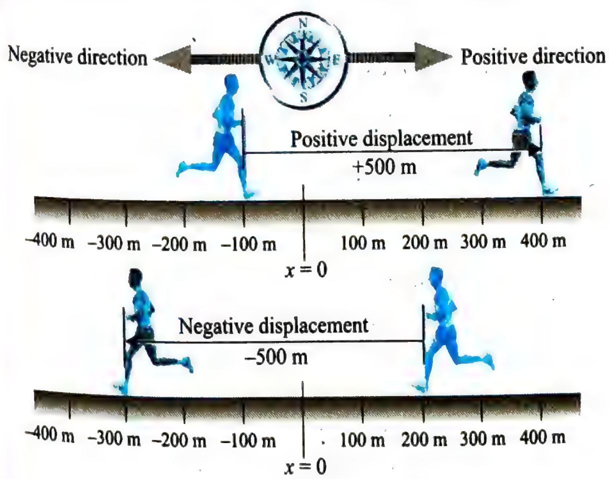
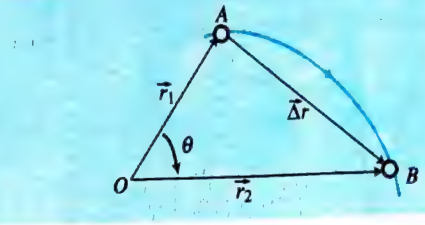


Illustration 4.3

A particle moves from position A to position B in a path as shown in figure. If the position vectors $\vec{r}_1$ and $\vec{r}_2$ making an angle θ between them are given, find the magnitude of displacement.

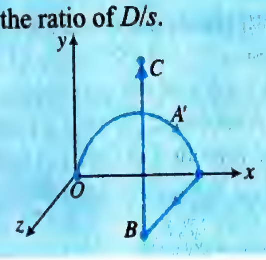


Sol. From triangle OAB: $\vec{r}_1 + \Delta\vec{r} = \vec{r}_2 \Rightarrow \vec{r}_2 - \vec{r}_1 = \Delta\vec{r}$

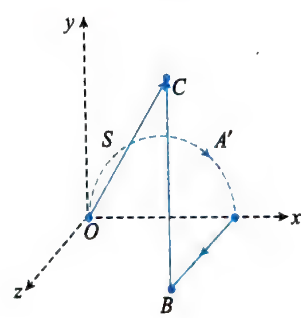
Hence, $|\vec{s}| = |\Delta\vec{r}| = |\vec{r}_2 - \vec{r}_1| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$

Illustration 4.4

A particle moves in a semicircular path of radius R from O to A. Then it moves parallel to z-axis covering a distance R upto B. Finally it moves along BC parallel to y-axis through a distance $2R$. Find the ratio of D/s.



Sol. The distance D, that is, length of the actual path covered by the particle ($OA'ABC$) as shown in figure.



D = length of the semicircle $OA'A$ + length AB + length BC. This gives $D = \pi R + R + 2R = (\pi + 3)R$.

Since $OA = 2R$, $AB = R$, and $BC = 2R$, the coordinates of C can be given as $C \equiv (2R, R, 2R)$. Then the position of C is expressed as:

$$\vec{r}_C = 2R\hat{i} + 2R\hat{j} + R\hat{k}$$

As $\vec{s}(=\vec{OC}) = \vec{r}_C - \vec{r}_0$, substituting $\vec{r}_C$ and $\vec{r}_0 = 0\hat{i} + 0\hat{j} + 0\hat{k}$, we obtain $\vec{s} = (2\hat{i} + 2\hat{j} + \hat{k})R$.

Its magnitude $|\vec{s}| = (\sqrt{2^2 + 2^2 + 1^2})R = 3R$

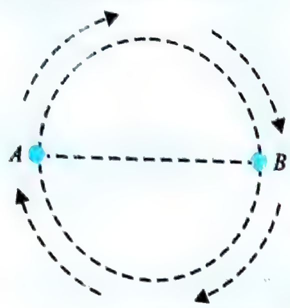
Hence, $\dfrac{D}{s} = \dfrac{(\pi+3)R}{3R} = \dfrac{\pi+3}{3}$

ILLUSTRATION 4.5

A particle is moving in a circle of radius R (figure below).

(a) What is its displacement when it covers (i) half the circle, (ii) full circle?

(b) What is its distance when it covers (i) half the circle and (ii) full circle?



Sol.

(a) Displacement is the vector drawn from the initial position to the final position, and its magnitude is equal to the shortest distance between the initial and final positions.

(i) Hence, displacement is equal to the diameter of the circle = $2R$.

(ii) As initial and final positions are same, i.e., displacement will be zero.

(b) Distance is the length of the path travelled by the particle. Hence, distance travelled in case (i) will be equal to πR. Hence, distance travelled in case (ii) will be equal to $2\pi R$.

Important Points:

- Distance is a scalar quantity and displacement is a vector quantity.
- Distance between a given set of initial and final positions can have infinite values but displacement is unique.
- Displacement can be negative, zero, or positive, but distance is never negative. When a body returns to its initial position of starting, its displacement is zero but distance or path length is not zero.
- The magnitude of displacement can never be greater than distance.
- In uniform motion, displacement is equal to distance.
- Displacement is zero if particle returns to initial position.
- Distance does not decrease with time and never be a zero for a moving body.
- Displacement can decrease with time, can be zero or even be negative if the body is returning to its initial position, got initial position, and moves back to the initial position. So the magnitude of displacement is not equal to distance; however, it can be so if the motion is along a straight line without change in direction. In general, distance ≥ displacement.

VELOCITY AND SPEED

To define the fastness of motion, we need to find how fast the space coordinates (position vector) of a particle relative to a fixed point changes with time. We call it velocity. Apart from this, we need to know the rate at which the particle changes the length of the path which gives us speed of the particle.

When a particle moves, its positive vector $\vec{r}$ must change (in magnitude or direction of both) whereas the distance (length of the path) covered by the particle always increases with time. We need to find the rate at which the position vector $\vec{r}$ and distance D change with time. The time rate of change of $\vec{r}$ and D are called velocity and speed, respectively.

INSTANTANEOUS SPEED

The magnitude of the velocity at any instant of time is known as *instantaneous speed* or simply *speed* at that instant of time. It is denoted by v.

Quantitatively: $\text{Speed} = \dfrac{\text{Distance}}{\text{Time}}$

Mathematically, it is the time rate at which distance is being travelled by the particle.

- Speed is a scalar quantity. It can never be negative.
- Instantaneous speed is the speed of a particle at a particular instant of time.

Uniform speed: A particle is said to be moving with uniform speed if it covers equal distances in equal intervals of time, however small these intervals may be.

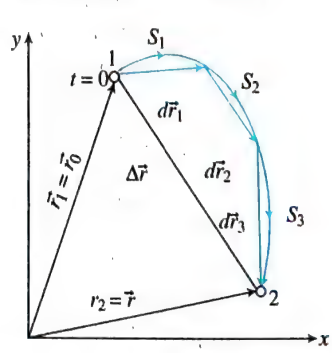
Uniform speed is the speed which always remains constant. Here the body may change its direction of motion.

Uniform velocity: A particle is said to be moving with uniform velocity if it covers equal displacements in equal intervals of time, however small these intervals may be. Uniform velocity is that velocity in which the body continuously moves in the same direction with constant speed. A particle moving with uniform velocity is said to be under uniform motion.

- Uniform motion is a straight line motion with constant velocity.
- In uniform motion, displacement and distance are equal.
- The average and instantaneous velocities have same values in uniform motion.
- No net force is required for an object to be in uniform motion.
- The velocity in uniform motion does not depend upon the time interval.
- The velocity in uniform motion is independent of choice of origin.

AVERAGE VELOCITY AND AVERAGE SPEED

Average velocity: If at any time t_1 position vector of the particle is $\vec{r}_1$ and at time t_2 position is $\vec{r}_2$.



Hence, on an average, the position vector changes with time at a rate $\Delta\vec{r}/\Delta t$.

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

To find the average velocity, we need to know only the total displacement from initial to final position and need not consider the nature of motion between initial and final positions.

Physical meaning of average velocity: It is that uniform velocity with which if the object is made to move, it will cover the same displacement in a given time as it does with its actual velocity in the same time.

Average speed: If the particle is going a distance D during the time $\Delta t(= t_2 - t_1)$, the average rate of distance covered by it is $D/\Delta t$ for this time interval. We can call this ratio, "average speed" denoted by v_{av}. It is the ratio of total distance d travelled by the particle to the total time taken t in which this distance is travelled.

$$v_{av} = \frac{\text{Total distance}}{\text{Total time taken}} = \frac{s_1 + s_2 + s_3}{\Delta t}$$

Note: If motion takes place in same direction, then average speed and average velocity are same.

In figure below, a particle goes from A to C. Distances, velocities and time taken are shown.

$$S_1 = v_1 t_1 \Rightarrow t_1 = \frac{S_1}{v_1}; S_2 = v_2 t_2 \Rightarrow t_2 = \frac{S_2}{v_2}$$

$$v_{av} = \frac{S_1 + S_2}{t_1 + t_2} = \frac{v_1 t_1 + v_2 t_2}{t_1 + t_2} = \frac{S_1 + S_2}{\frac{S_1}{v_1} + \frac{S_2}{v_2}}$$

Note:

- If $t_1 = t_2 = t$, then $v_{av} = \frac{v_1 + v_2}{2}$; average speed is equal to the arithmetical mean of individual speeds.
- If $S_1 = S_2 = S$, then $v_{av} = \frac{2v_1 v_2}{v_1 + v_2}$; average speed is equal to the harmonic mean of individual speeds.

ILLUSTRATION 4.6

A train travels from city A to city B with a constant speed of 10 m s^{-1} and returns back to city A with a constant speed of 20 m s^{-1}. Find its average speed during its entire journey.

Sol. Let the distance between the two cities A and B be x m.

Time taken by the train to travel from A to B $= \frac{x}{10} = t_1$ (say)

Time taken to come back from B to A $= \frac{x}{20} = t_2$ (say)

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{x + x}{t_1 + t_2} = \frac{2x}{\frac{x}{10} + \frac{x}{20}} = \frac{40}{3}\ \text{m s}^{-1}$$

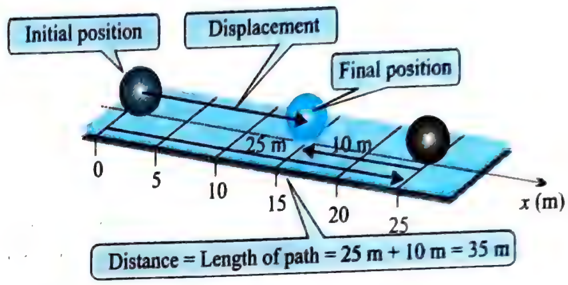
ILLUSTRATION 4.7

A particle starts from $x = 0$ along straight path, moves 25 m straight, then stops. Then turns around and moves 10 m back. Finally stops 15 m from the start point. Total time of motion is 10 sec. What is his average speed and average velocity?

Sol.

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time taken}} = \frac{d}{\Delta t} = \frac{(25+10)}{10} = \frac{35}{10} = 3.5\,\text{m/s}$$

Average speed of the particle is 3.5 m/s



According to definition of average velocity.

$$\vec{v}_{av} = \frac{\Delta \vec{x}}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{15 - 0}{10 - 0} = 1.5\,\text{m/s}$$

Average velocity of the particle is 1.5 m/s towards $+x$ direction.

ILLUSTRATION 4.8

A car moving along a straight line moves with a constant velocity v_1 for some time and with constant velocity v_2 for the next equal time. What is the average velocity of the car?

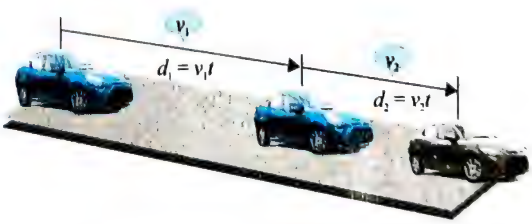
Sol.

Let equal time interval is t.

Displacement of the particle while moving with velocity v_1: $d_1 = v_1 t$

Displacement of the particle while moving with velocity v_2: $d_2 = v_2 t$

Total displacement of the particle, $D = d_1 + d_2 = v_1 t + v_2 t$ and the total time taken is $T = t + t = 2t$.

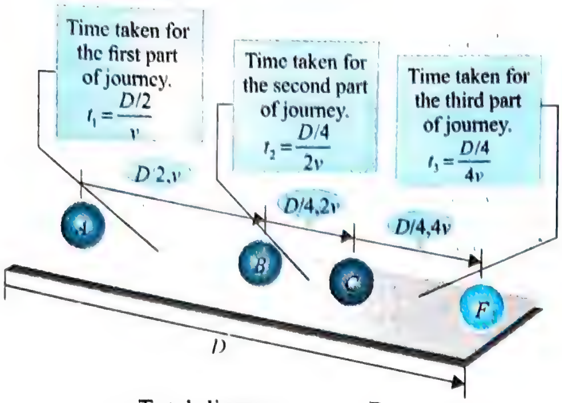


$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time taken}} = \frac{v_1 t + v_2 t}{2t} = \frac{v_1 + v_2}{2}$$

ILLUSTRATION 4.9

A particle covers half its journey with a constant speed of v, half the remaining part of journey with a constant speed of $2v$ and the rest of the journey with a constant speed of $4v$. Find its average speed during the entire journey.

Sol. Let D be the total distance covered by the particle between its initial (A) point and the final (F) point.



$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{D}{(t_1+t_2+t_3)}$$

$$= \frac{D}{\left(\frac{D}{2v}+\frac{D}{8v}+\frac{D}{16v}\right)} = \frac{v}{\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{16}\right)}$$

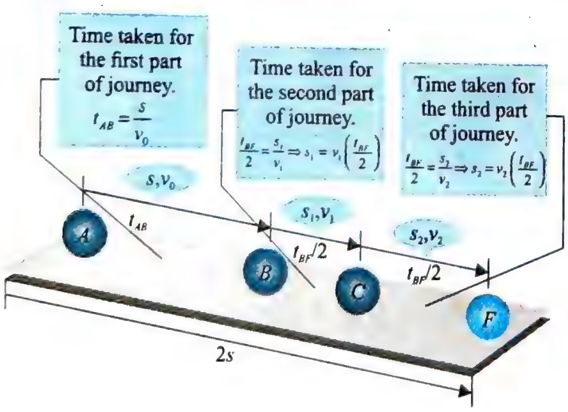
$$= \frac{v}{\left(\frac{8+2+1}{16}\right)} = \frac{v}{\left(\frac{11}{16}\right)} = \frac{16}{11}v$$

ILLUSTRATION 4.10

A point traversed half the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the average speed of the point average over the whole time of motion.

Sol.

Let $2s$ be the total distance covered by the particle between its initial (A) point and the final (F) point.



Now consider second part of journey $s = s_1 + s_2$

$$\Rightarrow \quad s = v_1 \cdot \left(\frac{t_{BF}}{2}\right) + v_2 \cdot \left(\frac{t_{BF}}{2}\right) = (v_1+v_2) \cdot \left(\frac{t_{BF}}{2}\right)$$

$$\Rightarrow \quad t_{BF} = \frac{2s}{(v_1+v_2)}$$

$$\text{Hence average velocity} = \frac{2s}{t_{AB}+t_{BF}} = \frac{2s}{\frac{s}{v_0}+\frac{2s}{(v_1+v_2)}}$$

$$\Rightarrow \quad \langle v_{AF} \rangle = \frac{2v_0(v_1+v_2)}{(v_1+v_2+2v_0)}$$

INSTANTANEOUS VELOCITY

When you move to some position and come back to same initial position, you undergo a zero displacement. Hence, your average velocity is zero. It does not mean that you were not moving. The average velocity cannot always define the motion of a particle. Then we require to define the instantaneous velocity. We need to know the instantaneous value of the velocity and speed of a particle to understand its motion.

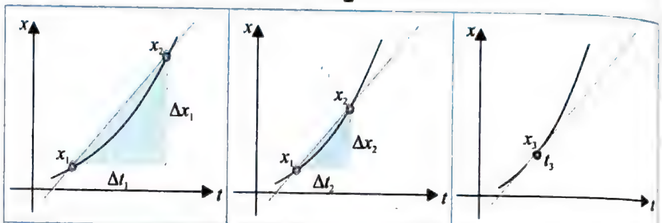
The word "instant" literally means a short time interval. How short? Shorter is the time interval, more accurate the meaning of instant will be. For this, we can write Δt tends to zero ($\Delta t \to 0$). In other words, "instant" may mean an infinitesimal time "dt" which replaced the term "$\Delta t \to 0$."

The time rate of change of position (x) or displacement (s) at any instant of time (t) is known as *instantaneous velocity* or simply *velocity* at that instant of time. It is denoted by v.

$$\text{Quantitatively: Velocity} = \frac{\text{Displacement}}{\text{Time}}$$

$$\text{Mathematically: } \vec{v} = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{x}}{\Delta t}\right) = \frac{d\vec{x}}{dt}$$

$$\text{or} \quad \vec{v} = \frac{d\vec{x}}{dt} \quad \left[\text{Also } \vec{v} = \left(\frac{d\vec{s}}{dt}\right)\right]$$



The average velocity is given by the displacement, Δx_1, divided by the time interval, Δt_1 or $\bar{v}_1 = \frac{\Delta x_1}{\Delta t_1}$	The average velocity is given by the displacement, Δx_2, divided by the time interval, Δt_2 or $\bar{v}_2 = \frac{\Delta x_2}{\Delta t_2}$.	The instantaneous velocity, $v(t_3) = \left.\frac{dx}{dt}\right\|_{t=t_3}$, is represented by the slope of the tangent to the curve at $t = t_3$.

The instantaneous velocity is defined as the limit of the average velocity as the time interval approaches 0:

$$v = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

We can now define the instantaneous velocity (or simply velocity) as $\vec{v} = \frac{d\vec{x}}{dt}$.

The notation $\lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t}$ means that the ratio $\Delta\vec{x}/\Delta t$ is defined by a limiting process in which smaller and smaller values of Δt are used, so small that they approach zero. As the values of Δt used are small, $\Delta \vec{x}$ also becomes small. However, the ratio $\Delta\vec{x}/\Delta t$ does not become zero but, rather, approaches the value of the instantaneous velocity. For brevity, we will use the word velocity to mean "instantaneous velocity" and speed to mean "instantaneous speed."

The instantaneous velocity v is tangent to the path at each point. Here v_1 and v_2 are the instantaneous velocities at points P_1 and P_2 shown in figure below.

In general, if the particle moves in space, then $\vec{r}$ will change and the time rate of change of position vector is known as velocity.

Thus,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\Rightarrow \quad \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

where $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$

and $v_z = \frac{dz}{dt}$

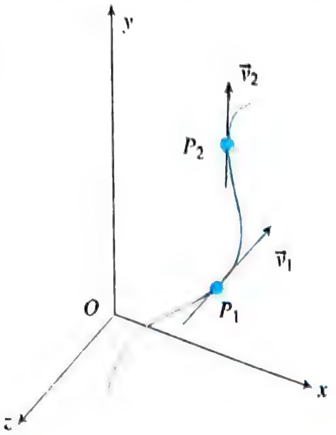


ILLUSTRATION 4.11

A particle moves along the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$, with constant speed v. Express its velocity vectorially as a function of (x, y).

Sol. Given $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Differentiating both sides w.r.t. time

$$\Rightarrow \quad \frac{1}{9}\frac{dx^2}{dt} + \frac{1}{4}\frac{dy^2}{dt} = \frac{d(1)}{dt}$$

$$\Rightarrow \quad \frac{1}{9}2x.\frac{dx}{dt} + \frac{1}{4}.2y.\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{2x}{9}\left[\frac{dx}{dt}\right] + \frac{2y}{4}\left[\frac{dy}{dt}\right] = 0$$

$$\Rightarrow \quad v_x = -\frac{9}{4}\frac{y}{x}v_y \qquad \left[\frac{dx}{dt} = v_x; \frac{dy}{dt} = v_y\right] \quad \text{...(i)}$$

As particle moves with constant velocity, then

$$v_x^2 + v_y^2 = v^2 \Rightarrow \left(-\frac{9}{4}\frac{y}{x}v_y\right)^2 + v_y^2 = v^2$$

$$\Rightarrow \quad v_y^2 = \frac{16x^2v^2}{16x^2 + 81y^2} \Rightarrow v_y = \frac{\pm 4xv}{\sqrt{16x^2 + 81y^2}} \quad \text{...(ii)}$$

From (i) and (ii), we get $v_x = \frac{\mp 9yv}{\sqrt{16x^2 + 81y^2}}$

So, velocity is given by $\vec{v} = v_x\hat{i} + v_y\hat{j} = \frac{(\mp 9y\hat{i} \pm 4x\hat{j})v}{\sqrt{16x^2 + 81y^2}}$

ILLUSTRATION 4.12

A particle moves so that its position vector varies with time as $\vec{r} = A\cos\omega t\,\hat{i} + A\sin\omega t\,\hat{j}$. Find the

(a) initial velocity of the particle,

(b) angle between the position vector and velocity of the particle at any time, and

(c) speed at any instant.

Sol. **(a)** Position at time t

$$\vec{r} = A\cos\omega t\,\hat{i} + A\sin\omega t\,\hat{j}$$

Instantaneous velocity, $\vec{v} = \frac{d\vec{r}}{dt}$

We have $\vec{v} = A\frac{d}{dt}(\cos\omega t)\hat{i} + A\frac{d}{dt}(\sin\omega t)\hat{j}$

$$= -A\omega\sin\omega t\,\hat{i} + A\omega\cos\omega t\,\hat{j}$$

At $t = 0$, $v = -A\omega\sin 0\,\hat{i} + A\omega\cos 0\,\hat{j} = A\omega\,\hat{j}$

(b) For calculating the angle between two vectors, we use the concept of dot product of the vectors. The angle θ between $\vec{r}$ and $\vec{v}$ can be given as

$$\theta = \cos^{-1}\frac{\vec{r}.\vec{v}}{|\vec{r}||\vec{v}|}$$

where

$$\vec{r}.\vec{v} = (A\cos\omega t\,\hat{i} + A\sin\omega t\,\hat{j}).(-A\omega\sin\omega t\,\hat{i} + A\omega\cos\omega t\,\hat{j})$$

$$= \omega A^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$$

Hence, $\theta = \cos^{-1} 0 = \pi/2$

That means $\vec{v} \perp \vec{r}$.

(c) Speed at any time is the magnitude of instantaneous velocity, i.e.,

$$v = |\vec{v}| = \sqrt{(-A\omega\sin\omega t)^2 + (A\omega\cos\omega t)^2} = A\omega$$

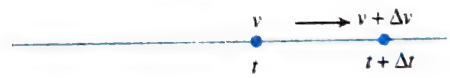
ACCELERATION

After knowing the concept of velocity, let us enquire how the velocity changes in order to analyze the nature of motion of a particle. If we want to find how rapidly the velocity is changing, we have to adopt two terms, i.e., the *rate of change in velocity over a finite and infinitesimal time interval.*

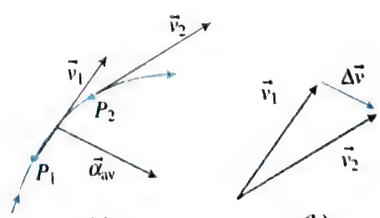
Average Acceleration

It is the ratio of the total change in velocity to the total time taken in which this change in velocity takes place.

$$\vec{a}_{av} = \frac{\text{Total change in velocity}}{\text{Total time taken}} = \frac{\Delta\vec{v}}{\Delta t}$$



To find average acceleration, we need to know only the total change in velocity from the initial position to final position. We need not consider how the motion takes place between these two points.



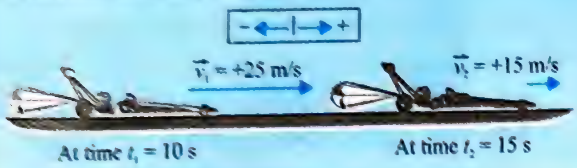
(a) (b)

If the velocity of a particle at instant t is $\vec{v}_1$ and at instant t_2 is $\vec{v}_2$, then the average acceleration is mathematically given by:

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

(Remember change in velocity = Final velocity – Initial velocity)

A drag racer crosses the finish line, and the driver deploys a parachute and applies the brakes to slow down, as Figure illustrates. The driver begins slowing down when $t_1 = 10$s and the car's velocity is $\vec{v}_1 = +25$ m/s. When $t_2 = 15$s, the velocity has been reduced to $\vec{v}_2 = +15$ m/s. What is the average acceleration of the dragster?



The average acceleration of an object is always specified as its change in velocity, $\vec{v}_2 - \vec{v}_1$, divided by the elapsed time, $(t_2 - t_1)$. The average acceleration is,

$$\bar{\vec{a}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{15\text{ m/s} - 25\text{ m/s}}{15\text{ s} - 10\text{ s}} = \boxed{-2\text{ m/s}^2}$$

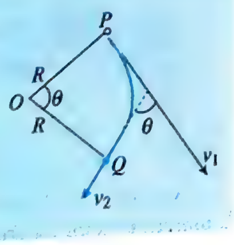
Here the acceleration calculated is negative, indicating that the acceleration points to the left in the drawing. As a result, the acceleration and the velocity point in opposite directions. *Whenever the acceleration and velocity vectors have opposite directions, the object slows down and is said to be "decelerating."*

Important Points:

- Note that we often refer to the deceleration of an object as a decrease in the speed the object over time 'the object slows down', which corresponds to acceleration in the opposite direction of the motion of the object.
- If the signs of the velocity and acceleration of a particle are the same, the speed of the particle increases. If the signs are opposite, the speed decreases, the object speeds up.

ILLUSTRATION 4.14

A particle describes an angle θ in a circular path with a constant speed v. Find the (a) change in the velocity of the particle and (b) average acceleration of the particle during the motion in the curve (circle).



(a) As the particle moves from P to Q, the velocity turns through an angle θ. Then,

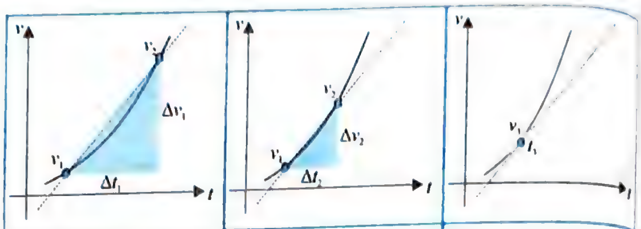
$$|\Delta\vec{v}| = \sqrt{v_1^2 + v_2^2 - 2v_1v_2\cos\theta}$$
$$= \sqrt{v^2 + v^2 - 2vv\cos\theta} = 2v\sin\frac{\theta}{2}$$

(b) The time of motion is $\Delta t = R\theta/v$. Then, average acceleration is $|\vec{a}_{av}| = \left|\frac{\vec{v}}{\Delta t}\right|$.

Substituting $|\Delta\vec{v}|$ and Δt, we have $|\vec{a}_{av}| = \dfrac{2v\sin\frac{\theta}{2}}{\frac{R\theta}{v}} = \dfrac{2v^2}{R\theta}\sin\dfrac{\theta}{2}$

INSTANTANEOUS ACCELERATION

If a particle has same velocity at any two points A and B, say, in its path, the change in its velocity is zero. Then the average acceleration of the particle over the time interval Δt is zero. Does it ($\vec{a}_{av}$) really confirm that the particle is not changing its velocity (accelerating) between A and B? In fact, the average acceleration deals with the net (total) change in velocity and the corresponding time interval. Hence, $\vec{a}_{av}$ cannot explain the change (variation) of velocity of the particle during any instant (short time) in the given finite time interval. For this, we need to establish the idea of instantaneous acceleration.



The average acceleration is given by the velocity change, Δv_1, divided by the time interval Δt_1, or $\bar{a}_1 = \dfrac{\Delta v_1}{\Delta t_1}$.	average acceleration in this case, $\bar{a}_2 = \dfrac{\Delta v_2}{\Delta t_2}$.	The instantaneous acceleration, $a(t_3) = \left.\dfrac{dv}{dt}\right\vert_{t=t_3}$, is represented by the slope of the tangent to the curve at $t = t_3$.

The instantaneous acceleration is defined as the limit of the average acceleration as the time interval approaches 0: $a = \lim\limits_{\Delta t\to 0}\bar{a} = \lim\limits_{\Delta t\to 0}\dfrac{\Delta v}{\Delta t} \equiv \dfrac{dv}{dt}$.

We can now define the instantaneous acceleration (or simply acceleration) as $\vec{a} = \dfrac{d\vec{v}}{dt}$

In words, we can say the acceleration of a particle at any instant is the rate at which its velocity is changing at that instant. Graphically, the acceleration at any point is the slope of the curve of $v(t)$ at that point.

In general: $\vec{v} = v_x\hat{i} + v_y\hat{i} + v_z\hat{k}$

Differentiating $\vec{v}$ w.r.t. time, we get acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right]$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

where $a_x = \dfrac{dv_x}{dt}$, $a_y = \dfrac{dv_y}{dt}$, and $a_z = \dfrac{dv_z}{dt}$.

Magnitude of acceleration: $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Note:

- $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$
- Acceleration is also defined as the second time derivative of position.
- Also $a = \dfrac{dv}{dt} = \dfrac{dv}{dt}\left(\dfrac{dx}{dx}\right) = \dfrac{dx}{dt}\left(\dfrac{dv}{dx}\right) = v\left(\dfrac{dv}{dx}\right)$
- Acceleration is a vector quantity. It can be positive, zero, or negative.

ILLUSTRATION 4.15

A particle starts moving rectilinearly at time $t = 0$ such that its velocity v changes with time t according to the equation $v = t^2 - t$, where t is in seconds and v in m s^{-1}. Find the time interval for which the particle retards.

Sol. Acceleration of the particle, $\vec{a} = \frac{d\vec{v}}{dt} = 2t - 1$

The particle retards when acceleration is opposite to velocity. Hence, acceleration vector and velocity vector should be opposite to each other or the dot product of $\vec{a}$ and $\vec{v}$ should be negative.

$\vec{a} \cdot \vec{v} < 0$

$\Rightarrow \quad (2t - 1)(t^2 - t) < 0$

$\Rightarrow \quad t(2t - 1)(t - 1) < 0$

t is always positive.

$\therefore \quad (2t - 1)(t - 1) < 0$

$\Rightarrow$ Either $2t - 1 < 0$ or $t - 1 > 0$

$\Rightarrow \quad t < \frac{1}{2}$ s and $t > 1$s. This is not possible.

or $\quad 2t - 1 > 0$ and $t - 1 < 0 \Rightarrow t > \frac{1}{2}$ s and $t < 1$s. Hence, the required time interval is $\frac{1}{2}$ s $< t < 1$s.

ILLUSTRATION 4.16

A particle's position on the x axis is given by $x = 4 - 27t + t^3$, with x in meters and t in seconds.

(a) Find the particle's velocity function $v(t)$ and acceleration function $a(t)$.

(b) Is there ever a time when $v = 0$?

(c) Describe the particle's motion for $t \geq 0$.

Sol.

(a) Because position x depends on time t, the particle must be moving

- To get the velocity function $v(t)$, we differentiate the position function $x(t)$ with respect to time.
- To get the acceleration function $a(t)$, we differentiate the velocity function $v(t)$ with respect to time.

Differentiating the position function, we find:

$$v = -27 + 3t^2 \qquad \text{... (i)}$$

with v in meters per second.

Differentiating the velocity function then gives us

$$a = +6t \qquad \text{...(ii)}$$

with a in meters per second squared.

(b) Setting $v(t) = 0$ yields, $0 = -27 + 3t^2$ which has the solution.

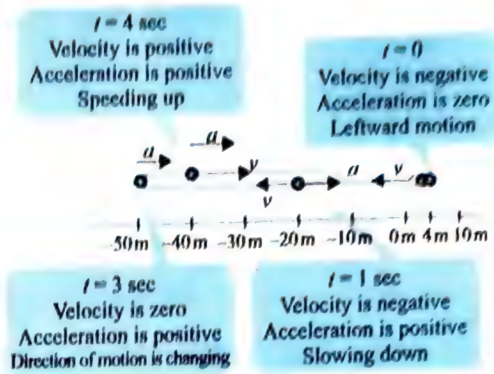
$$t = \pm 3\text{s}$$

Thus, the velocity is zero both 3 s before and 3 s after the clock reads 0.

(c) For describing the particle's motion for $t \geq 0$, we need to examine the expressions for $x(t)$, $v(t)$, and $a(t)$.

At $t = 0$, the particle is at $x(0) = +4$m and is moving with a velocity of $v(0) = -27$ m/s. That is, in the negative direction of the x axis. Its acceleration is $a(0) = 0$ because just then the particle's velocity is not changing.

For $0 < t < 3$ s, the particle still has a negative velocity, so it continues to move in the negative direction. However, its acceleration is no longer zero but is increasing and positive. Because the signs of the velocity and the acceleration are opposite, the particle must be slowing.



Indeed, we already know that it stops momentarily at $t = 3$ s. Just then the particle is as far to the left of the origin in figure as it will ever get. Substituting $t = 3$ s into the expression for $x(t)$, we find that the particle's position just then is $x = -50$ m. Its acceleration is still positive.

For $t > 3$ s, the particle moves to the right on the axis. Its acceleration remains positive and grows progressively larger in magnitude. The velocity is now positive, and it too grows progressively larger in magnitude.

ILLUSTRATION 4.17

The position of a particle moving along x-axis is related to time t as follow: $x = 2t^2 - t^3$, where x is in meters and t is in seconds.

(a) What is the maximum positive displacement of the particle along the x axis and at what instant does it attain it?

(b) Describe the motion of the particle.

(c) What is the distance covered in the first three seconds?

(d) What is its displacement in the first four seconds?

(e) What is the particle's average speed and average velocity in the first 3 seconds?

(f) What is the particles instantaneous acceleration at the instant of its maximum positive x displacement?

(g) What is the average acceleration between the interval $t = 2$ s to $t = 4$ s?

Sol. The derivative of x w.r.t. time will give velocity of the particle in relation with time.

$$v = \frac{dx}{dt} = 4t - 3t^2 \qquad \text{...(i)}$$

The velocity of the particle will be zero

$4t - 3t^2 = 0 \Rightarrow t(4 - 3t) = 0$

$\Rightarrow \quad t = 0$ and $t = 4/3$ sec

The double derivative of x w.r.t. time will give acceleration of the particle in relation with time

$$a = \frac{d^2x}{dt^2} = 4 - 6t \qquad \text{...(ii)}$$

(a) For minimum and maximum displacement, $dx/dt = 0$.

Thus, $\frac{dx}{dt} = 4t - 3t^2 = 0$

$$t(4-3t)=0 \Rightarrow \text{ either } t=0 \text{ or } t=\frac{4}{3}\text{ s}$$

Also, for maxima, double derivative of x should be negative, i.e., $\frac{d^2x}{dt^2}<0$.

$$\frac{d^2x}{dt^2}=4-6t$$

At $t=0$: $\frac{d^2x}{dt^2}=4-6(0)=4$ (positive)

And at $t=\frac{4}{3}$ s: $\frac{d^2x}{dt^2}=4-6\left(\frac{4}{3}\right)=4-8=-4$ (negative)

Therefore, the particle is at maximum displacement at $t=\frac{4}{3}$ s and the corresponding displacement is

$$x_{max}=2\left(\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^3=\frac{32}{27}\text{ m}$$

The maximum particle displacement of the particle is $x_{max}=\frac{32}{27}$m and it will occur at $t=\frac{4}{3}$s.

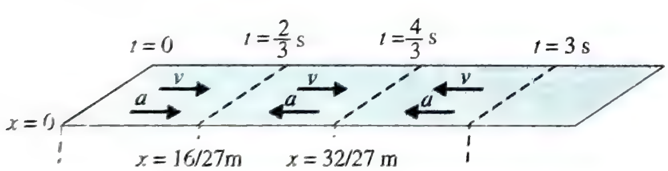
(b) Velocity is positive when $v>0$ or $4t-3t^2>0$ or $t(4-3t)>0$

i.e., $t>0$ and $t<\frac{4}{3}$s

For acceleration to be positive, $a>0 \Rightarrow 4-6t>0$ or $t<\frac{2}{3}$s

From $t=0$ to $t=\frac{2}{3}$s, the velocity $(\vec{v})$ and $(\vec{a})$ both are in the same direction. Hence, velocity of the particle increases continuously during the time interval $t=0$ to $t=\frac{2}{3}$s.

From $t=\frac{2}{3}$s onward, the acceleration $(\vec{a})$ acts in negative x-direction. Hence, the velocity of the particle decreases and becomes zero at $t=\frac{4}{3}$s. Then after, the particle moves in negative x-direction.



(c) The velocity is positive between the instants 0 and 4/3 s and is negative at instant 4/3 s to the remaining 3 s.

Total distance

$$=\left|\text{Displacement from 0 to }\frac{4}{3}\text{s}\right|+\left|\text{Displacement from }\frac{4}{3}\text{s to 3s}\right|$$

i.e., $\left|x\left(\frac{4}{3}\right)-x(0)\right|+\left|x(3)-x\left(\frac{4}{3}\right)\right|$

$$=\left|2\left(\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^3-(0)\right|+\left|\{2(3)^2-(3)^3\}-\left\{2\left(\frac{4}{3}\right)^2-\left(\frac{4}{3}\right)^3\right\}\right|$$

$$=\frac{16}{9}\left(2-\frac{4}{3}\right)+\left|9(2-3)-\frac{16}{9}\left(2-\frac{4}{3}\right)\right|=\frac{307}{27}\text{ m}$$

(d) The required displacement is difference in position at $t=4$ s and $t=0$.

$$\Delta x=x_{t=4}-x_{t=0}=[2(4)^2-4^3]-[0]=-32\text{ m}$$

(e) For calculating average speed in first 3 s, we need to calculate the distance travelled by the particle in 3 s.

$$\text{Average speed}=\frac{\text{Total distance}}{\text{Total time}}=\frac{307}{27\times 3}=\frac{307}{81}\text{ m s}^{-1}$$

For calculating average velocity in first 3 s, we need to calculate displacement of the particle in 3 s.

$$\text{Average velocity}=\frac{\text{Total displacement}}{\text{Total time}}$$

$$\Rightarrow \frac{x(3)-x(0)}{3-0}=\frac{[2(3)^2-(3^3)]-0}{3}=-3\text{ m s}^{-1}$$

(f) Instantaneous acceleration, $a=\frac{dv}{dt}=4-6t$

The particle is at its maximum displacement at $t=4/3$ s.

$\therefore$ Acceleration at $t=\frac{4}{3}\text{s}=4-6\left(\frac{4}{3}\right)=-4\text{ m s}^{-2}$

(g) Average acceleration

$$=\frac{\text{Total change in velocity}}{\text{Total time}}=\frac{v(4)-v(2)}{4-2}$$

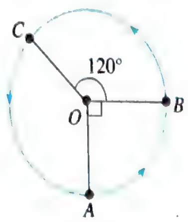
$$=\frac{\left[4(4)-3(4)^2\right]-\left[4(2)-3(2)^2\right]}{2}=-14\text{ m s}^{-2}$$

CONCEPT APPLICATION EXERCISE 4.1

1. Two cars are moving in the same direction in parallel lanes along a highway. At some instant, the velocity of car A exceeds the velocity of car B. Does that mean that the acceleration of car A is greater than that of car B? Explain.
2. Say Yes or No.
   (a) Can an object moving towards north have acceleration towards south?
   (b) Can an object reverse the direction of its motion even though it has constant acceleration?
   (c) Can an object reverse the direction of its acceleration even though it continues to move in the same direction?
   (d) Average speed is the magnitude of average velocity.
   (e) At any instant of time, the directions of change in velocity and acceleration are different.
3. Can a body have
   (a) Zero instantaneous velocity and yet be accelerating?
   (b) Zero average speed but non-zero average velocity?
   (c) Negative acceleration and yet be speeding up?
   (d) Magnitude of average velocity be equal to average speed?
4. A body moves at a speed of 100 ms^{-1} for 10 s and then moves at a speed of 200 ms^{-1} for 20 s along the same direction. The average speed is ________.

5. A body moves in the southern direction for 10 s at the speed of 10 ms^{-1}. It then starts moving in the eastern direction at the speed of 20 ms^{-1} for 10 s. The magnitude of the average velocity is________. The average speed is________. The total displacement will be ________.

6. Figure shows a particle starting from point A, travelling up to B with a speed s, then up to point C with a speed $2s$, and finally upto A with a speed of $3s$. Determine its average speed.



7. A particle moving in a straight line covers half the distance with a speed of 3 ms^{-1}. The other half of the distance is covered in two equal time intervals with speeds of 4.5 ms^{-1} and 7.5 ms^{-1}, respectively. Find the average speed of the particle during this motion.

8. A car runs at a constant speed on a circular track of radius 200 m, taking 62.8 s on each lap. Find the average velocity and average speed on each lap.

9. An athlete swims the length of 50 m pool in 20 s and makes the return trip to the starting position in 22 s. Determine his average velocity in
   (a) The first half of the swim
   (b) The second half of the swim
   (c) The round trip

10. You move along $+x$-direction through a distance of 10 m and then move back through a distance of 4 m. If you repeat it four times during ten minutes, find the (a) total (i) distance, (ii) displacement and (b) average (i) speed, (ii) velocity.

11. A student starts from his house with a speed of 12 kmh^{-1} and reaches the school 3 min late. Next day he increases speed by 1 kmh^{-1} and reaches the school 3 min earlier. Find the distance between his house and office.

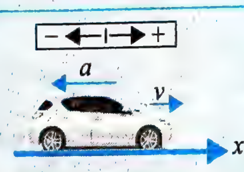
ANSWERS

1. No
2. (a) Yes (b) Yes (c) Yes (d) No (e) No
3. (a) Yes (b) No (c) Yes (d) Yes
4. $\frac{500}{3} ms^{-1}$
5. (1) $5\sqrt{5}\ ms^{-1}$ (2) 15 m s^{-1} (3) $100\sqrt{5}$ m
6. 1.8 s **7.** 4 m s^{-1}
8. Average velocity zero, Average speed = 20 m s^{-1}
9. (a) 2.5 m s^{-1} (b) 2.27 m s^{-1}
(c) v_{av} is zero for round trip
10. (a) (i) 56 m (ii) +24 m
(b) (i) 5.6 m/minute (ii) +2.4 m/minute
11. 0.6 km

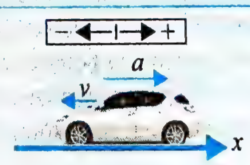
MOTION IN A STRAIGHT LINE

While moving in a straight line, we have two possible directions for acceleration, velocity, and displacement of the particle. Hence, for the sake of simplicity, we need not mention them with their usual vector symbols. Rather we prefer to write them like scalars a, v, s and r and use plus (+) and minus (–) to treat them as vectors keeping their senses (directions) in our minds. Hence, we call the given quantities (s, v, and a) as "algebraic scalars." Conventionally, when $\vec{a}, \vec{v}$, and $\vec{s}$, are directed along positive s-axis we write them as $+a$, $+v$, and $+s$, respectively. If they are directed along negative s-axis, we can write them as negative quantities such as $-a$, $-v$, and $-s$, respectively, as shown in figure.

The velocity vector and acceleration vector points to the right which is the positive direction. Hence all are positive. The car is speeding up as the acceleration vector points in the direction of velocity vector.



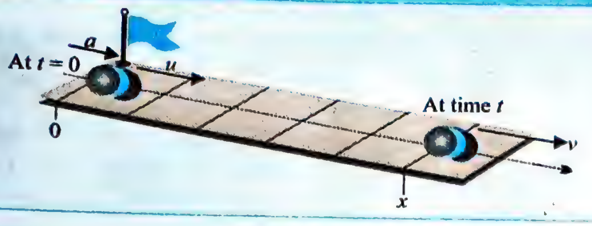
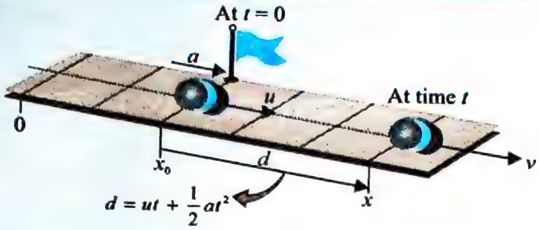
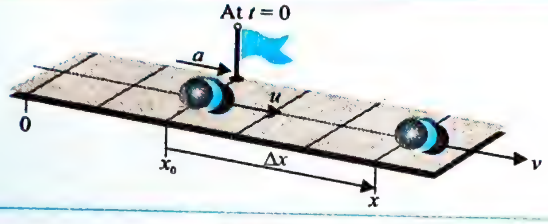
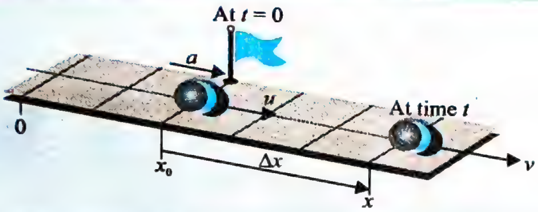
The velocity vector points to the right. Hence, positive and acceleration vector is pointing to left. Hence, negative. The car is "decelerating" as the acceleration vector points opposite to the velocity vector.



The velocity vector is pointing to left. Hence, negative and acceleration vector points to the right which is the positive direction. Hence, positive. The car is "decelerating" as the acceleration vector points opposite to the velocity vector.

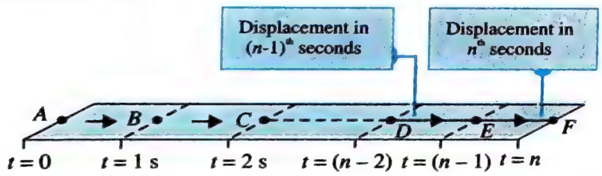
FORMULAE FOR UNIFORMLY ACCELERATED MOTION IN A STRAIGHT LINE

- Acceleration is said to be constant when both the magnitude and direction of acceleration remain constant.
- There will be one-dimensional motion if initial velocity and acceleration are parallel or anti-parallel to each other.

Situation	Kinematic equation
	Particle starts moving along x-axis with constant acceleration a Initial velocity $= u$ The velocity of the particle at any time t: $v = u + a.t$
	The particle starts moving from initial position $x = x_0$ along x-axis The position of the object at any time t: $x = x_0 + u.t + \frac{1}{2}at^2$
	The velocity of the particle in relation with displacement $v^2 = u^2 + 2a.\Delta x$
	The displacement of the particle $\Delta x = \left(\frac{u+v}{2}\right).t = \left[\frac{v_{\text{initial}} + v_{\text{final}}}{2}\right] \times \Delta t$ Average velocity $= \left[\frac{v_{\text{initial}} + v_{\text{final}}}{2}\right]$

Displacement of a Particle in *n*th Second of its Motion in Uniformly Accelerated Motion

Let a particle starts from a point A at $t = 0$ and travels along the straight line $ABC...DEF$. At $t = 1$ s, the particle arrives at point B; at $t = 2$ s, the particle arrives at point C; and in the last, the particle arrives at point F at $t = n$ s, as shown in figure. Our aim is to calculate the displacement in nth second (say D_n) and that is equal to EF here.



Let a = uniform acceleration of the particle, u = initial velocity of object at point A. Now,

$$EF = AF - AE \quad \text{...(i)}$$

AF is displacement travelled in $t = n$ seconds, so

$$AF = un + \frac{1}{2}an^2$$

AE is displacement travelled in $t = (n - 1)$ seconds, so

$$AE = u(n-1) + \frac{1}{2}a(n-1)^2$$

Put the values of AF and AE in (i), we get

$$EF = \left[un + \frac{1}{2}an^2\right] - \left[u(n-1) + \frac{1}{2}a(n-1)^2\right]$$

After simplifying, we get

$$EF = u + \frac{a}{2}(2n-1) \Rightarrow D_n = u + \frac{a}{2}(2n-1)$$

Important Points:

- The displacement in the nth second will be same as the distance if the particle moves along straight line without turning back.
- If the particle turns its direction of motion during nth second, the distance and displacement will not be equal. In that situation, we cannot use the formula of nth second.

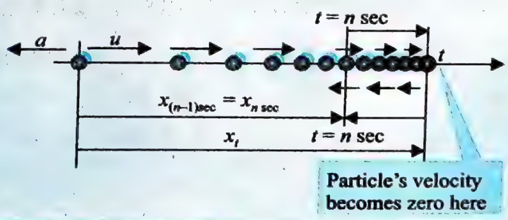
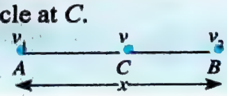


ILLUSTRATION 4.18

A particle moving with uniform acceleration from A to B along a straight line has velocities v_1 and v_2 at A and B, respectively. If C is the mid-point between A and B, then determine the velocity of the particle at C.



Sol. Let v be the velocity of the particle at C. Assume the acceleration of the particle to be a and distance between A and B to be x. To find the velocity at point C, consider the motion from A to C:

Applying $v^2 - u^2 = 2as$, we get $v^2 - v_1^2 = 2a\dfrac{x}{2}$

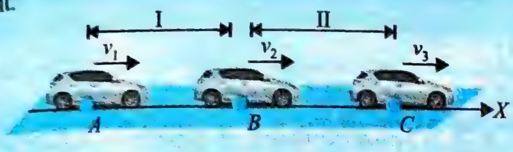
$\Rightarrow \quad v^2 - v_1^2 = ax$ ...(i)

Apply the same equation from C to B, we get

$$v_2^2 - v^2 = 2a\frac{x}{2} \Rightarrow v_2^2 - v^2 = ax \quad \text{...(ii)}$$

From (i) and (ii), we get $v = \sqrt{\dfrac{v_1^2 + v_2^2}{2}}$

Important Points:
When the motion of an object is divided into segments, the final velocity of one segment is the initial velocity for the next segment.



Here the final velocity v_2 of segment I is the initial velocity for segment II

ILLUSTRATION 4.19

A man loses 20% of his velocity after running through 108 m. Prove that he cannot run more than 192 m. further, if his retardation is uniform.

Sol. Let the initial velocity be u. Since the man loss 20% of his velocity after running through 108 m,

Hence, velocity at this moment = 80% of $u = 0.8u$

Now, initial velocity = u;

Final velocity = $0.8u$ and $s = 108$ m; $a = ?$

Using $v^2 = u^2 + 2a.s \Rightarrow (0.8u)^2 = u^2 + 2a.(108)$

$$216a = 0.64u^2 - u^2 = -0.36u^2 \Rightarrow a = -\frac{0.36}{216}u^2 = -\frac{1}{600}u^2$$

Now, let the man run a further distance of x m. In second segment of his motion: Initial velocity = $0.8u$, final velocity = 0; $s = x$;

Again using $v^2 = u^2 + 2a.s \Rightarrow 0 = (0.8u)^2 + 2a.x$

$$0 = 0.64u^2 + 2\left(-\frac{1}{600}u^2\right)x \Rightarrow \left(\frac{1}{300}u^2\right)x = 0.64u^2$$

$\Rightarrow x = 0.64 \times 300 = 192\,\text{m}$

ILLUSTRATION 4.20

Two trains P and Q are moving along parallel tracks with same uniform speed of 20 m s^{-1}. The driver of train P decides to overtake train Q and accelerates his train by 1 m s^{-2}. After 50 s, train P crosses the engine of train Q. Find out what was the distance between the two trains initially, provided the length of each train is 400 m.

Sol. Let the initial distance between two trains be x.

Distance travelled by train P in 50 s:

$$S_P = ut + \frac{1}{2}at^2 = 20 \times 50 + \frac{1}{2} \times 1 \times 50^2 = 2250 \text{ m}$$

Distance travelled by train Q in 50 s:

$$S_Q = ut = 20 \times 50 = 1000 \text{ m}$$

Now $S_P - S_Q = 2250 - 1000 = 1250$ m. This distance must be equal to the initial distance between the trains plus the sum of the lengths of the two trains.

$$x + 800 = 1250 \quad \Rightarrow \quad x = 450 \text{ m}$$

ILLUSTRATION 4.21

Consider a particle initially moving with a velocity of 5 m s^{-1} starts decelerating at a constant rate of 2 m s^{-2}.

(a) Determine the time at which the particle becomes stationary.

(b) Find the distance travelled in the second second.

(c) Find the distance travelled in the third second.

Sol.

(a) Here $u = 5$ m s^{-1}, $a = -2$ m s^{-2}, $v = 0$, $t = ?$

Using $v = u + at$, where

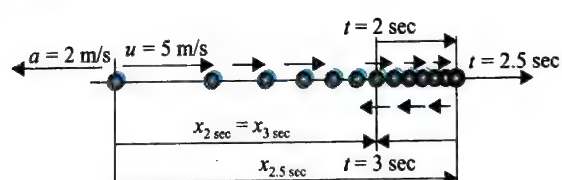
$$0 = 5 - 2t \Rightarrow t = 2.5 \text{ s}$$

(b) Here $u = 5$ m s^{-1}, $a = -2$ m s^{-2}, $n = 2$.

Using $x_n = u + \dfrac{a}{2}(2n - 1) = 5 - \dfrac{2}{2}[2(2) - 1)] = 2\,\text{m}$

(c) Here, if we use the above formula, we will get $x_n = 0$, but in reality it is not zero. This formula is not applicable for the third second because velocity becomes zero in the third second, i.e., at $t = 2.5$ s.

The particle has a turning point at $t = 2.5$ s. We have to indirectly calculate the distance travelled in this particular second. That is, we have to determine the distance travelled between 2 s $\le t \le$ 2.5 s and 2.5 s $\le t \le$ 3 s, and then add the two.



Displacement of the particle at $t = 2.5$ s is

$$x_{2.5} = \frac{u^2}{2a} = \frac{(5)^2}{2(2)} = 6.25 \text{ m}$$

Due to symmetry, the displacement of the particle at $t = 2$ s and $t = 3$ s are same, i.e.,

$$x_3 = x_2 = 5(2) + \frac{1}{2}(-2)(2)^2 = 6 \text{ m}$$

Thus, the distance travelled in the third second is

$$x = (x_{2.5} - x_2) + (x_{2.5} - x_3)$$
$$= (6.25 - 6) + (6.25 - 6) = 0.5 \text{ m}$$

ILLUSTRATION 4.22

In a car race, car A takes a time of t s, less than car B at the finish and passes the finishing point with a velocity v more than car B. Assuming that the cars start from rest and travel with constant accelerations a_1 and a_2, respectively, show that $v = \sqrt{a_1 a_2}\,t$.

Sol. In the car race, the cars start from zero velocity and accelerate with constant accelerations. Here we'll discuss an important concept of uniformly accelerated motion. If a body starts from rest, i.e., with zero initial velocity and accelerates with an acceleration a after travelling distance s, its velocity can be given by speed equation $v^2 = u^2 + 2as$.

As we have $u = 0 \Rightarrow v = \sqrt{2as}$

For the time taken to travel this distance s, we use equation as $s = ut + \frac{1}{2}at^2$

Here again we have $u = 0$, $s = \frac{1}{2}at^2$ or $t = \sqrt{\frac{2s}{a}}$

In the questions given, car A starts with acceleration a_1 and car B with acceleration a_2. If car B reaches the finishing point at time T and with speed u, car A will reach at time $T - t$ and with speed $u + v$ as given in the question.

As both the cars start from rest and cover same distance, say s, we have

For car A: $v + u = \sqrt{2a_1 s}$ and $T - t = \sqrt{\frac{2s}{a_1}}$

For car B: $u = \sqrt{2a_2 s}$ and $T = \sqrt{\frac{2s}{a_2}}$

From above equations, eliminating the terms of u and T, we get

$$v = \sqrt{2a_1 s} - \sqrt{2a_2 s}$$

$$t = \sqrt{\frac{2s}{a_2}} - \sqrt{\frac{2s}{a_1}}$$

Dividing the above equations, we get $v = \sqrt{a_1 a_2}\,t$

Important Points:

- If a body starts from rest and moves with uniform acceleration, then distance covered by the body in t seconds is proportional to t^2 (i.e. $s \propto t^2$).
  So we can say that the ratio of distance covered in 1 s, 2 s, and 3 s is or 1 : 4 : 9.
- If a body starts from rest and moves with uniform acceleration, then distance covered by the body in nth second is proportional to $(2n - 1)$ [i.e., $s_n \propto (2n - 1)$].
  So we can say that the ratio of distance covered in I second, II second, and III second is 1 : 3 : 5.
- A body moving with a velocity u is stopped by the application of brakes after covering a distance s. If the same body moves with velocity nu and same braking force is applied on it, then it will come to rest after covering a distance of $n^2 s$.
  As $v^2 = u^2 - 2as$
  $\Rightarrow 0 = u^2 - 2as \Rightarrow s = u^2/2a \Rightarrow s \propto u^2$ [since a is constant]
  So we can say that if u becomes n times, then s becomes n^2 times that of previous value.
- A particle moving with uniform acceleration from A to B along a straight line has velocities v_1 and v_2 at A and B, respectively. If C is the mid-point between A and B, then velocity of the particle at C is equal to $v = \sqrt{\frac{v_1^2 + v_2^2}{2}}$.

Concept Application Exercise 4.2

1. A particle starts moving in a straight line, with constant acceleration a and initial velocity u. Find the average velocity of a particle moving in first t seconds.
2. Two cars start off to race with velocities 4 m/s and 2 m/s and travel in straight line with uniform accelerations 1 m/s² respectively. If they reach the final point at the same instant, then find the length of the path.
3. A car is moving towards check post with velocity 54 kmh^{-1}. When car is at 400 m from check post, driver applied brake which is caused of deceleration of 0.3ms^{-2}. Find the distance of car from check post for 2 min after applying the brakes.
4. A car starts from rest and accelerates uniformly to a speed of 50 ms^{-1} over a distance of 250 m.
   (a) Find the acceleration and time taken by the car to cover this distance.
   (b) The speed of car is increased to 60 ms^{-1} within 10 s, if the acceleration is further increased. Find this acceleration and the distance travelled by car in 10 s.
   (c) To stop the car in 6 s, now the brakes are applied. Find the distance travelled by the car during retardation.
5. A 200 m long train starts from rest at $t = 0$ with constant acceleration 4 cms^{-2}. The head light of its engine is switched on at $t = 60$ s and its tail light is switched on at $t = 120$ s. Find the distance between these two events for an observer standing on platform.
6. A car travelling at 108 kmh^{-1} has its speed reduced to 36 kmh^{-1} after travelling a distance of 200 m. Find the retardation (assumed uniform) and time taken for this process.
7. A car starts form rest and accelerates uniformly for 10 s to a velocity of 8 ms^{-1}. It then runs at a constant velocity and is finally brought to rest in 64 m with a constant retardation. The total distance covered by the car is 584 m. Find the value of acceleration, retardation, and total time taken.
8. A body covers 10 m in the seconds second and 25 m in fifth second of its motion. If the motion is uniformly accelerated, how far will it go in the seventh second?
9. A body moving with uniform acceleration in a straight line describes 25 m in the fifth second and 33 m in the seventh second. Find its initial velocity and acceleration.
10. Two trains, each of length 100 m, moving in opposite directions along parallel lines, meet each other with speeds of 50 kmh^{-1} and 40 kmh^{-1}. If their accelerations are 30 cms^{-2} and 20 cms^{-2}, respectively, find the time they will take to pass each other.
11. A train accelerates from rest for time t_1 at a constant rate α and then it retards at the constant rate β for time t_2 and comes to rest. Find the ratio t_1/t_2.

Answers

1. $u + \frac{1}{2}at$ **2.** 24 m **3.** 25 m

4. (a) 10 s, 5 ms^{-2} **(b)** 1 ms^{-2}, 550 m **(c)** 180 m

5. 16 m **6.** 2 ms^{-2}, 10 s **7.** 0.8 ms^{-2}, –0.5 ms^{-2}, 86 s

8. 35 m **9.** $u = 7\,\text{ms}^{-1}, a = 4\,\text{ms}^{-2}$

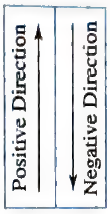
10. $10\sqrt{33} - 50$ s **11.** $\frac{\beta}{\alpha}$

ONE-DIMENSIONAL MOTION IN A VERTICAL LINE (MOTION UNDER GRAVITY)

When the motion takes place under the effect of *gravitational attractive force* only, the motion is known as free fall. Here free fall does not mean that the particle is falling down only. Even if the particle is rising up or may be momentarily at rest at highest point, but if only gravitational force is acting on it, then also motion will be called free fall.

The acceleration of the body can be approximated as a constant. In this section, we will consider the straight line motion of a particle near earth's surface. It is experimentally verified that all bodies accelerate towards the center of earth with an acceleration of 9.81 m s^{-2} "near earth's surface." When viewed near the "earth," its surface appears as flat. Hence, any particle appears to accelerate vertically downwards with an acceleration of 9.81 m s^{-2} and sometimes to make the calculations easy we take $g = 10$ m s^{-2}.

Sign convention: Any vector quantity directed upward will be taken as positive and directed downward will be taken as negative. According to this sign convention,



1. Displacement will be taken as positive if the final position lies above initial position and negative if the final position lies below initial position.
2. Velocity (initial or final) will be taken as positive if it is upward and negative if it is downward.
3. Acceleration a is always taken to be $-g$.

In equations of motions, when we replace a by $-g$ (minus sign, because acceleration is always directed downward), we get

$$v = u - gt \Rightarrow s = ut - \frac{1}{2}gt^2 \Rightarrow v^2 = u^2 - 2gs$$

Important Points:

- The motion is independent of the mass of the body, as in any equation of motion, mass is not involved. That is why a heavy and light body when released from the same height, reach the ground simultaneously and with same velocity, i.e., $t = \sqrt{(2h/g)}$ and $v = \sqrt{2gh}$.
- In case of motion under gravity, time taken to go up is equal to the time taken to fall down through the same distance.
  Time of descent (t_1) = Time of ascent $(t_2) = u/g$
  $\therefore$ Total time of flight. $T = t_1 + t_2 = \dfrac{2u}{g}$
- In case of motion under gravity, the speed with which a body is projected up is equal to the speed with which it comes back to the point of projection.
  As well as the magnitude of velocity at any point on the path is same whether the body is moving in upwards or downward direction.
- A body is thrown vertically upwards. If air resistance is to be taken into account, then the time of ascent is less than the time of descent $(t_2 > t_1)$.
  Let u be the initial velocity of body, then time of ascent

$$t_1 = \frac{u}{g+a} \text{ and } h = \frac{u^2}{2(g+a)}$$

where g is acceleration due to gravity and a is retardation by air resistance and for upward motion both will act vertically downward.

For downward motion, a and g will act in opposite direction because a always acts in direction opposite to motion and g always acts vertically downward.

So $\quad h = \frac{1}{2}(g-a)t_2^2$

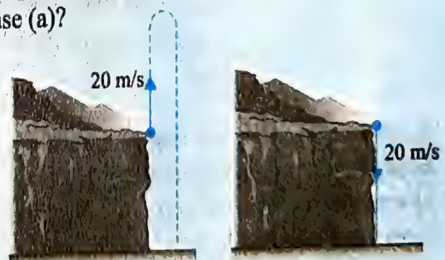
$$\Rightarrow \quad \frac{u^2}{2(g+a)} = \frac{1}{2}(g-a)t_2^2 \Rightarrow t_2 = \frac{u}{\sqrt{(g+a)(g-a)}}$$

Comparing t_1 and t_2 we can say that $t_2 > t_1$ since $(g + a) > (g - a)$.

- The motion of an object which is thrown upward and eventually returns to earth has a symmetry that is useful to keep in mind from the point of view of problem solving. The calculations just completed indicate that a time symmetry exists in free-fall motion, in the sense that the time required for the object to reach maximum height equals the time for it to return to its starting point. Let us learn this point through coming illustration.

ILLUSTRATION 4.23

Consider two cases, in case (a) a ball that has been thrown straight upward from the edge of a cliff. The initial speed of the ball is 20 m/s. It goes up and then falls back down, eventually hitting the ground beneath the cliff. In case (b) the ball has been thrown straight downward at the same initial speed. In the absence of air resistance, would the ball in case (b) strike the ground with (i) smaller speed than in case (a), (ii) same speed as in case (a), or (iii) greater speed than the ball in case (a)?



Sol. We know in the absence of air resistance and the motion is that of free-fall type we can analyze the motion by symmetry consideration. In figure we can observe that the ball after it has been thrown upward has fallen back down to its starting point. Symmetry suggests that the speed in figure is the same as in case (a)—namely, 20 m/s, as is also the case when the ball has been actually thrown downward. Consequently, whether the ball is thrown as in case (a) or (b), it begins to move downward from the cliff edge at a speed of 20 m/s. In either case, there is the same acceleration due to gravity and the same displacement from the cliff edge to the ground below. Under these conditions, the ball reaches the ground with the same speed in both cases (a) and (b).

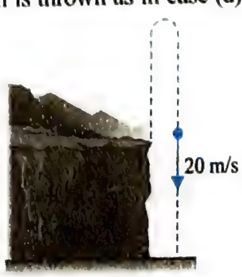


ILLUSTRATION 4.24

A cricket match customarily begins with a coin toss. The umpire tosses the coin up with an initial speed of 5.0 m/s. In the absence of air resistance,

(a) How high does the coin go above its point of release?

(b) What is the total time the coin is in the air before returning to its release point?

Sol.

(a) The statement "how high does the coin go" means the maximum height, which occurs when the final velocity of the coin in the vertical direction is zero or $v = 0$ m/s.

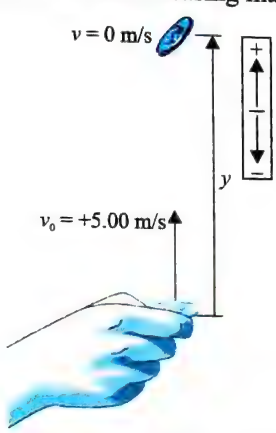
Taking upward direction as positive, as the coin is tossed upward, hence the initial velocity $v_0 = +5.0$ m/s. The velocity of the coin is momentarily zero when the coin reaches its maximum height.

There are three parts to the motion of the coin, in which air resistance is being ignored.

(i) On the way up, the coin has an upward-pointing velocity vector with a decreasing magnitude.

(ii) At the top of its path, the velocity vector of the coin is momentarily zero.

(iii) On the way down, the coin has a downward-pointing velocity vector with an increasing magnitude.



In any of the cases, the object always experiences the same downward acceleration due to gravity, During the upward and downward parts of the motion, and also at the top of the path, the acceleration due to gravity has a constant downward direction and a constant magnitude of 10 m/s². In other words, the acceleration vector of the particle does not behave as the velocity vector does. In particular, the acceleration vector is not zero at the top of the motional path just because the velocity vector is zero there. Acceleration is the rate at which the velocity is changing, and the velocity is changing at the top even though at one instant it is zero.

Using $v^2 = v_0^2 + 2a.s$

$\Rightarrow \quad 0 = (+5)^2 + 2(-10).y$ which gives $y = 1.25$ m

(b) During the time the coin travels upward, gravity causes its speed to decrease to zero. On the way down, however, gravity causes the coin to regain the lost speed. Thus, the time for the coin to go up is equal to the time for it to come down. In other words, the total travel time is twice the time for the upward motion.

We can use equation $v = u + a.t$ to find the upward travel time.

The total up-and-down time is twice this value, or 1.0 s.

It is possible to determine the total time by another method. When the coin is tossed upward and returns to its release point, the displacement for the *entire trip* is $y = 0$ m.

With this value for the displacement, we can use equation $\left(y = v_0.t + \frac{1}{2}a.t^2\right)$ to find the time for the entire trip directly.

$$0 = (+5).T + \frac{1}{2}(-10).T^2 \Rightarrow T = 1\text{s}$$

ILLUSTRATION 4.25

A particle is projected up with initial speed $u = 10$ m/s from the top of a building at time $t = 0$. At time $t = 5$ sec, the particle strikes the ground. Find the height of the building.

Sol.

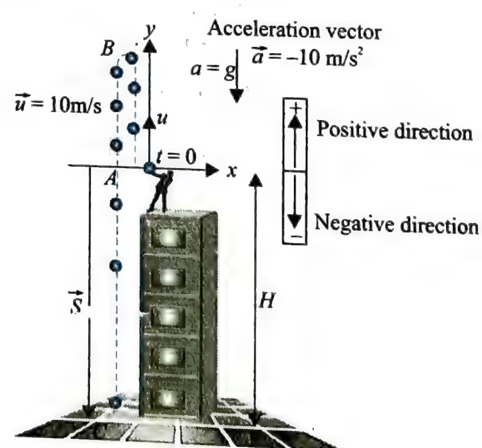
The particle is "freely falling" always. The expression "freely falling" does not necessarily mean an object is falling down. A freely falling object is any object moving either upward or downward under the influence of gravity alone.

The particle starts from A and finally reaches at C.

Let us take origin at A. Upward direction is taken as positive and downward direction is taken as negative.

The particle moves in gravitational field where acceleration due to the gravity is always acting in downward direction whether it is moving upward or downward.

Hence acceleration vector $\vec{a}$ will always be -10 m/s², as its magnitude as well as direction remains constant always throughout the motion.



During complete motion of the particle,
• the acceleration of the particle is '$-g$'
• the displacement of the particle is '$-H$'

Hence acceleration, $\vec{a} = -10\text{m/s}^2$

Initially at $t = 0$, the particle is projected in upward direction. Hence, initial velocity $\vec{u} = 10\,\text{m/s}^2$

The particle moves from A to B (upward) and then B to C (downward).

The motion of the particle goes from A to B and then again passes point A. The net displacement of the particle upto this instant is zero. Then particle crosses point A and finally reaches

to C. We know that net displacement is equal to the difference of final position vector and initial position vector.

Hence net displacement of the particle during motion (t = 5sec) is $-H$ (m).

Using $\vec{s}=\vec{u}t+\frac{1}{2}\vec{a}t^2$

$\Rightarrow \quad \vec{s}=(10)\times 5+\frac{1}{2}(-10)(5)^2=50-125=75(\text{m})$

Hence $H=75$ m

ILLUSTRATION 4.26

A ball is projected vertically up such that it passes through a fixed point after a time t_1 and t_2, respectively. Find

(a) The height at which the point is located with respect to the point of projection.

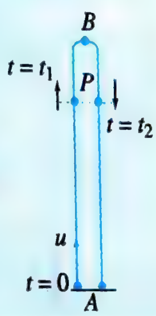
(b) The speed of projection of the ball.

(c) The velocity of the ball at the time of passing through point P.

(d) (i) The maximum height reached by the ball relative to the point of projection A and

(ii) maximum height reached by the ball relative to point P under consideration.

(e) The average speed and average velocity of the ball during the motion from A to P for the time t_1 and t_2, respectively.



Sol.

(a,b) Let the ball be projected up with an initial velocity u. It passes through point P at $t=t_1$ during its ascent and at $t=t_2$ during its descent.

For the motion of the ball from A to P,

$s=+h,\ v_0=u,\ a=-g$, and $t=t_1$ and t_2

Substituting the above value, in $s=v_0t+\frac{1}{2}at^2$, we get

$$\left.\begin{aligned} h &= ut_1-\frac{1}{2}gt_1^2 \text{ (upward motion)} \\ &= ut_2-\frac{1}{2}gt_2^2 \text{ (downward motion)} \end{aligned}\right] \quad \ldots\text{(i)}$$

Hence, from (i), $ut_1-\frac{1}{2}gt_1^2=ut_2-\frac{1}{2}gt_2^2$

$$u(t_2-t_1)=\frac{1}{2}g(t_2-t_1)(t_2+t_1) \Rightarrow u=\frac{1}{2}g(t_2+t_1)$$

Substituting the value of u in (i) we get $h=\frac{1}{2}gt_1t_2$

(c) Using the relation $\vec{v}=\vec{u}+\vec{a}t$, we get

$$v_P=\left(\frac{g(t_2+t_1)}{2}\right)-gt_1=\frac{g}{2}(t_2-t_1)$$

(d)(i) Maximum height reached by ball from the point of projection.

Using $v^2=u^2+2as$

$$0=\left(\frac{g}{2}(t_2+t_1)\right)^2-2gH \Rightarrow H_{\max}=\frac{g}{8}(t_1+t_2)^2$$

(ii) Height reached by ball from point P,

$$h'=H_{\max}-h=\frac{g}{8}(t_1+t_2)^2-\frac{1}{2}gt_1t_2=\frac{g}{8}(t_2-t_1)^2$$

(e)(i) A to P for time t_1

The average speed and average velocity from A to P will be same as the particle is moving in straight line without changing the direction.

$$\langle v\rangle=\frac{\Delta y}{\Delta t}=\frac{h}{t_1}=\frac{\frac{1}{2}gt_1t_2}{t_1}=\frac{1}{2}gt_2$$

(ii) Motion from A to P for time t_2.

$$\text{Average speed}=\frac{\text{Total distance}}{\text{Total time}}$$

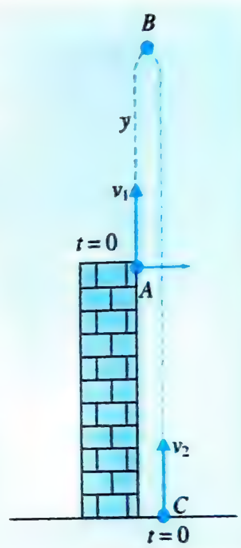
$$=\frac{h+2h'}{t_2}=\frac{\frac{1}{2}gt_1t_2+2\left(\frac{g}{8}(t_2-t_1)^2\right)}{t_2}=\frac{g(t_1^2+t_2^2)}{4t_2}$$

Motion from A to P for time t_2

$$\text{Average velocity } \langle\vec{v}\rangle=\frac{\Delta\vec{y}}{\Delta\vec{t}}=\frac{h}{t_2}=\frac{\frac{1}{2}gt_1t_2}{t_2}=\frac{1}{2}gt_1$$

ILLUSTRATION 4.27

Two particles 1 and 2 are projected simultaneously with velocities v_1 and v_2, respectively. Particle 1 is projected vertically up from the top of a cliff of height h and particle 2 is projected vertically up from the bottom of the cliff. If the bodies meet (a) above the top of the cliff, (b) between the top and bottom of the cliff, and (c) below the bottom of the cliff, find the time of meeting of the particles.



Sol.

(a) The point of collision above the top of the cliff. Let the particle meet after time t.

For first particle, $s=s_1,\ v_0=v_1,\ a=-g$.

Then, $s_1=v_1t-\frac{1}{2}gt^2 \quad \ldots\text{(i)}$

For second particle,

$s=s_2,\ v_0=v_2,\ a=-g$.

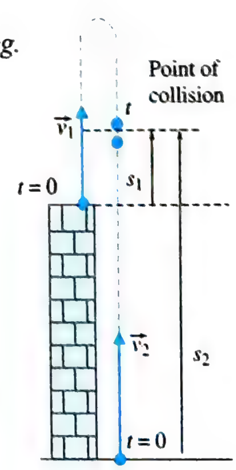
Then, $s_2=v_2t-\frac{1}{2}gt^2 \quad \ldots\text{(ii)}$

Referring to figure,

$s_2-s_1=h \quad \ldots\text{(iii)}$

Substituting s_1 from (i), s_2 from (ii) in (iii), we have

$$t=\frac{h}{(v_2-v_1)}$$



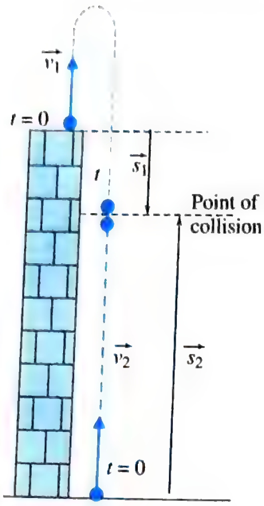
(b) The point of collision of the particle in between the top and bottom of the cliff.

For the first particle, $s = -s_1$, $v_0 = v_1$, and $a = -g$

Position–time relation for first particle,

$$-s_1 = v_1 t - \frac{1}{2}gt^2$$

This gives $s_1 = \frac{1}{2}gt^2 - v_1 t$ ...(i)



Similarly, for the second particle,

$s = s_2$, $v_0 = v_2$, and $a = -g$.

Then $s_2 = v_2 t - \frac{1}{2}gt^2$ ...(ii)

Referring to figure, $s_1 + s_2 = h$. ...(iii)

Substituting s_1 from (i), s_2 from (ii) in (iii), we have

$$t = \frac{h}{(v_2 - v_1)}$$

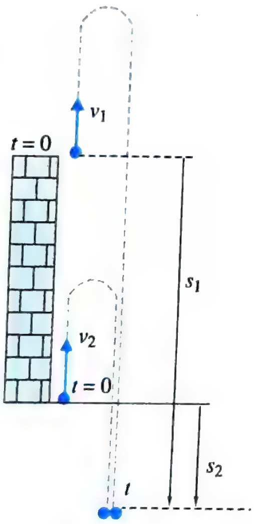
(c) The point of collision of the particles below the bottom of the cliff (let us assume a ditch at the base of the cliff).

For particle 1, $s = -s_1$, $v_0 = v_1$, and $a = -g$.

Position–time relation for first particle

$$-s_1 = v_1 t - \frac{1}{2}gt^2$$

This gives $s_1 = \frac{1}{2}gt^2 - v_1 t$ ...(i)



Point of collision

Similarly, for particle 2, $s = -s_2$, $v_0 = v_2$, and $a = -g$.

We have $s_2 = \frac{1}{2}gt^2 - v_2 t$ ...(ii)

From Figure, $s_1 - s_2 = h$ ...(iii)

Substituting s_1 from (i), s_2 from (ii) in (iii), we have

$$t = \frac{h}{v_2 - v_1}$$

ILLUSTRATION 4.28

A body is thrown vertically upwards from A, the top of a tower. It reaches the ground in time t_1. If it is thrown vertically downwards from A with the same speed it reaches the ground in time t_2. If it is allowed to fall freely from A, then find the time it takes to reach the ground.

Sol.

Suppose the body be projected vertically upwards from A wi[th] a speed u_0.

Using equation $s = ut + \left(\frac{1}{2}\right)at^2$

For first case $-h = u_0 t_1 - \left(\frac{1}{2}\right)gt_1^2$ ...(i)

For second case, $-h = -u_0 t_2 - \left(\frac{1}{2}\right)gt_2^2$ ...(ii)

From (i) – (ii) $\Rightarrow$ $0 = u_0(t_2 + t_1) + \left(\frac{1}{2}\right)g(t_2^2 - t_1^2)$

or $u_0 = \left(\frac{1}{2}\right)g(t_1 - t_2)$ ...(iii)

Put the value of u_0 in (ii), we get

$$-h = -\left(\frac{1}{2}\right)g(t_1 - t_2)t_2 - \left(\frac{1}{2}\right)gt_2^2$$

$\Rightarrow$ $h = \frac{1}{2}gt_1 t_2$ ...(iv)

For third case, $u = 0$, $t = ?$

$$-h = 0 \times t - \left(\frac{1}{2}\right)gt^2$$

or $h = \left(\frac{1}{2}\right)gt^2$ ...(v)

Combining (iv) and (v), we get

$$\frac{1}{2}gt^2 = \frac{1}{2}gt_1 t_2 \quad \text{or} \quad t = \sqrt{t_1 t_2}$$

ILLUSTRATION 4.29

A body is projected upwards with a velocity u. It passes through a certain point above the ground after t_1. Find the time after which the body passes through the same point during the return journey.

Sol.

Suppose v be the velocity attained by the body after time t_1. Then

$v = u - gt_1$ ...(i)

Let the body reaches the same point at time t_2. Now velocity will be downward with same magnitude v, then

$-v = u - gt_2$ ...(ii)

(i) – (ii) $\Rightarrow 2v = g(t_2 - t_1)$

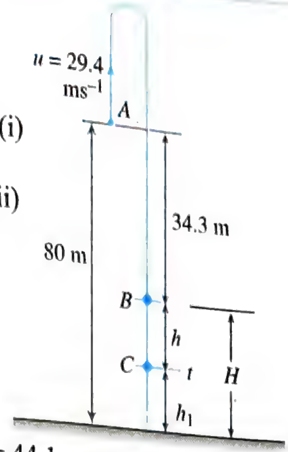
or $t_2 - t_1 = \frac{2v}{g} = \frac{2}{g}(u - gt_1) = 2\left(\frac{u}{g} - t_1\right)$

ILLUSTRATION 4.30

From a point A, 80 m above the ground, a particle is projected vertically upwards with a velocity of 29.4 m s^{-1}. Five seconds later, another particle is dropped from a point B, 34.3 m vertically below A. Determine when and where one overtakes the other. Take $g = 9.8$ m s^{-2}.

Sol.

Let at time t, they reach at level C.



For A: $-(h+34.3) = 29.4t - \frac{1}{2}gt^2$ ...(i)

For B: $-h = -\frac{1}{2}g(t-5)^2$ ...(ii)

(ii) – (i) $\Rightarrow 34.3$

$$= -29.4t + \frac{1}{2}g[t^2 - (t-5)^2]$$

$\Rightarrow \quad 7 = -6t + (2t-5)5 \Rightarrow t = 8\text{ s}$

Now from (ii), $h = \frac{1}{2}(9.8)(8-5)^2 = 44.1\text{ m}$.

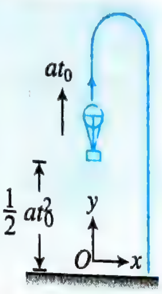
From figure,

$$H = 80 - 34.3 = 45.7\text{ m}.$$

So finally, $h_1 = H - h = 45.7 - 44.1 = 1.6\text{ m}$

ILLUSTRATION 4.31

A balloon starts rising upwards with constant acceleration a and after time t_0 second, a packet is dropped from it which reaches the ground after t seconds of dropping. Determine the value of t.



Sol.

$t = 0$ is the time when the balloon started rising up. At $t = t_0$, when the packet is dropped, the balloon is moving up with velocity $v = 0 + at_0 = at_0$. Hence, initial velocity of the packet will be $v_0 = at_0$ (upward). As the balloon has started rising upwards with constant acceleration a, so after t_0 seconds, its height from the ground is $y_0 = \frac{1}{2}at_0^2$.

For packet: $s = ut - \frac{1}{2}gt^2$

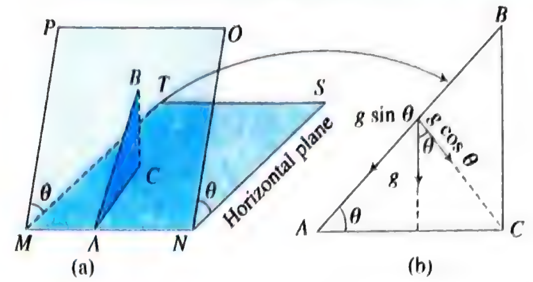
$$\Rightarrow \quad -\frac{1}{2}at_0^2 = at_0t - \frac{1}{2}gt^2 \Rightarrow gt^2 - 2at_0t - at_0^2 = 0$$

Solving the quadratic equation, we get $t = \frac{at_0}{g}\left[1 + \sqrt{1 + \frac{g}{a}}\right]$

MOTION UPON AN INCLINED PLANE

Any plane inclined to the horizontal at a definite angle is said to be an inclined plane, and the corresponding angle is known as the angle of inclination (or simply inclination).

Figure shows a horizontal plane $MNST$ being cut by an inclined plane $MNOP$. Along the line MN, consider any point B upon the inclined. From B, drop perpendiculars BA and BC to the line MN and horizontal plane $MNST$, respectively. Then $\angle BAC$ is said to be the *angle of inclination of the inclined plane.*



Line AB is said to be the line of greatest slope. If a body is allowed to move freely along the inclined plane, it always chooses the line of greatest slope to move. However, it can be made to move along other line by allowing it to slide along a groove in the desired direction. Figure (b) shows an inclined plane AB with angle of inclination θ. θ is such that $\sin\theta = l/n$. Usually l is chosen as 1.

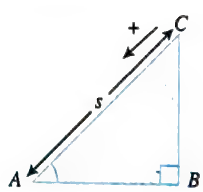
The inclination is often referred to as "l in n." It implies that a "1 in n" inclination is such that, there is a vertical rise of 1 unit for every n unit distance travelled along the plane up.

ILLUSTRATION 4.32

(a) Show that the velocity acquired by a particle in sliding down an inclined plane is the same as that acquired by a particle falling freely from rest through a distance equal to the height of the inclined plane. (b) Find the time taken in sliding a particle down the whole length of the incline.

Sol.

(a) Let a particle sliding from C to A along the inclined plane CA acquire a final velocity v_1, covering a distances.



If θ is the angle of inclination, then $\sin\theta = \frac{h}{s}$

Now for the sliding particle, $s = s$, $u = 0$, $a = g\sin\theta$, $v = v_1$.

[Taking the direction C to A as positive]

Using, $v^2 = u^2 + 2as$

$$\Rightarrow \quad v_1^2 = 2(g\sin\theta)s = 2g\left[\frac{h}{s}\right]s = 2gh$$

$$\therefore \quad v_1 = \sqrt{2gh} \qquad \text{...(i)}$$

Again, let a particle fall from rest along a distance h and acquire a velocity v_2, then from $v^2 = u^2 + 2as$,

$$v_2^2 = 0 + 2gh \Rightarrow v_2 = \sqrt{2gh} \qquad \text{...(ii)}$$

From (i) and (ii), we get the same result.

(b) Time taken in sliding a particle down the entire length of the incline:

For the motion of particle from C to A (Fig), $u = 0$, $s = s$, $a = g\sin\theta$, $t = ?$.

Using $s = ut + \frac{1}{2}at^2$, we get $s = \frac{1}{2}(g\sin\theta)t^2$

$$\Rightarrow \quad t^2 = \frac{2s}{g\sin\theta} = \frac{2h}{g\sin^2\theta} \qquad \left[\because \sin\theta = \frac{h}{s}\right]$$

$$\therefore \quad t = \sqrt{\frac{2h}{g}}\operatorname{cosec}\theta$$

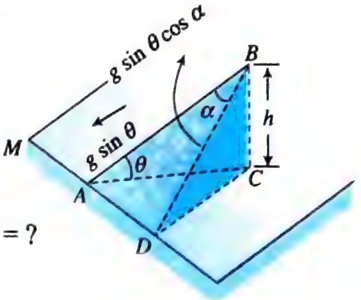
Thus, the time taken varies directly as the cosecant of the angle of inclination. Now, since cosecant is a decreasing function in the first quadrant with an increase in θ, time t would decrease.

Motion Along Other Slope Lines

Time of fall: To find the time taken by the particle to slide down a height h along line BD (other than the line of greatest slope).

Let AB be the line of greatest slope and BD make an angle α with it. The component of acceleration acting along BD is $g\sin\theta\cos\alpha$.

From the right-angled triangle BAD, we have



$$\cos\alpha = \frac{AB}{BD}$$

$$\Rightarrow \quad BD = AB\sec\alpha$$

But $AB = h\operatorname{cosec}\theta$

$$\therefore \quad AB = h\operatorname{cosec}\theta\sec\alpha$$

Now, $u = 0$, $a = g\sin\theta\cos\alpha$, $t = ?$

$$s = h\operatorname{cosec}\theta\sec\alpha$$

Using $s = ut + \frac{1}{2}at^2$, $h\operatorname{cosec}\theta\sec\alpha = \frac{1}{2}(g\sin\theta\cos\alpha)t^2$

$$\Rightarrow \quad t^2 = \frac{2h}{g}\operatorname{cosec}^2\theta\sec^2\alpha$$

$$\Rightarrow \quad t = \left(\sqrt{\frac{2h}{g}}\operatorname{cosec}\theta\right)\cdot\sec\alpha$$

Again, since $\sec\alpha > 1$, $\qquad$ [for $0 < \alpha < 90°$]

$$t > \sqrt{\frac{2h}{g}}\operatorname{cosec}\theta$$

i.e., $t >$ time of sliding along BD

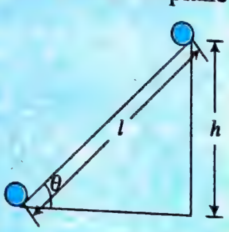
In other words, the time taken to slide down an inclined plane is least along the line of greatest slope. Hence, the line of greatest slope is sometimes referred to as the line of quickest descent.

Illustration 4.33

Ball 1 is released from the top of a smooth inclined plane, and at the same instant ball 2 is projected from the foot of the plane with such a velocity that they meet halfway up the incline. Determine:

(a) the velocity with which balls are projected and

(b) the velocity of each ball when they meet.



Sol. For ball '1': $u_1 = 0$, $a = g\sin\theta = g\left(\frac{h}{l}\right)$ and $s = l/2$

Using $v^2 = u^2 + 2as \Rightarrow v^2 = 0 + 2g\left(\frac{h}{l}\right)\cdot\left(\frac{l}{2}\right)$

$$\Rightarrow \quad v = \sqrt{gh} \qquad \text{...(i)}$$

Again using $v = u + at \Rightarrow v = 0 + g\left(\frac{h}{l}\right)\cdot t$ $\qquad$ ...(ii)

$$\sqrt{gh} = g\left(\frac{h}{l}\right)t \Rightarrow t = \frac{l}{\sqrt{gh}}$$

For ball '2': $u = u_2$, $v = v_2$ and $a = -g\sin\theta = -g\left(\frac{h}{l}\right)$

Using $v^2 = u^2 + 2as \Rightarrow v_2^2 = u_2^2 + 2\left(-g\frac{h}{l}\right)\cdot\left(\frac{l}{2}\right)$ $\qquad$ ...(iii)

Again using $v = u + at \Rightarrow v_2 = u_2 - \left(g\frac{h}{l}\right)\cdot\frac{l}{\sqrt{gh}}$ $\qquad$ ...(iv)

From (iii) and (iv) $u_2 = \sqrt{gh}$ and $v_2 = 0$

Hence velocity of ball '2' at the time of throwing will be $\sqrt{gh}$ and when it meets ball '1' its velocity becomes zero.

The velocity of ball '1' when it meets ball '2' is $\sqrt{gh}$.

Concept Application Exercise 4.3

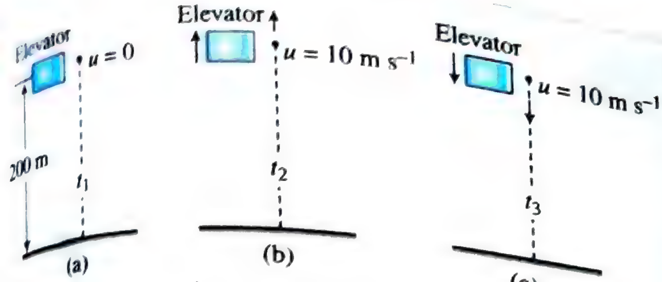
1. **(a)** Mark the following statements as true or false.
   (i) A ball thrown vertically up takes more time to go up than to come down.
   (ii) If a ball starts falling from the position of rest, then it travels a distance of 25 m during the third second of its fall.
   (iii) A packet dropped from a rising balloon first moves upwards and then moves downward as observed by a stationary observer on the ground.
   (iv) In the absence of air resistance, all bodies fall on the surface of earth at the same rate.

   (b) Fill in the blanks.
   (i) When a body is thrown vertically upwards, at the highest point _____ (both velocity and acceleration are zero/only velocity is zero/only acceleration is zero).
   (ii) If air drag is not neglected, then which is greater: time of ascent or time of descent?
   (iii) A body is projected upward. Up to the maximum height, time taken will be greater to travel _____ (first half/second half).

2. A ball thrown up from the ground reaches a maximum height of 20 m. Find:
   (a) Its initial velocity.
   (b) The time taken to reach the highest point.
   (c) Its velocity just before hitting the ground.
   (d) Its displacement between 0.5 s and 2.5 s.
   (e) The time at which it is 15 m above the ground.

3. A body is projected from the bottom of a smooth inclined plane with a velocity of 20 m s^{-1}. If it is just sufficient to carry it to the top in 4 s, find the inclination and height of the plane.

4. A ball is dropped from an elevator at an altitude of 200 m. How much time will the ball take to reach the ground if the elevator is



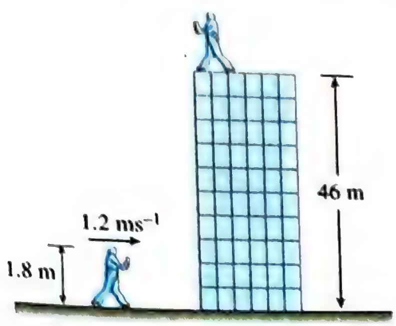
(a) Stationary?
(b) Ascending with velocity 10 m s^{-1}?
(c) Descending with velocity 10 m s^{-1}?

5. A balloon rises from rest on the ground with constant acceleration $g/8$. A stone is dropped from the balloon when the balloon has risen to a height of H. Find the time taken by the stone to reach the ground.

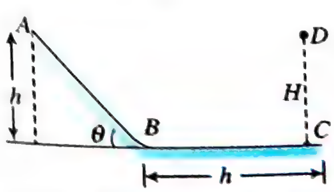
6. A parachutist, after bailing out, falls 50 m without friction. When the parachute opens, it decelerates at 2 m s^{-2}. He reaches the ground with a speed of 3 m s^{-1}. At what height did he bail out?

7. A ball is dropped from the top of a tower of height h. It covers a distance of $h/2$ in the last second of its motion. How long does the ball remain in air?

8. You are on the roof of the physics building, 46.0 m above the ground. Your physics professor, who is 1.80 m tall, is walking alongside the building at a constant speed of 1.20 m s^{-1}. If you wish to drop a flower on your professor's head, where should the professor be when you release the flower? Assume that the flower is in free fall. (Take g = 9.8 m/s²)



9. Two particles are simultaneously released from points A and D as shown in figure. How should the value of H be adjusted in order that the two particles collide? Neglect dissipative forces.



10. A ball is thrown from the top of a tower in vertically upward direction. Velocity at a point h m below the point of projection is twice of the velocity at a point h m above the point of projection. Find the maximum height reached by the ball above the top of tower.

11. A ball is thrown upward from the ground with an initial speed of 25m/s at the same instant, another ball is dropped from a building 15 m high. After how long will the balls be at the same height above the ground?

12. At time $t = 0$, a student throws a set of keys vertically upward to her sorority sister, who is in a window at distance h above. The second student catches the keys at time t. (a) With what initial velocity were the keys thrown? (b) What was the velocity of the keys just before they were caught?

ANSWERS

1. (a) (i) False (ii) True (iii) True (iv) True
(b) (i) Only velocity is zero (ii) Time of descent (iii) Second half

2. (a) 20 m s^{-1} **(b)** 2 s **(c)** 20 m s^{-1} **(d)** 10 m **(e)** 1 s, 3 s

3. 30°, 20 m **4. (a)** $\sqrt{40}$ s **(b)** $1+\sqrt{41}$ s **(c)** $-1+\sqrt{41}$ s

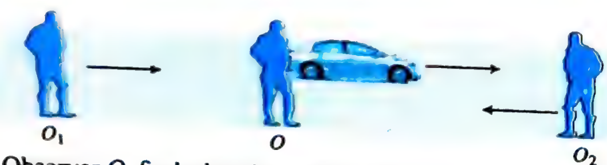
5. $2\sqrt{\dfrac{H}{g}}$ **6.** 293 m **7.** $2+\sqrt{2}$ s **8.** 3.60 m

9. $\dfrac{h}{4\sin^2\theta}(2+\sin\theta)^2$ **10.** $H = \dfrac{5}{3}h$ **11.** 0.60 s

12. (a) $\dfrac{h}{t} + \dfrac{gt}{2}$ **(b)** $\dfrac{h}{t} - \dfrac{gt}{2}$

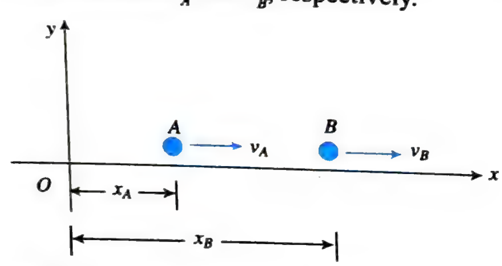
RELATIVE MOTION IN ONE DIMENSION

Consider the motion of a car moving towards right and two observers O_1 and O_2 are coming from opposite directions as shown in figure.



Observer O_1 finds that the car is moving slower while observer O_2 finds that the car is moving faster in comparison to when observer is at rest. The motion of same object looks different for two different observers. To understand such observations, there is need of introduction of the concept of relative velocity.

In figure, two particles A and B are moving with velocities v_A and v_B and accelerations a_A and a_B, respectively.



If $\bar{x}_A$ and $\bar{x}_B$ are their respective displacements with respect to the fixed origin, then

- The relative displacement of B w.r.t A is defined as

$$\bar{x}_{BA} = \bar{x}_B - \bar{x}_A \qquad \text{...(i)}$$

Differentiating (i) w.r.t time.

- The relative velocity of B with respect to A is defined as

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A \quad ...(ii)$$

Differentiating (ii) w.r.t time.

- The relative acceleration of B with respect to A is defined as

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A$$

The equations of motion in one dimension are modified as:

$$\vec{v}_{BA} = \vec{u}_{BA} + \vec{a}_{BA}t \Rightarrow \vec{v}_{rel} = \vec{v}_{rel} + \vec{v}_{rel}t$$

$$\vec{x}_{BA} = \vec{u}_{BA}t + \frac{1}{2}\vec{a}_{BA}t^2 \Rightarrow \vec{x}_{rel} = \vec{u}_{rel}t + \frac{1}{2}\vec{a}_{rel}t^2$$

$$v_{rel}^2 - u_{rel}^2 = 2a_{rel}x_{rel}$$

The relative velocity of a particle A with respect to B is defined as the velocity with which A appears to move if B is considered to be at rest. In other words, it is the velocity with which A appears to move as seen by B considering itself to be at rest.

Note: All velocities are relative and have no significance unless observer is specified. However, when we say "velocity of A," what we mean is that the velocity of A w.r.t. ground which is assumed to be at rest.

$$\vec{v}_{\text{Object, Observer}} = \vec{v}_{\text{Object}} - \vec{v}_{\text{Observer}}$$

or $\vec{v}_{rel} = \vec{v}_{\text{Object}} - \vec{v}_{\text{Observer}}$

Velocity of Approach/Separation

It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

If the separation is decreasing, we say it is the velocity of approach and if the separation is increasing, then we say it is the velocity of separation.

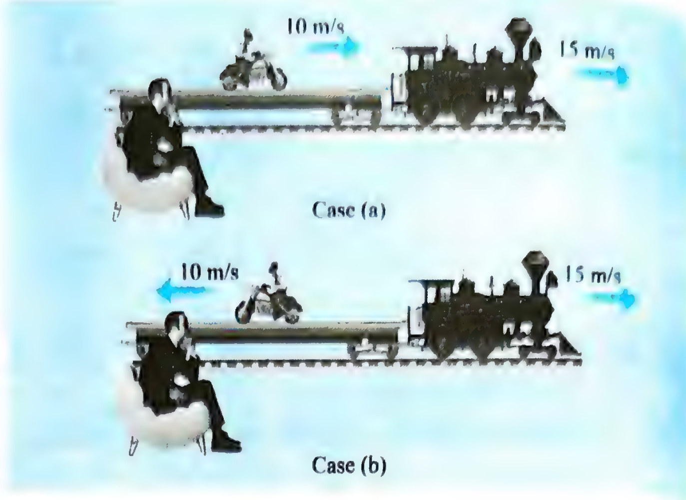
In one dimension, since relative velocity is along the line joining A and B, hence the velocity of approach/separation is simply equal to the magnitude of relative velocity of A w.r.t. B.

Note:
Problem-solving tips for relative velocity

- If the velocity is mentioned without specifying the frame, assume it is with respect to ground.
- In many cases, a body travels on water or in air. Depending on the context you will have to figure out whether the velocity is with respect to water/air or with respect to ground.
- In some situations you have to presume velocities. For example, if the problem says that a man can walk at a maximum of 8 km h^{-1} and if it asks you to find the velocity on a train, then you have to assume that the velocity of the man with respect to the surface he is on (in this case the train is 8 km h^{-1}). Similarly, the velocity of a bullet is always measured with respect to the gun. If the gun is mounted on a truck, the bullet will have a different velocity.

ILLUSTRATION 4.34

A railroad flatcar is travelling to the right at a speed of 15 m/s relative to an observer sitting on the ground. Someone is riding a scooter on the flatcar (as shown in figure). Velocity of scooter with respect to car is 10 m/s. What is the velocity of the scooter relative to the observer on the ground in case (a) and case (b) as shown in figure?



Sol. **Case(a):** Taking right direction as positive.

Velocity of rail road car, $\vec{v}_{car} = 15\,(\text{m/s})$

Velocity of scooter with respect to car $\vec{v}_{scooter,car} = 10\,(\text{m/s})$

If velocity of scooter is

$$\vec{v}_{scooter} = \vec{v}_{scooter,car} + \vec{v}_{car} = 15 + 10 = 25\,(\text{m/s})$$

The velocity of the scooter relative to the observer on the ground is 25m/s in right direction.

Case (b): Taking right direction as positive.
Velocity of rail road car, $\vec{v}_{car} = 15\,(\text{m/s})$
Velocity of scooter with respect to car, $\vec{v}_{scooter,car} = -10\,(\text{m/s})$
If velocity of scooter is

$$\vec{v}_{scooter} = \vec{v}_{scooter,car} + \vec{v}_{car} = -10 + 15 = 5\,(\text{m/s})$$

The velocity of the scooter relative to the observer on the ground is 5m/s in right direction.

ILLUSTRATION 4.35

Two parallel rail tracks run north south. Train A moves north with a speed of 54 km/h and train B moves south with a speed of 90 km/hr. A monkey running on the roof of the train A against its motion (with its velocity of 18 km/h with respect to the train A). What is the velocity of the monkey

(a) as observed by an observer on ground?

(b) as observed by an observer moving in train B?

Sol. Taking North direction as positive.

The velocity of train A is 54 km/hr towards North.

$$\vec{v}_A = 54\,\text{km/hr} = 54 \times \frac{5}{18} = 15\,\text{m/s}$$

The velocity of train B is 90 km/hr towards South.

$$\vec{v}_B = -90\,\text{km/hr} = -90 \times \frac{5}{18} = -25\,\text{m/s}$$

The velocity of monkey is B w.r.t train A is 18 km/hr towards south.

$$\vec{v}_{\text{Monkey},A} = -18\,\text{km/hr} = -18 \times \frac{5}{18} = -5\,\text{m/s}$$

(a) The velocity of a monkey as observed by an observer on ground

$$\vec{v}_{\text{Monkey},A} = \vec{v}_{\text{Monkey}} - \vec{v}_A \Rightarrow \vec{v}_{\text{Monkey}} = \vec{v}_{\text{Monkey},A} + \vec{v}_A$$

$$\vec{v}_{\text{Monkey}} = (-5) + 15 = 10\,\text{m/s}$$

For an observer standing on ground the monkey will appear to move with 10 m/s towards north.

(b) The velocity of a monkey as observed by an observer moving in train B

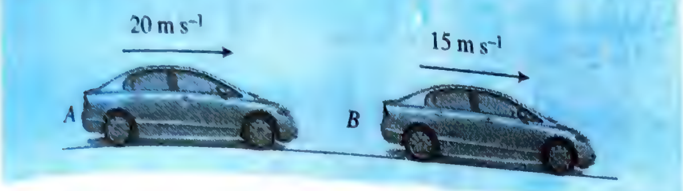
$$\vec{v}_{\text{Monkey},B} = \vec{v}_{\text{Monkey}} - \vec{v}_B = 10 - (-25) = 35\,\text{m/s}$$

Hence velocity of monkey will be 36 km/hr towards north.

For an observer moving in train B the monkey will appear to move with 35 m/s towards north.

ILLUSTRATION 4.36

A car A moves with velocity 20 m s^{-1} and car B with velocity 15 ms^{-1} as shown in figure. Find the relative velocity of B w.r.t. A and A w.r.t. B.



Sol. Let us assume that the right direction is positive.

We are given: $v_A = 20$ m s^{-1}, $v_B = 15$ m s^{-1},

Relative velocity of B w.r.t A:

$$v_{B/A} = v_B - v_A = 15 - 20 = -5\ \text{m s}^{-1}$$

(Negative sign indicates that this relative velocity is in the left direction.)

Relative velocity of A w.r.t B:

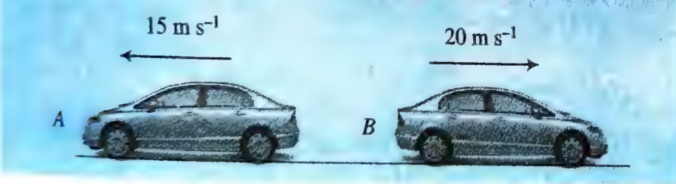
$$v_{A/B} = v_A - v_B = 20 - 15 = 5\ \text{m s}^{-1}$$

(Positive sign indicates that this relative velocity is in the right direction.)

Here the separation between A and B is decreasing with time, hence the velocity of approach of A w.r.t B is 5 m s^{-1}.

ILLUSTRATION 4.37

A car A moves with velocity 15 m s^{-1} and B with velocity 20 m s^{-1} are moving in opposite directions as shown in figure. Find the relative velocity of B w.r.t. A and A w.r.t. B.



Sol. Here also let us assume right direction is positive, then left direction will be negative.

Given: $v_A = -15$ m s^{-1}, $v_B = 20$ m s^{-1},

Relative velocity of B w.r.t A:

$$v_{B/A} = v_B - v_A = 20 - (-15) = 35\ \text{m s}^{-1}$$

(Positive sign indicates that this relative velocity is in right direction.)

Relative velocity of A w.r.t B:

$$v_{A/B} = v_A - v_B = -15 - 20 = -35\ \text{m s}^{-1}$$

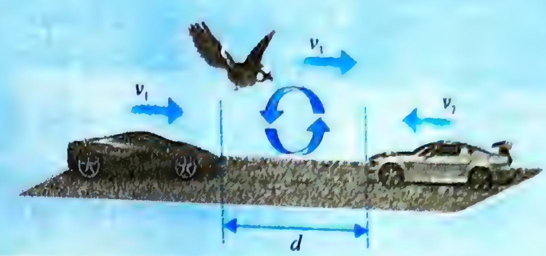
(Negative sign indicates that this relative velocity is in backward direction.)

Here the separation between A and B is increasing with time, hence the velocity of separation is 35 m s^{-1}.

Note: $\vec{v}_{B/A}$ and $\vec{v}_{A/B}$ are of equal magnitude but have opposite directions.

ILLUSTRATION 4.38

A bird flies to and fro between two cars which move with velocities v_1 and v_2. If the speed of the bird is v_3 and the initial distance of separation between them is d, find the total distance covered by the bird till the cars meet.



Sol. The magnitude of the relative speed of car '1' w.r.t. car '2'

$$v_{12} = v_1 + v_2$$

Time taken to cover distance d

$$t = \frac{d}{v_{12}} = \frac{d}{v_1 + v_2}$$

Since the speed of bird is v_3, therefore, distance covered by bird in time t

$$v_3 \times t = \frac{d}{v_1 + v_2} v_3$$

ILLUSTRATION 4.39

A person walks up a stationary escalator in t_1 second. If he remains stationary on the escalator, then it can take him up in t_2 second. If the length of the escalator is L, then

(a) Determine the speed of man with respect to the escalator.

(b) Determine the speed of the escalator.

(c) How much time would it take him to walk up the moving escalator?

Sol.

(a) As the escalator is stationary, so the distance covered in t_1 seconds is L which is the length of the escalator.

Speed of man w.r.t. the escalator, $v_{me} = \dfrac{L}{t_1}$

(b) When the man is stationary, by taking man as reference point the distance covered by escalator is L in time t_2.

Speed of escalator, $v_e = \dfrac{L}{t_2}$

(c) Speed of man w.r.t. the ground, $v_m = v_{me} + v_e$

$$\Rightarrow v_m = \frac{L}{t_1} + \frac{L}{t_2} = L\left[\frac{1}{t_1} + \frac{1}{t_2}\right] = L\left[\frac{t_1 + t_2}{t_1 t_2}\right]$$

$$\therefore \quad t = \frac{L}{v_m} = \left[\frac{t_1 t_2}{t_1 + t_2}\right]$$

which is the time taken by the man to walk up the moving escalator.

ILLUSTRATION 4.40

Suppose you are riding a bike with a speed of 10 m s^{-1} due east relative to a person A who is walking on the ground towards east. If your friend B walking on the ground due west measures your speed as 15 m s^{-1}, find the relative velocity between two reference frames A and B.

Sol. Taking east direction as positive (positive east)

$$\vec{v}_{\text{you},A} = 10\ \text{m s}^{-1}$$

$$\vec{v}_{A,B} = \vec{v}_{A,\text{you}} + \vec{v}_{\text{you},B} \quad \text{...(i)}$$

$$\vec{v}_{\text{you}} = \vec{v}_{\text{you},A} + \vec{v}_{A} \quad \text{...(ii)}$$

As the direction of $\vec{v}_{\text{you},A}$ and $\vec{v}_A$ is both toward east. Hence, your direction should be toward east.

$$\vec{v}_{\text{you},B} = \vec{v}_{\text{you}} - \vec{v}_{B} \quad \text{...(iii)}$$

As B is moving toward west and you are moving towards east. Hence, your direction w.r.t B should be toward east.

$$\vec{v}_{\text{you},B} = 15\ \text{m s}^{-1}\ \text{(towards east)}$$

Now from (i), $\vec{v}_{A,B} = -10 + 15 = 5\ \text{m s}^{-1}$

Hence, velocity of A w.r.t B will be 5 m s^{-1} toward east.

ILLUSTRATION 4.41

Two towns A and B are connected by a regular bus service with a bus leaving in either direction every T min. A man cycling with a speed of 20 km h^{-1} in the direction A to B notices that a bus goes past him every 18 min in the direction of his motion, and every 6 min in the opposite direction. What is the period T of the bus service and with what speed (assumed constant) do the buses ply on the road?

Sol. Let speed of each bus = v km h^{-1}

The distance between the nearest buses plying on either car = vT km ...(i)

For buses going from town A to B:

Relative speed of bus in the direction of motion of man = $(v - 20)$

Buses plying in this direction go past the cyclist after every 18 min. Therefore, separation between the buses $= (v-20)\times\frac{18}{60}$

From (i), $(v-20)\times\frac{18}{60} = vT$ ...(ii)

For buses coming from B to A:

The relative velocity of bus with respect to man = $(v + 20)$

Buses coming from town B past the cyclist after every 6 min,

$$\therefore \quad (v+20)\times\frac{6}{60} = vT \quad \text{...(iii)}$$

Solving (ii) and (iii), we get

$$v = 40\ \text{km h}^{-1} \text{ and } T = \frac{3}{20}\text{h} = 9\text{min}$$

ILLUSTRATION 4.42

A police van moving on a highway with a speed of 30 km h^{-1} fires a bullet at a thief's car speeding away in a same direction with a speed of 192 km h^{-1}. If the muzzle speed of the bullet is 150 m s^{-1}, with what speed does the bullet hit the thief's car?

Sol. Speed of police van $= 30\times\frac{5}{18} = \frac{25}{3}\ \text{m s}^{-1}$

The muzzle velocity, that is, the velocity of bullet with respect to van is

$$[\vec{v}_{\text{bullet}}]_{\text{van}} = [\vec{v}_{\text{bullet}}]_{\text{ground}} - [\vec{v}_{\text{van}}]_{\text{ground}}$$

$$[\vec{v}_{\text{bullet}}]_{\text{ground}} = [\vec{v}_{\text{bullet}}]_{\text{van}} + [\vec{v}_{\text{van}}]_{\text{ground}}$$

$$= 150 + \frac{25}{3} = \frac{475}{3}\ \text{m s}^{-1}$$

Speed of thief's car $= 192\times\frac{5}{18} = \frac{160}{3}\ \text{m s}^{-1}$

Now velocity of bullet with respect to the thief's car

$$[\vec{v}_{\text{bullet}}]_{\text{car}} = [\vec{v}_{\text{bullet}}]_{\text{ground}} - [\vec{v}_{\text{car}}]_{\text{ground}}$$

$$= \frac{475}{3} - \frac{160}{3} = 105\ \text{m s}^{-1}$$

Hence, the bullet hits the thief's car with speed 105 m s^{-1}.

ILLUSTRATION 4.43

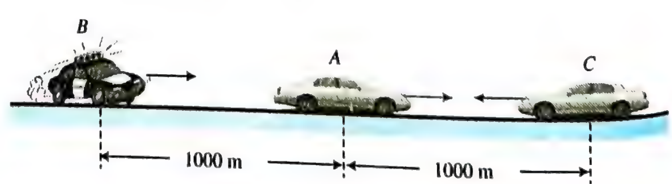
On a two lane road, car A is travelling with a speed of 36 km h^{-1}. Two cars B and C approach car A in opposite directions with a speed of 54 km h^{-1}. At a certain instant, when the distance AB is equal to AC, both 1 km, B decided to overtake A before C does. What minimum acceleration of car B is required to avoid an accident?

Sol. At the instant when car B decides to overtake car A, the velocities of cars are:

$$v_A = 36\times\frac{5}{18} = 10\ \text{m s}^{-1}$$

$$v_B = 54\times\frac{5}{18} = 15\ \text{m s}^{-1}$$

and $$v_C = -54\times\frac{5}{18} = -15\ \text{m s}^{-1}$$



Velocity of car B relative to A,

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 15 - 10 = 5\ \text{m s}^{-1}$$

Velocity of car C relative to A,

$$\vec{v}_{CA} = \vec{v}_C - \vec{v}_A = -15 - 10 = -25\ \text{m s}^{-1}$$

Time required by car C to just cross A

$$= \frac{1000}{v_{CA}} = \frac{1000}{25} = 40 \text{ s}$$

In order to avoid accident, car B must overtake A in this time. So

$$1000 = v_{BA}t + \frac{1}{2}a_{BA}t^2$$

$$1000 = 5 \times 40 + \frac{1}{2}a_{BA} \times 40^2$$

$$a_{BA} = 1 \text{ m s}^{-2}$$

Thus, the minimum acceleration that car B requires to avoid an accident is 1 ms^{-2}.

ILLUSTRATION 4.44

A car travelling at 60 km h^{-1} overtake another car traveling at 42 km h^{-1}. Assuming each car to be 5.0 m long, find the time taken during the overtake and the total road distance used for the overtake.

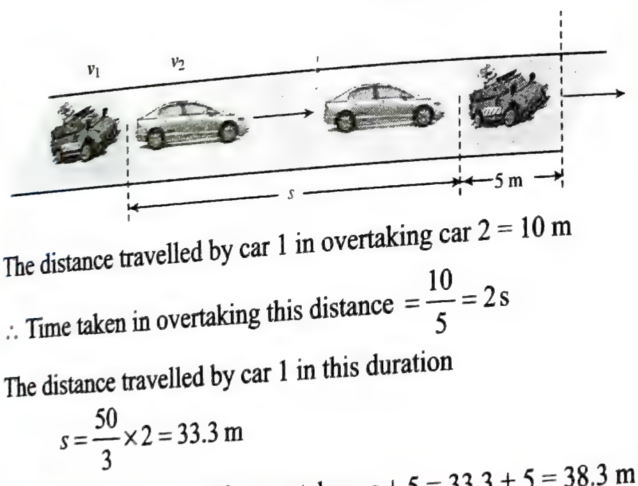
Sol.

The velocity of car which is overtaking $v_1 = 60 \times \frac{5}{18} = \frac{50}{3}$ m s^{-1}

and the velocity of car to be overtaken, $v_2 = 42 \times \frac{5}{18} = \frac{35}{3}$ m s^{-1}

The relative velocity between them $v_{12} = v_1 - v_2$

$$= \frac{50}{3} - \frac{35}{3} = 5 \text{ m s}^{-1}$$



The distance travelled by car 1 in overtaking car 2 = 10 m

$\therefore$ Time taken in overtaking this distance $= \frac{10}{5} = 2$ s

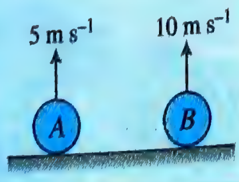
The distance travelled by car 1 in this duration

$$s = \frac{50}{3} \times 2 = 33.3 \text{ m}$$

The total distance used for overtake = s + 5 = 33.3 + 5 = 38.3 m

ILLUSTRATION 4.45

Two particles A and B are thrown vertically upward with velocity, 5 m s^{-1} and 10 m s^{-1}, respectively (g = 10 m s^{-2}). Find separation between them after 1 s.



Sol. **Method-1:** Position of A after 1 s

$$S_A = ut - \frac{1}{2}gt^2$$

$$= 5t - \frac{1}{2} \times 10 \times t^2 = 5 \times 1 - 5 \times 1^2 = 5 - 5 = 0$$

i.e., the particle will return to ground at t = 1 s

The position of B after 1 s

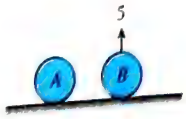
$$S_B = ut - \frac{1}{2}gt^2 = 10 \times 1 - \frac{1}{2} \times 10 \times 1^2 = 10 - 5 = 5 \text{ m}$$

Hence, separation between A and B, $S_B - S_A = 5$ m

Method-2: Relative velocity method:

Acceleration of B w.r.t A

$$\vec{a}_{BA} = \vec{a}_B - \vec{a}_A = (-10) - (-10) = 0$$



Initial relative velocity

Also $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = 10 - 5 = 5$ m s^{-1}

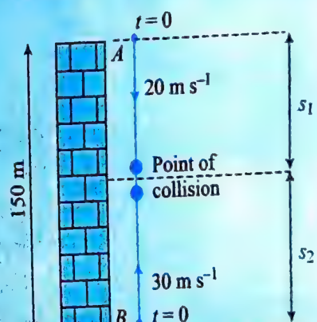
Hence, relative separation between particles

$\therefore$ $\vec{s}_{BA}$ (in 1 s) = $\vec{v}_{BA} \times t = 5 \times 1 = 5$ m

$\therefore$ Distance between A and B after 1 s = 5 m

ILLUSTRATION 4.46

A ball is thrown downwards with a speed of 20 m s^{-1} from the top of a building 150 m high and simultaneously another ball is thrown vertically upwards with a speed of 30 m s^{-1} from the foot of the building. Find the time after which both the balls will meet. (g = 10 m s^{-2})



Sol. **Method-1:** Let the first ball move down distance S_1 and second ball moves up a distance S_2 before they meet.

Let us take downward direction as positive.

Then $S_1 = 20t + 5t^2$

$S_2 = 30t - 5t^2$

But, $S_1 + S_2 = 150$

$\Rightarrow$ $150 = 50t$

$\Rightarrow$ $t = 3$ s

Method-2: Let us solve this problem by using the method of relative velocity.

Relative acceleration of both is zero since both have same acceleration in downward direction.

$$\vec{a}_{AB} = \vec{a}_A = -\vec{a}_B = g - g = 0$$

Initial relative velocity, $\vec{v}_{BA} = 30 - (-20) = 50$ m s^{-1}

Relative separation between the particles $s_{BA} = v_{BA} \times t$

Hence, required time, $t = \frac{s_{BA}}{v_{BA}} = \frac{150}{50} = 3$ s

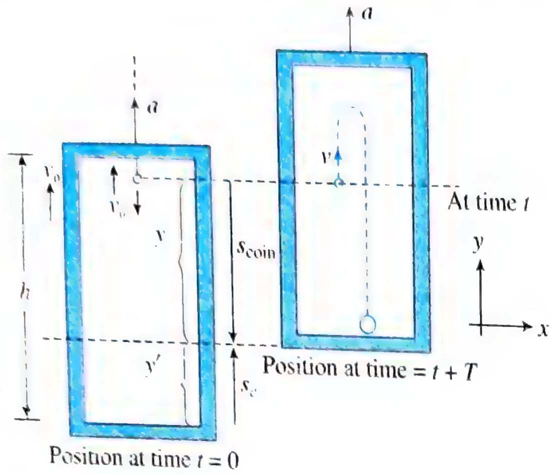
ILLUSTRATION 4.47

An elevator is moving with an upward acceleration a. A coin is dropped from rest from the roof of the elevator, relative to you. After what time the coin will strike the base of the elevator?

Sol. **Method 1:** Observation from ground frame

Let us assume that the elevator has a velocity v_0 when the coin loses contact with the elevator at time $t = 0$. Let the coin strike the base of the elevator after time $t = T$, just below the point of loosing contact with roof of the coin.

In time T, the elevator moves up the displacement y' (upward) and the coin has net displacement $(h - y') = y$ (downward)



For motion of coin

Displacement of coin $= -y = -(h - y')$

Using equation $S = ut + \frac{1}{2}at^2$,

$$-(h - y') = v_0T - \frac{1}{2}gT^2 \quad \text{...(i)}$$

For motion of elevator:

Displacement of elevator during time $T = +y'$

$$+y' = v_0T + \frac{1}{2}aT^2 \quad \text{...(ii)}$$

Adding (i) and (ii), we have $h = \frac{1}{2}(g + a)T^2$

This yields $T = \sqrt{\frac{2h}{g+a}}$

Method 2: Analyze the motion of coin with respect to the observer standing in the elevator. As the coin releases from rest inside elevator, its velocity with respect to ground is equal to the velocity of elevator.

Initial relative velocity of coin w.r.t. observer in the elevator,

$$[\vec{u}_{coin}]_{elevator} = [\vec{u}_{coin}]_{ground} - [\vec{u}_{elevator}]_{ground} = v_0 - v_0 = 0$$

and acceleration of coin with respect to the observer in the elevator,

$$[\vec{a}_{coin}]_{elevator} = [\vec{a}_{coin}]_{ground} - [\vec{a}_{elevator}]_{ground}$$
$$= -g - (+a) = -(g + a)$$

Now using second equation for relative motion

$$[\vec{s}_{coin}]_{elevator} = [\vec{u}_{coin}]_{elevator}[\vec{u}_{coin}]_{elevator}t + \frac{1}{2}[\vec{a}_{coin}]_{elevator}t^2$$

$$-h = 0 - \frac{1}{2}(g + a)t^2$$

This yields $t = \sqrt{\frac{2h}{g+a}}$

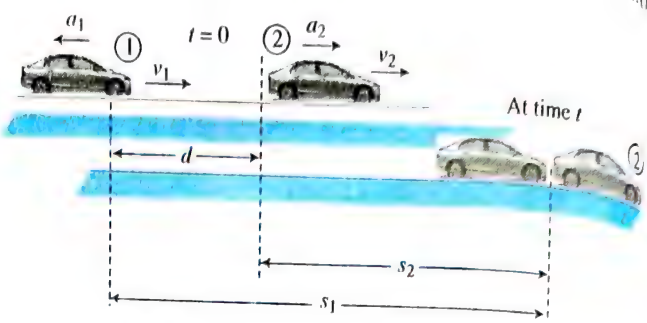
ILLUSTRATION 4.48

Two cars 1 and 2 move with velocities v_1 and v_2, respectively, on a straight road in same direction. When the cars are separated by a distance d, the driver of car 1 applies brakes and the car moves with uniform retardation a_1. Simultaneously, car 2 starts accelerating with a_2. If $v_1 > v_2$, find the minimum initial separation between the cars to avoid collision between them.

Sol. **Method 1:** Let us assume that the cars undergo displacements s_1 and s_2 after a time t from the instant of braking.

From figure, $s_1 - s_2 = d$ ...(i)

where $s_1 = v_1t - \frac{1}{2}a_1t^2$ ...(ii)



and $s_2 = v_2t + \frac{1}{2}a_1t^2$ ...(iii)

Using (i), (ii), and (iii), we have

$$\left(v_1t - \frac{1}{2}a_1t^2\right) - \left(v_2t + \frac{1}{2}a_1t^2\right) = d$$

or $(a_1 + a_2)t^2 - 2(v_1 - v_2)t + 2d = 0$

which is quadratic equation in t, solving for t

This gives $t = \frac{(v_1 - v_2) \pm \sqrt{(v_1 - v_2)^2 - 2(a_1 + a_2)d}}{a_1 + a_2}$

If cars do not collide, $b^2 - 4ac < 0$

$$(v_1 - v_2)^2 - 2(a_1 + a_2)d < 0$$

$$\frac{(v_1 - v_2)^2}{2(a_1 + a_2)} < d$$

Hence, $d_{min} = \frac{(v_1 - v_2)^2}{2(a_1 + a_2)}$

Method 2: Relative velocity approach

Using $v_{rel}^2 = u_{rel}^2 + 2a_{rel} \cdot s_{rel}$, we can obtain the same result.

Substitute: $v_{rel} = v'_{12} = v'_1 - v'_2$, $u_{rel} = v_1 - v_2$,

$a_{rel} = a_{12} = (-a_1) - a_2 = -(a_1 + a_2)$ and $s_{rel} = s_{12} = -d$,

to obtain $(v'_1 - v'_2)^2 = (v_1 - v_2)^2 - 2(a_1 + a_2)d$

Since $v'_1 < v'_2$ to avoid collision, we have

$$d = \frac{(v_1 - v_2)^2 - (v'_1 - v'_2)^2}{2(a_1 + a_2)}$$

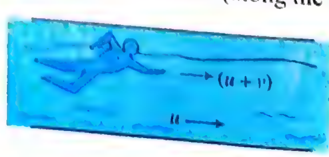
Maximum distance d for collision is possible when $v'_1 = v'_2$,

then, $d_{max} = \frac{(v_1 - v_2)^2}{2(a_1 + a_2)}$

RIVER–MAN PROBLEM IN ONE DIMENSION

The velocity of river water current is u and that of man in still water is v, i.e., man can swim in water with velocity v.

Case 1: Man swimming downstream (along the direction of river flow)

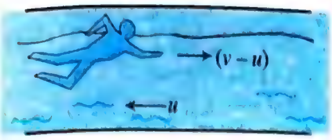


In this case, velocity of river $v_R = +u$

Velocity of man w.r.t. river $v_{mR} = +v$

Now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = u + v$

Case 2: Man swimming upstream (opposite to the direction of river flow)



In this case, velocity of river $\vec{v}_R = -u$

Velocity of man w.r.t. river $\vec{v}_{mR} = +v$

Now $\vec{v}_m = \vec{v}_{mR} + \vec{v}_R = (v-u)$

ILLUSTRATION 4.49

A swimmer capable of swimming with velocity v relative to water jumps in a flowing river having velocity u. The man swims a distance d down stream and returns back to the original position. Find out the time taken in complete motion.

Sol. Total time = Time of swimming downstream + Time of swimming upstream

Velocity of the man during swimming downstream = $v + u$

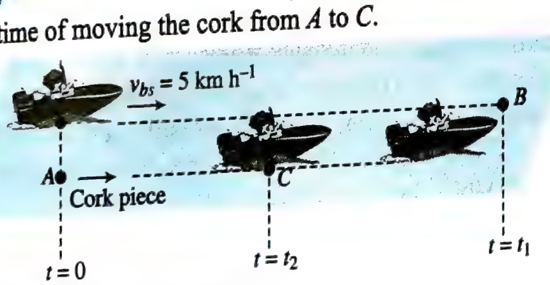
Velocity of the man during swimming upstream = $v - u$

$$t = t_{\text{down}} + t_{\text{up}} = t_{\text{down}} + t_{\text{up}} = \frac{d}{v+u} + \frac{d}{v-u} = \frac{2dv}{v^2 - u^2}$$

ILLUSTRATION 4.50

Let us consider a boat which moves with a velocity $v_{bw} = 5$ km h^{-1} relative to water. At time $t = 0$, the boat passes through a piece of cork floating in water while moving downstream. If it turns back at time $t = t_1$, when and where does the boat meet the cork again? Assume $t_1 = 30$ min.

Sol. Time of travelling of boat from A to B (t_1) and then B to C (t_1') = time of moving the cork from A to C.



Velocity of boat from A to B

$$\vec{v}_{b,w} + \vec{v}_w = (5 + u) \text{ km h}^{-1}$$

And velocity of boat from B to C

$$\vec{v}_{b,w} + \vec{v}_w = (5 - u) \text{ km h}^{-1}$$

Distance moved by boat in time t_1

$$AB = (5 + u)t_1$$

And distance moved by boat in time $t_1' = BC = (5 - u)t_1'$

Distance moved by cork during this time

$$AC = u(t_1 + t_1')$$

But $AB = AC + BC$

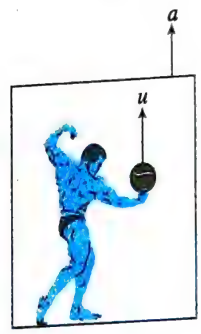
$$(5+u)t_1 = u(t_1 + t_1') + (5-u)t_1'$$

$$5t_1 = 5t_1' \Rightarrow t_1' = t_1 = 30 \text{ min}$$

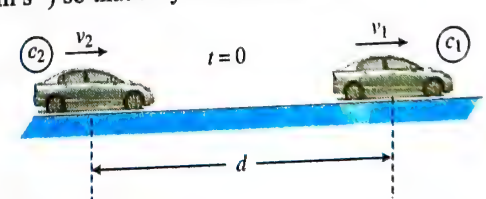
Hence, the cork meets the boat again after 1 h.

CONCEPT APPLICATION EXERCISE 4.4

1. A train 200 m long is moving with a velocity of 72 km h^{-1}. Find the time taken by the train to cross the bridge 1 km long.
2. Two cars A and B are moving on the straight parallel paths with speeds 36 km h^{-1} and 72 km h^{-1}, respectively, starting from the same point in the same direction. After 20 min, how much behind is car A from car B?
3. Two trains 110 m and 90 m long, respectively, are running in opposite directions with velocities 36 km h^{-1} and 54 km h^{-1}. Find the time taken by the two trains to completely cross each other.
4. A moving sidewalk in an airport terminal building moves at a speed of 1.0 m s^{-1} and is 35.0 m long. If a woman steps on at one end and walks at 1.5 m s^{-1} relative to the moving sidewalk, then find the time that she requires to reach the opposite end (a) when she walks in the same direction the sidewalk is moving and (b) when she walks in the opposite direction.
5. A lift is moving up with acceleration a. A person inside the lift throws the ball upwards with a velocity u relative to hand.



   (a) What is the time of flight of the ball?
   (b) What is the maximum height reached by the ball in the lift?
6. Two cars C_1 and C_2 moving in the same direction on a straight single lane road with velocities $v_1 = 12$ m s^{-1} and $v_2 = 10$ m s^{-1}, respectively. When the separation between the two was $d = 200$ m, C_2 started accelerating to avoid collision. What is the minimum acceleration of car C_2 (in cm s^{-2}) so that they do not collide?



7. Two boys enter a running escalator at the ground floor in a shopping mall and they do some fun on it. The first boy repeatedly follows $p_1 = 1$ step up and then $q_1 = 2$ steps down whereas the second boy repeatedly follows $p_2 = 2$ steps up and then $q_2 = 1$ step down. Both of them move relative to escalator with speed $v_r = 50$ cm s^{-1}. If the first boy takes $t_1 = 250$ s and the second boy takes $t_2 = 50$ s to reach the first floor, how fast is the escalator running?

8. The speed of a motor launch with respect to water in a stream is 8 $m s^{-1}$, while water current's speed is 3 $m s^{-1}$. When the launch began travelling upstream, a float was dropped from it. After travelling a distance of 4.8 km upstream, the launch turned back and caught up with the float. What is the total time which elapsed during the process?

9. Two boats A and B moved away from a buoy anchored in the middle of a river. A moved along the river and B at right angle to it. Having moved off equal distances from the buoy, the boats returned. Find the ratio of the times of motion of the boats, if the velocity of each boat with respect to still water is η times greater than the velocity of water current.

10. A ship of length l = 150 m moving with velocity v_s = 36 $km h^{-1}$ on the sea, suddenly discovered straight ahead a sinking boat people having met an accident. A rescue boat has been lowered from the mid of the ship, which went to the sinking boat with speed v_b = 72 $km h^{-1}$. When the rescue boat overtakes the leading edge of the ship, the sinking boat was x_0 = 3.0 km away. The rescue boat reaches the sinking boat, spends t_0 = 1.0 min there to take the people on board, and then returned with the same speed to the mid of the ship where it was lowered. Determine the time taken in the whole rescue operation from the moment the rescue boat was lowered to the moment the rescue boat returned to the ship.

11. A 10-km long straight road connects two towns A and B. Two cyclists simultaneously start one from town A and the other from town B. On reaching the opposite town, a cyclist immediately returns to his starting town whereas the other cyclist takes some rest and then returns to his starting town. Both of them can ride at speed 20 $km h^{-1}$ in absence of wind but during their whole journey uniform wind from town A to B increases the speed of the cyclist going into the wind by the same amount as it decreases the speed of the cyclist going against the wind. Both the cyclists meet twice, first at 2 km and then 6 km away from one of the towns. For what period does a cyclist rest?

Answers

1. 60 s **2.** 12 km **3.** 8 s **4. (a)** 14 s **(b)** 70 s

5. (a) $\dfrac{2u}{(g+a)}$ **(b)** $\dfrac{u^2}{2(g+a)}$ **6.** 1 $cm s^{-2}$

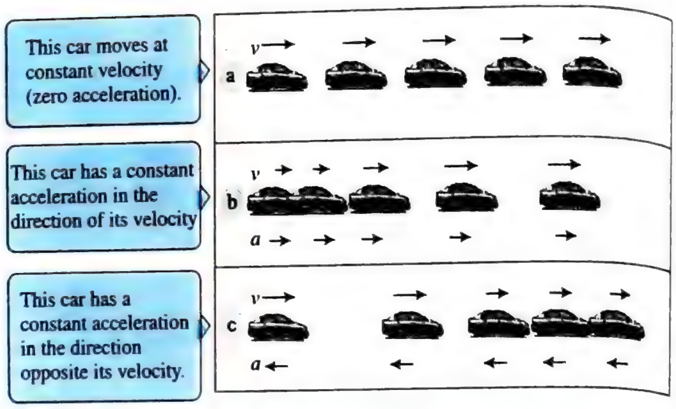
7. 25 $cm s^{-1}$ **8.** 32 min **9.** $\dfrac{n}{\sqrt{n^2-1}}$ **10.** 250 s

11. 18.75 min

GRAPHS IN MOTION IN ONE DIMENSION

Motion Diagrams

The concepts of velocity and acceleration are often confused with each other, but in fact they are quite different quantities. In forming a mental representation of a moving object, a pictorial representation called a *motion diagram* is sometimes useful to describe the velocity and acceleration while an object is in motion.



Graphical Representation for Motion

Graphical analysis is a very effective method of studying the motion of a particle. Indeed the method of analyzing situations graphically can be effectively applied not only to motion, but to any field. The variations of two quantities (related) with respect to each other can be demonstrated by means of a graph. The greatest advantage of depicting variables by means of curves lies in the fact that the whole situation can be understood at any instant. In other words, graphs reveal much more than that revealed by a table. In this section, we will study and interpret various types of graphs related to motion such as displacement–time graph, velocity–time graph, etc.

For graphical representation, we require two coordinate (reference) axes, one variable being taken along one axis. The usual practice is to take the independent variable along x-axis and the dependent on along y-axis. In general cases, involving time as one of the variables, time being independent, is usually taken along x-axis.

Graphs play very important role in analyzing a motion. Some times it becomes difficult to solve the problems analytically. But with the help of graphs, we can solve the problems easily and without much calculation.

How to Analyze and Draw the Graphs

In one-dimensional motion, generally, we come across *position–time* (or *displacement–time* graph), *velocity–time* graph, *acceleration–time* graph, etc.

Whenever we draw a graph, we need an equation involving the variables between which we have to draw the graph. For example, to draw position–time graph, we generally use the equation $x = ut + \frac{1}{2}at^2$, to draw velocity–time graph, we generally use the equation $v = u + at$, etc. Note that we can use these relations only when acceleration is constant.

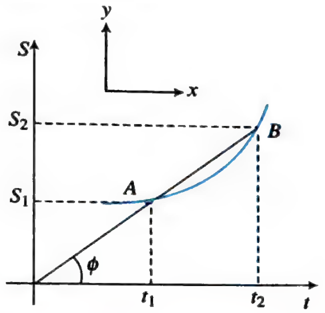
POSITION–TIME RELATION

Position–time graphs are the most basic form of graphs in kinematics, which allow us to describe the motion of objects. In these graphs, the vertical axis represents the position of the object while the horizontal axis represents the time elapsed: the dependent variable, position, depends on the independent variable, time. In this way, the graph tells us where the particle can be found after some amount of time. These graphs help us visualize the trajectory of objects. A lot of conclusions can be derived by studying a position–time graph for an object, as long as we know how to properly analyze them.

Average velocity As discussed earlier. The average velocity of a particle over a time interval is the total displacement divided by total time taken. In the above figure, average velocity of car while moving from position Ⓐ to position Ⓑ is

$$v_{\text{average}} = \frac{\Delta x}{\Delta t}$$

Let us take a general motion of a particle and plot its position–time graph. From the graph, the position of the particle in time interval from t_1 to t_2 changes from s_1 to s_2.

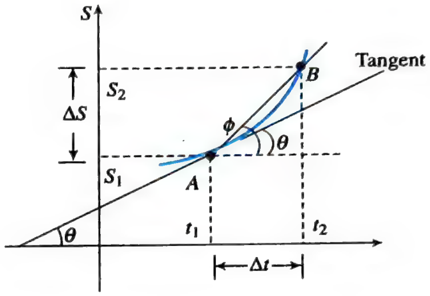


Hence, average velocity, $v_{av} = \dfrac{\Delta s}{\Delta t} = \tan\phi$,

$$v_{av} = \frac{\Delta s}{\Delta t} = \text{Slope of the line joining the points in } s\text{–}t \text{ graph.}$$

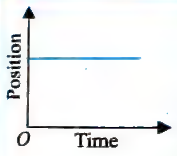
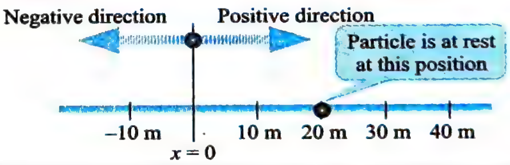
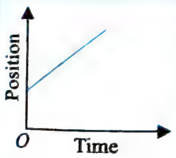
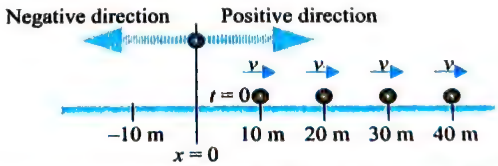
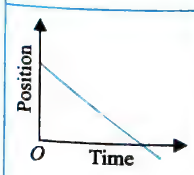
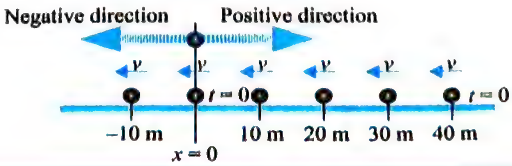
Note: The slope of a straight line connecting any two points in s–t graph gives the velocity averaged over the time interval $(t_2 - t_1)$.

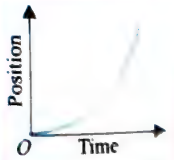
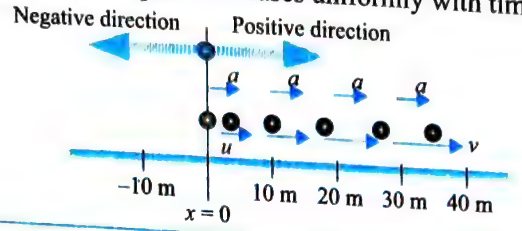
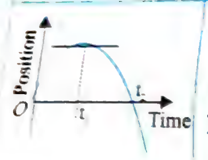
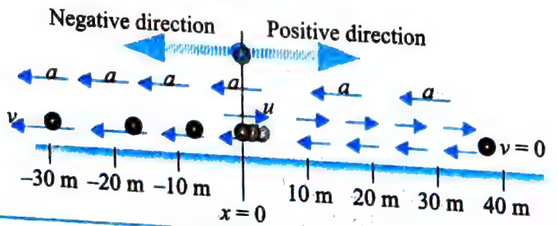
Instantaneous velocity In figure, $\Delta t \to 0, \tan\phi \to \tan\theta$, as making the line AB to coincide with tangent at A.



Hence, instantaneous velocity at A is equal to $v_A = ds/dt = \tan\theta$. We can say instantaneous velocity at any point is equal to the slope of s–t graph. If the slope is positive, v is positive (the particle moves along positive axes) and when the slope is negative, v is negative (directed along negative axes).

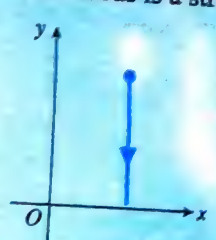
Table: Different Cases in Position–Time Graph

Graph	Analysis	Description
	The slope of position-time graph is zero, $\dfrac{dx}{dt} = 0$.	The particle is not moving 
	The slope of position-time graph is constant and positive, $\dfrac{dx}{dt}$ = constant and positive	The line with constant slope represents uniform velocity of the particle. The particle is moving from positive side of a reference point and keep on moving in positive x-direction. 
	The slope of position-time graph is constant and negative, $\dfrac{dx}{dt}$ = constant and positive	The line with constant slope represents uniform velocity of the particle. The particle is moving from positive side of a reference point and moving towards reference point in negative x-direction (negative displacement). 

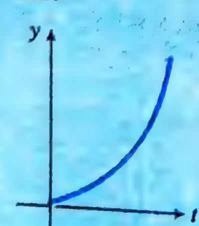
 Parabola opening up	The slope of position-time graph is not constant but positive and increasing with time.	• The positive slope of tangent at $t = 0$ represents the initial velocity of the particle is positive. • As slope of the curve is increasing with time, it means velocity is increasing. The particle is moving with positive acceleration. • The particle start moving from positive initial velocity and keep on moving in positive x-direction and the speed increases uniformly with time. 
 Parabola opening down	The slope of position-time graph is not constant and changing its sign from positive to negative.	• The positive slope of tangent at $t = 0$ represents the initial velocity of the particle is positive. • In time interval 0 to t_1, the slope of the curve is positive but decreasing with time, it means the particle is moving towards right and speed is decreasing. The particle is moving with negative acceleration. • At time t_1, the slope of the curve is zero, it means the particle momentary comes into rest. • After time t_1, the slope of the curve is negative. It means the particle is moving towards left and speed is increasing. 

Important Points:

When the displacement of a particle along y-axis is expressed as the function of displacement of the particle along x-axis, we call it "locus equation" or "equation of trajectory." But when the displacement of a particle along y-axis is expressed as the function of time, it is displacement–time graph. For example, when we drop a stone from rest, it moves down in a straight line. Hence, the locus is a straight line.



Trajectory of a particle

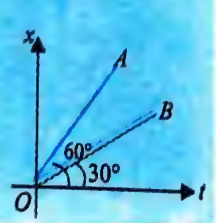


Position–time graph

If you plot the displacement versus time you will get a parabola. We call it displacement (or position)–time (r–t) graph which shows the variation of position with time.

ILLUSTRATION 4.51

The Position-time graphs for the two cars A and B moving along a straight road are shown in figure. What conclusions may be obtained from the graph ?



Sol.

- Both the cars move with uniform velocity i.e., with zero acceleration.
- The velocity of car A is more than that of B, and the ratio of velocities is given by

$$\frac{v_A}{v_B} = \frac{\tan 60°}{\tan 30°} = \frac{\sqrt{3}}{1/\sqrt{3}} = 3$$

ILLUSTRATION 4.52

The given figure is an x-t graph of the motion of a particle.

(i) At which of the points P, Q, R and S is the velocity v_x positive?

(ii) At which points is the velocity negative?

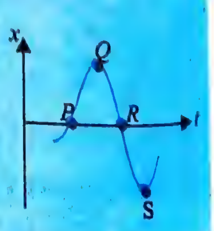
(iii) At which points is the velocity zero?

(iv) At which of the points P, Q, R, and S is the x-acceleration a_x positive?

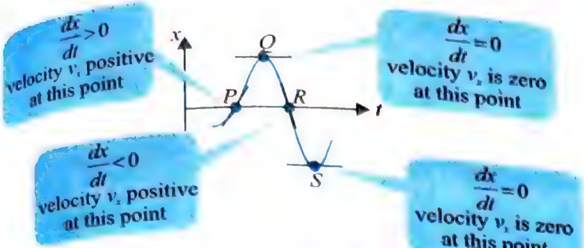
(v) At which point is the a_x negative?

(vi) At which points does the x-acceleration appear to be zero?

(vii) At each point, state whether the speed is increasing, decreasing, or not changing.



In position time graph velocity, $v_x = \frac{dx}{dt}$ = slope of tangent at that point



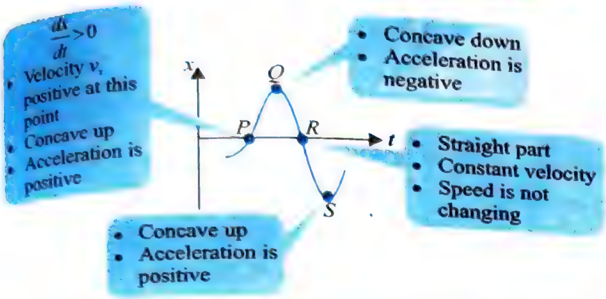
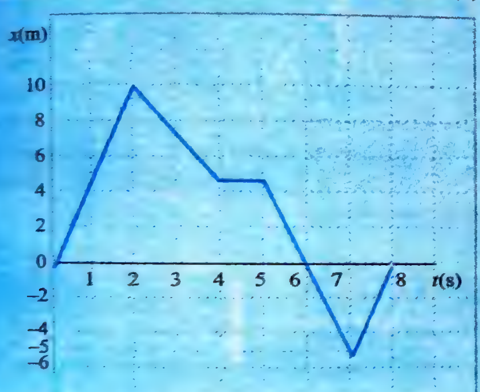


ILLUSTRATION 4.53

The position versus time graph for a certain particle moving along the x-axis is shown in figure. Find the average velocity in the time intervals (a) 0 to 2 s, (b) 2 s to 4 s, and (c) 4 s to 7 s.



Sol. We are given the values of time, but the values of the position are to obtained from the graph corresponding to the given time interval under consideration. We know that the slope of the displacement–time graph represents velocity.

Formula to be used: $v_{av} = \frac{x_2 - x_1}{t_2 - t_1}$, where x_1 and x_2 are the initial and final positions, respectively, and t_1 and t_2 are the initial and final values of time, respectively.

(a) Here $x_1 = 0$ m, $x_2 = 10$ m, $t_1 = 0$ s, $t_2 = 2$ s.
Therefore, the required average velocity is given by

$$v_{av} = \frac{10-0}{2-0} = 5 \text{ m s}^{-1}$$

(b) Here $x_1 = 10$ m, $x_2 = 5$ m, $t_1 = 2$ s, $t_2 = 4$ s.
So the required value of average velocity is given by

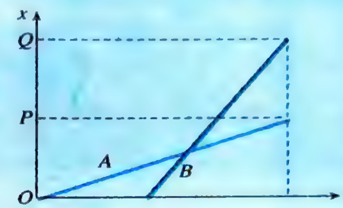
$$v_{av} = \frac{5-10}{4-2} = -2.5 \text{ m s}^{-1}$$

(c) Here $x_1 = 5$ m, $x_2 = -5$ m, $t_1 = 4$ s, $t_2 = 7$ s.
So the required value of average velocity is given by

$$v_{av} = \frac{-5-5}{7-4} = \frac{-10}{3} = -3.3 \text{ m s}^{-1}$$

ILLUSTRATION 4.54

The position–time (x–t) graphs for two children A and B returning from their school O to their homes P and Q, respectively, are shown in figure. Choose the correct entries in the brackets below.



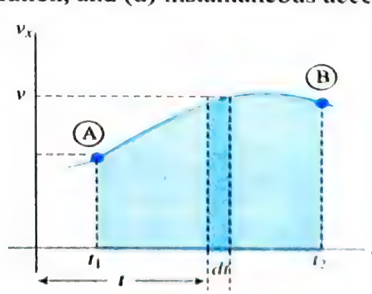
(a) (A/B) lives closer to school than (B/A).
(b) (A/B) starts from the school earlier than (B/A).
(c) (A/B) walks faster than (B/A).
(d) A and B reach home at the (same/different) time.
(e) (A/B) overtakes on the road (once/twice).

Sol.

(a) It is clear from the graph that $OP < OQ$. A lives closer to the school than B.
(b) As A starts from $t = 0$ while B starts little later. So A starts from the school earlier than B.
(c) The slope of x–t for motion of $B >$ slope of x–t of A. Hence B walks faster than A.
(d) The value of t corresponding to positions P and Q of there homes is same, so A and B reach home at the same time.
(e) It is clear from the graph that B overtakes A once on the road.

VELOCITY–TIME GRAPH

From v–t graph, we can find (a) displacement, (b) distance, (c) average acceleration, and (d) instantaneous acceleration.

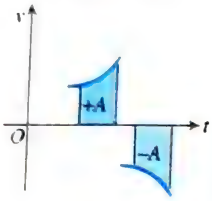


Displacement: When the velocity of a particle is given as the function of time, i.e., $v = f(t)$, we find the total displacement s by summing up (integrating) the elementary displacements ds

The elementary displacement $d\vec{s} = \vec{v}\,dt$ can be given as the area of the shaded elementary strip. Summing up all the elementary areas between $t = t_1$ and $t = t_2$, we have the total area under v–t graph enclosed within the time interval which gives the total displacement of the particle during the time interval $t(= t_2 - t_1)$.

$$\vec{s} = \int \vec{v}\, dt = A (= \text{Area under } v\text{–}t \text{ graph})$$

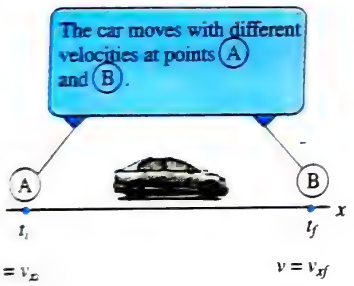
When the area lies above the time axis, it is said to be positive. A positive area reveals a positive displacement (displacement directed along positive directions of coordinate system) and negative area signifies a negative displacement.



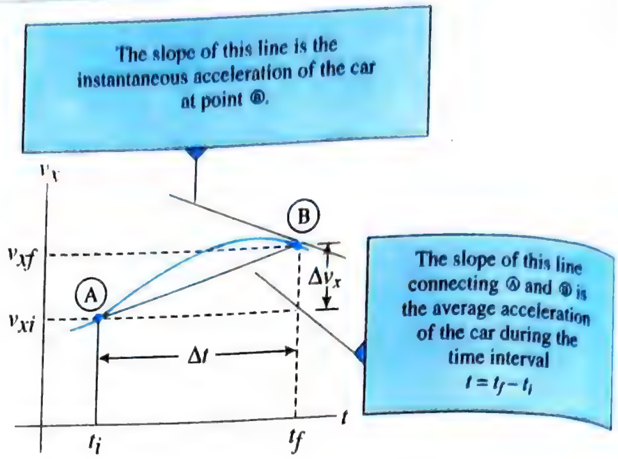
Distance: Since the distance covered D is given as $D = \int |\vec{v}|\, dt$, it can be given graphically as the sum of the magnitude of positive and negative areas under v–t graph.

$$D = \int |\vec{v}|\, dt = |A_+| + |A_-|$$

Acceleration:



a



b

Average acceleration: As derived earlier, the average acceleration

$$\vec{a}_{\text{av}} = \frac{\Delta \vec{v}}{\Delta t}$$

Then, $\vec{a}_{\text{av}}$ = slope of the line joining A and B in v–t graph as shown in figure.

The slope of the straight line AB that joins two points of time t_i and t_f under consideration gives the average acceleration over the time interval $\Delta t = t_f - t_i$.

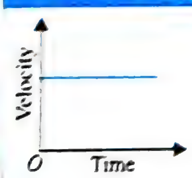
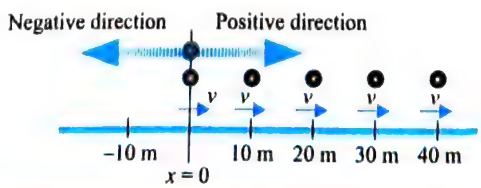
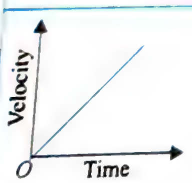
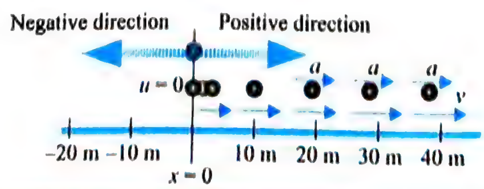
Instantaneous acceleration: The instantaneous acceleration is given as $\vec{a} = \dfrac{d\vec{v}}{dt}$

Then, $\vec{a}$ = slope of $v - t$ graph.

As we reduce the time interval Δt, the slope of the straight line AB tends to be equal to the slope of the tangent drawn at B. The slope of a line on v–t graph, that is (dv/dt) gives the acceleration.

If the slope of v–t graph is positive at any point of time on the graph, the acceleration of the particle is positive at that time. If the slope of v–t graph is zero, it signifies that the acceleration is zero. When the slope of v–t graph is negative, it means negative acceleration (or retardation).

Table: Different Cases in Velocity–Time Graphs

Graph	Analysis	Description
	The slope of velocity-time graph is zero, $\dfrac{dv}{dt} = 0$	The acceleration of moving particle is zero. The particle keep on moving in positive x-direction with constant velocity. 
	• The slope of velocity-time graph is constant and positive, $\dfrac{dv}{dt}$ = constant • Initial velocity of the particle is zero.	The acceleration is constant and in the direction of velocity. The particle start moving from rest and keep on moving in positive x-direction and the speed increases uniformly with time. 

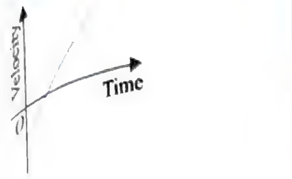
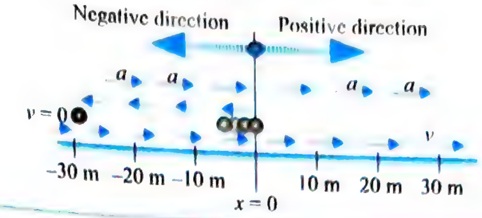
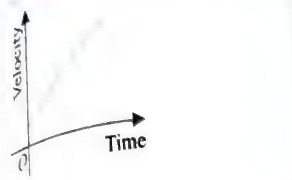
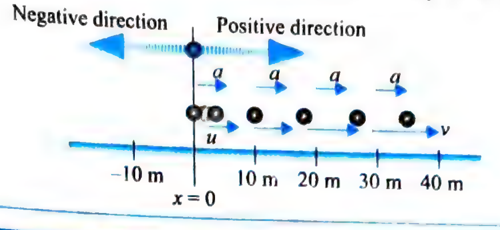
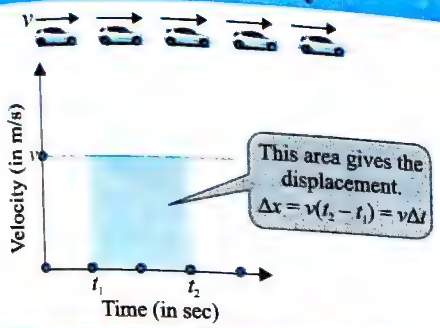
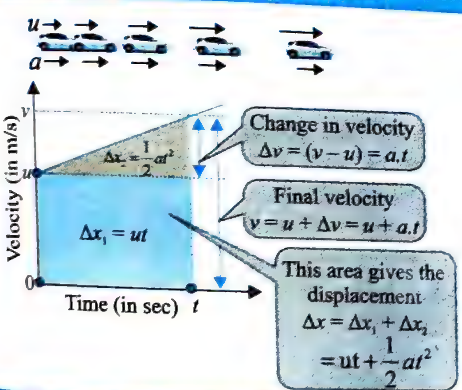
	• The slope of velocity-time graph is constant and positive, $\frac{dv}{dt} = \text{constant}$ • Initial velocity of the particle is negative.	The particle first moves in negative x-direction, the acceleration acts in positive x-direction. The speed of the particle decreases and becomes zero after some time. Then the particle starts moving in positive x-direction and keeps on moving in positive x direction. The speed of the particle is increasing uniformly with time. 
	• The slope of velocity-time graph is constant and positive, $\frac{dv}{dt} = \text{constant}$. • Initial velocity of the particle is positive.	The acceleration is constant and in the direction of velocity. The particle starts moving from positive initial velocity and keeps on moving in positive x-direction and the speed increases uniformly with time. 

Table: Kinematics Equations from Graph

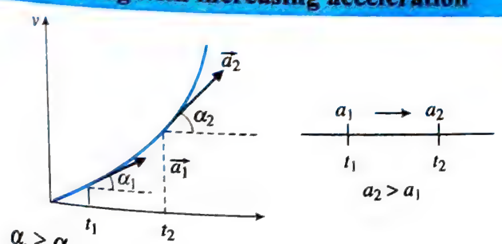
Particle moving with constant velocity



Particle moving with constant acceleration



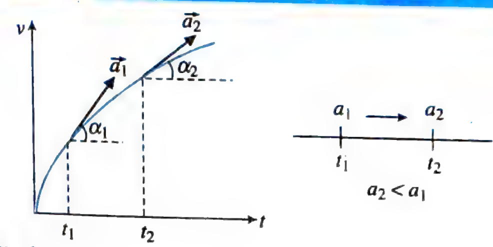
Particle is moving with increasing acceleration



$\alpha_2 > \alpha_1$

$\Rightarrow$ $\tan\alpha_2 > \tan\alpha_1$ $\Rightarrow$ $\alpha_2 > \alpha_1$

Particle is moving with decreasing acceleration



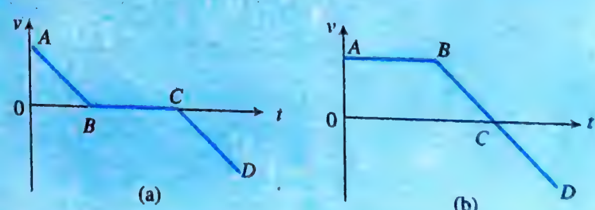
$\alpha_2 < \alpha_1$

$\Rightarrow$ $\tan\alpha_2 < \tan\alpha_1$

$\Rightarrow$ $\alpha_2 < \alpha_1$

ILLUSTRATION 4.55

Describe the motion shown by the following velocity–time graphs.



Sol.

For figure (a)

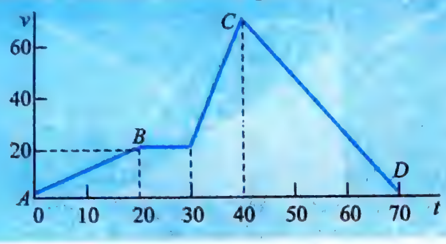
During interval AB	Velocity is +ve, so the particle is moving in +ve direction, but it is slowing down as acceleration (slope of v–t curve) is negative.
During interval BC	Particle remains at rest as the velocity is zero. Acceleration is also zero.
During interval CD	Velocity is –ve, so the particle is moving in –ve direction and is speeding up as acceleration is also negative.

For figure (b)

During interval AB	Particle is moving in +ve direction with constant velocity and acceleration is zero.
During interval BC	Particle is moving in +ve direction as velocity is +ve, but it slows down until it comes to rest as acceleration is negative.
During interval CD	Velocity is –ve so the particle is moving in –ve direction and is speeding up as acceleration is also negative.

ILLUSTRATION 4.56

The velocity versus time curve of a moving point is shown in figure. Find the retardation of the particle for the portion CD.

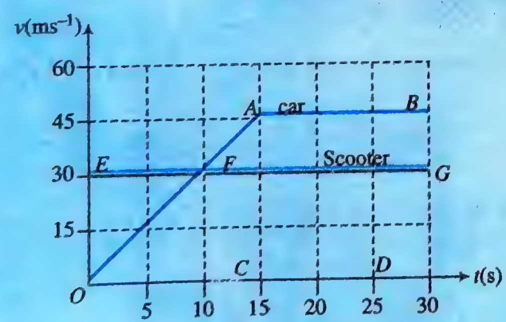


Sol. The slope of the velocity–time graph represents acceleration or retardation of the particle during motion. If the slope is positive, it represents acceleration and if the slope is negative, it represents retardation. The section CD of the graph represents retardation and magnitude of retardation is

$$|\vec{a}| = \frac{\text{Change in velocity}}{\text{Time taken}} = \frac{60}{(70-40)} = 2\ \text{ms}^{-2}$$

ILLUSTRATION 4.57

As soon as a car just starts from rest in a certain direction, a scooter moving with a uniform speed overtakes the car. Their velocity–time graph is shown in figure. Calculate



(a) The difference between the distances travelled by the car and the scooter in 15 s.

(b) The distance of car and scooter from the starting point at that instant when car catches scooter.

Sol.

(a) The distance travelled by car in 15 s = Area of ΔOAC

$$= \frac{1}{2}\times 15\times 45 = 337.5\ \text{m}$$

Distance travelled by scooter in 15 s

= Area of rectangle $OCEF$

$= 15 \times 30 = 450$ m

Thus, difference between distance travelled by them

$= 450\ \text{m} - 337.5\ \text{m} = 112.5\ \text{m}$

(b) Let after time t from start car will catch up the scooter. In time t, the distance travelled by them are equal.

Distance travelled by car $= \frac{1}{2}\times 15\times 45 + 45(t-15)$

Distance travelled by scooter $= 30t$

$$\frac{1}{2}\times 15\times 45 + 45(t-15) = 30t$$

which gives $t = 22.5$ s

Distance travelled by car or scooter in 22.5 s $= 30 \times 22.5$

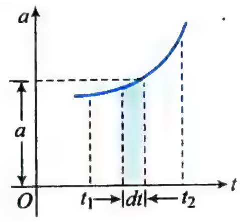
$= 675$ m

So the car catches the scooter when both are at 67.5 m from the starting point.

ACCELERATION–TIME GRAPH OF VARIOUS TYPES OF MOTIONS OF A PARTICLE

Let the acceleration be given as the function of time, i.e., $a = f(t)$. Then the elementary change in velocity during a time dt is: $d\vec{v} = \vec{a}\,dt$.

We can observe that in a–t graph, $\vec{a}\,dt$ is equal to the area of the shaded elementary strip (rectangle of sides a and dt; Fig.). Summing up all the elementary areas dA, we have the total area



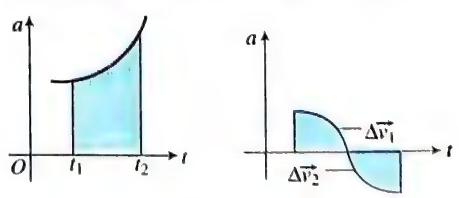
$$A = \int dA = \int \vec{a}\,dt$$

Since $\int_{t1}^{t_2} \vec{a}\,dt = \int_{v_1}^{v_2} d\vec{v} = \Delta\vec{v}$

we have $\Delta\vec{V} = \int \vec{a}\,dt = A$

= Area under $a-t$ graph

That means, the change in velocity $\Delta\vec{v}$ over any time interval is represented by the algebraic sum of all positive and negative areas which we call "area under a–t graph" over that time interval: $A = \Sigma A_+ + \Sigma A_-$. If the area is negative, $\Delta\vec{v}$ is negative; $\Delta\vec{v}$ is directed in negative direction of coordinate axes. When area is zero, $\Delta\vec{v} = 0$, which means no change in velocity.



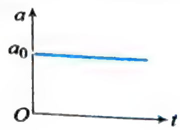
$$\Delta\vec{V} = \int \vec{a}\,dt = \text{Area under } a-t \text{ graph}$$

$A_+ \rightarrow \Delta\vec{v}_1$ is +ve; $A_- \rightarrow \Delta\vec{v}_2$ is –ve

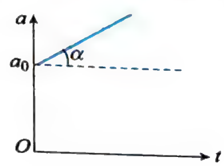
When $\Delta\vec{V}$ is positive, you should not immediately accept it as $\Delta|\vec{V}| > 0$, because $\Delta\vec{V}$ is a vector quantity. Hence, positive area means a positive $\Delta\vec{v}$ but not positive $\Delta|\vec{v}|(\neq \Delta V)$.

Different Cases in Acceleration–Time Graph

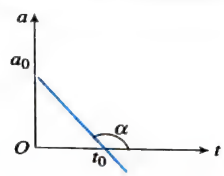
Particle is moving with constant acceleration: Let at time $t = 0$, acceleration is a_0. Since acceleration is constant, so acceleration at any further time will remain a_0. Hence, acceleration–time graph for a particle moving with constant acceleration is a straight line parallel to time axis.



Particle is moving with increasing acceleration at constant rate: Acceleration–time graph for a particle moving with increasing acceleration at constant rate is a straight line making an acute angle α with time axis. Graph will be straight line because acceleration is increasing at constant rate.



Particle is moving with decreasing acceleration at constant rate: Acceleration–time graph for a particle moving with decreasing acceleration at constant rate is a straight line making an obtuse angle α with time axis.

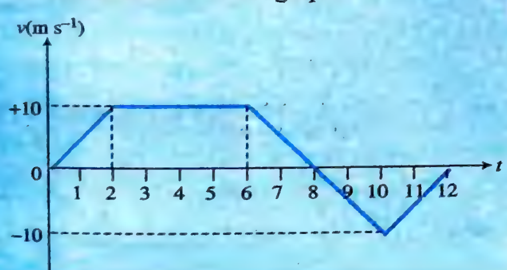


The graph will be a straight line because acceleration is decreasing at constant rate.
Let at $t = 0$, acceleration is a_0. At some time $t = t_0$, acceleration becomes zero and then it becomes negative.

ILLUSTRATION 4.58

The velocity–time graph of a body moving along a straight line is given below. Find:

(a) Average velocity in whole time of motion
(b) Average speed in whole time of motion
(c) Draw acceleration vs time graph



Sol. (a) Area of graph above time axis

$$A = \frac{1}{2}\times(8+4)\times 10 = 60\,\text{m}$$

Area of graph below time axis

$$B = \frac{1}{2}\times 4\times 10 = 20\,\text{m}$$

Average velocity in whole time of motion

$$\vec{v} = \frac{\text{Displacement}}{\text{Total time}}$$

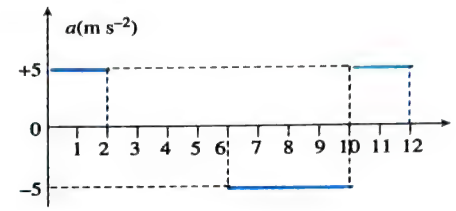
$$= \frac{\text{Area } A - \text{Area } B}{\text{Total time}} = \frac{60-20}{12} = 3.33\ \text{m s}^{-1}$$

(b) Average speed in whole time of motion

$$v_{av} = \frac{\text{Total distance}}{\text{Total time}}$$

$$= \frac{\text{Area } A + \text{Area } B}{\text{Total time}} = \frac{60+20}{12} = 6.67\ \text{m s}^{-1}$$

(c) Acceleration:



From 0 to 2 s,

$$a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{10-0}{2-0} = 5\ \text{m s}^{-2}$$

From 2 to 6 s, $a = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{10-10}{6-2} = 0$

From 6 to 10 s, $a = \dfrac{v_2 - v_1}{t_2 - t_1} = \dfrac{-10-10}{10-6} = -5\ \text{m s}^{-2}$

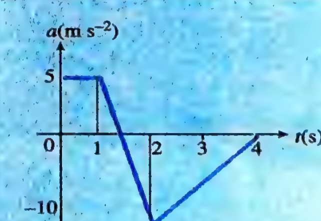
From 10 to 12 s,

$$a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{0-(-10)}{12-10} = 5\ \text{m s}^{-2}$$

Acceleration vs time graph is drawn in figure.

ILLUSTRATION 4.59

A particle moves along x-axis with an initial speed $v_0 = 5\ \text{m s}^{-1}$. If its acceleration varies with time as shown in a–t graph in figure,



(a) Find the velocity of the particle at $t = 4$s.
(b) Find the time when the particle starts moving along $-x$ direction.

Sol. The velocity of the particle at $t = 4$ s can be given as

$$\vec{v}_4 = \vec{v}_0 + \Delta\vec{v} \quad \text{...(i)}$$

where $\Delta\vec{v} \equiv A$
(= area under a–t graph during first four seconds)
Referring to a–t graph (figure), we have

$$A = A_1 + A_2 - A_3 - A_4 \quad \text{...(ii)}$$

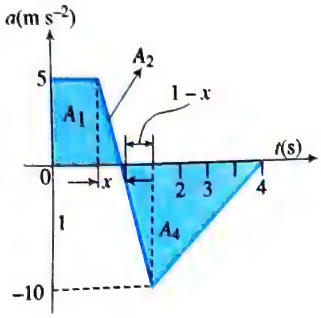
where $A_1 = 5\times 1 = 5$, $A_2 = \dfrac{1}{2}\times x\times 5$,

$$A_3 = \frac{1}{2}\times(1-x)\times 10, \text{ and } A_4 = \frac{1}{2}\times 2\times 10 = 10$$

We can find x as following:

Using properties of similar triangles, we have $\frac{x}{5} = \frac{1-x}{10}$

This yields $x = \frac{1}{3}$.



Substituting $x = \frac{1}{3}$ in $A_2 = \frac{1}{2} \times x \times 5$

and $A_3 = \frac{1}{2}(1-x) \times 10,$

we have $A_2 = \frac{5}{6}$ and $A_3 = \frac{10}{3}$.

Then substituting A_1, A_2, A_3, and A_4 in (ii), we have $A = -7.5$. Negative area tells us that change in velocity is along $-x$ direction

$$\Delta\vec{v} = -7.5 \text{ m s}^{-1}$$

Hence, substituting in (i),

$$\vec{v}_0 = 5 \text{ m s}^{-1} \text{ and } \Delta\vec{v} = -7.5 \text{ m s}^{-1},$$

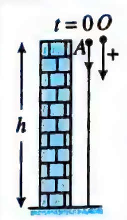
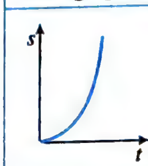
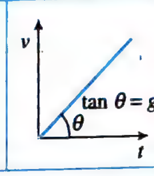
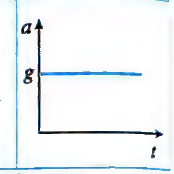
we have

$$\vec{v}_4 = \vec{v}_0 + \Delta\vec{v} = 5 - 7.5 = -2.5 \text{ m s}^{-1}.$$

Graphical Representation of Motion of A Particle Moving Under Gravity

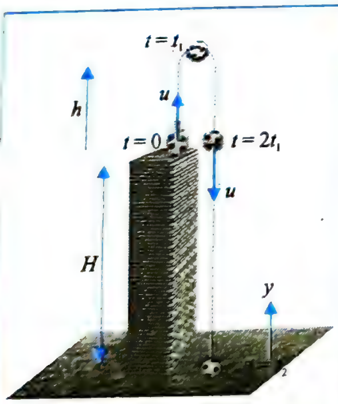
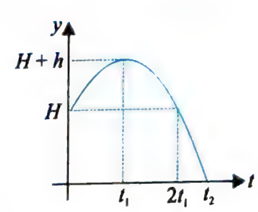
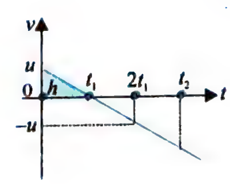
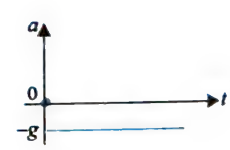
If a body is dropped from some height (initial velocity zero)

Taking initial position as origin and direction of motion (i.e., downward direction) as a positive.

	Position–time graph	Velocity–time graph	Acceleration–time graph
			

If a body is projected vertically upward from some height

Taking initial position as origin and direction of motion (i.e., downward direction) as a positive

	Position–Time Graph	Velocity–Time Graph	Acceleration–Time Graph
			

If a body is projected vertically upward

Taking initial position as origin and direction of motion (i.e., vertically up) as positive

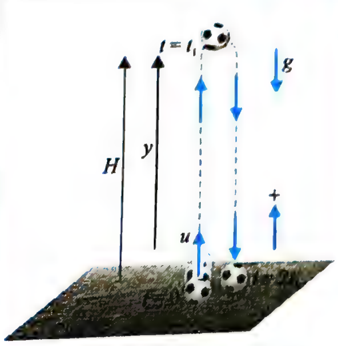
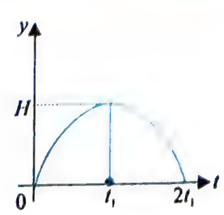
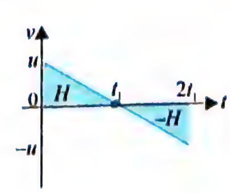
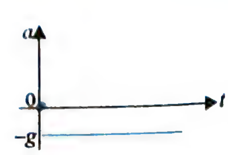
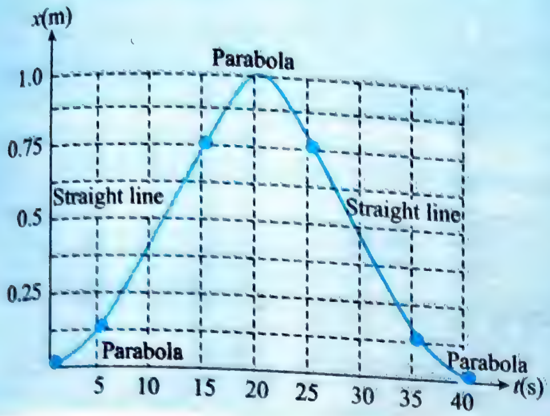
	Position–Time Graph	Velocity–Time Graph	Acceleration–Time Graph
			

ILLUSTRATION 4.60

The given figure is a graph of the coordinate of a spider crawling along the x-axis.

(a) Graph its velocity and acceleration as functions of time.

(b) In a motion diagram, show the position, velocity, and acceleration of the spider at the five times: $t = 2.5$ s, $t = 10$ s, $t = 20$ s, $t = 30$ s, and $t = 37.5$ s.



Sol. v_x is the slope of the x versus t curve and a_x is the slope of the v_x versus t curve.

$t = 0$ to $t = 5$ s: x versus t is a parabola, so a_x is a constant. The curvature is positive so a_x is positive. v_x versus t is a straight line with positive slope. $v_{0x} = 0$.

$t = 5$ s to $t = 15$ s: x versus t is a straight line, so v_x is constant and 0. $a_x = 0$. The slope of x versus t is positive, so v_x is positive.

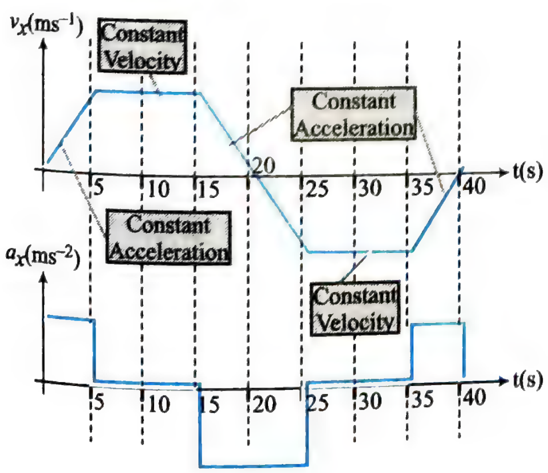
$t = 15$ s to $t = 25$ s: x versus t is a parabola with negative curvature, so a_x is constant and negative. v_x versus t is a straight line with negative slope. The velocity is zero at 20 s, positive to 15 s to 20 s, and negative to 20 s to 25 s.

$t = 25$ s to $t = 35$ s: x versus t is a straight line, so v_x is constant and $a_x = 0$. The slope of x versus t is negative, so v_x is negative.

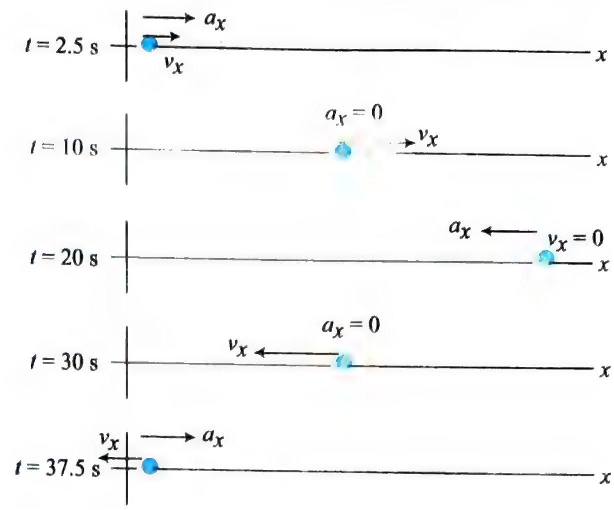
$t = 35$ s to $t = 40$ s: x versus t is a parabola with positive curvature, so a_x is constant and positive.

v_x versus t is a straight line with positive slope. The velocity reaches zero at $t = 40$ s.

The graphs of $v_x(t)$ and $a_x(t)$ are sketched in figure.



b. The motions diagrams are sketched in figure.



The spider speeds up for the first 5 s, since v_x and a_x are both positive. Starting at $t = 5$ s, the spider starts to slow down, stops momentarily at $t = 20$ s, and then moves in the opposite direction. At $t = 35$ s, the spider starts to slow down again and stops at $t = 40$ s.

ILLUSTRATION 4.61

A local train leaves station A; it gains speed at the rate of 1 m s^{-2} for first 6 s and then at the rate of 1.5 m s^{-2} until it has reached the speed of 12 m s^{-1}. The train maintains the same speed until it approaches station B; brakes are then applied, giving the train a constant deceleration and bringing it to a stop in 6 s. If the total running time of train is 40 s. Find (a) the distance between stations A and B. (b) Draw acceleration–time, velocity–time, and position–time relation of motion.

Sol.

Time interval	*a–t* graph	*v–t* graph	*x–t* graph
$0 < t < 6$ s $a_1 = 1$ m s^{-2}	Straight line parallel with time axis and above time axis.	Straight line with (+) slope passing through origin	Parabola open up and passing through origin
$6\text{ s} < t < t_1$ $a_2 = 1.5$ m s^{-2}	Straight line parallel with time axis and above time axis.	Straight line with (+) slope	Parabola open up
$t_1 < t < t_2$ Constant speed	Straight line parallel along time axis	Straight line parallel with time axis	Straight line with (+) slope
$t_2 < t < 40$ s Retardation	Straight line parallel with time axis and below time axis.	Straight line with (–) slope	Parabola open down

Analysis from acceleration–time curve: We have change in velocity = area under acceleration–time curve.

Denoting the velocity at $t = 6$ s as v_6,

For $0 < t < 6$ s: $v_6 - 0 = (6) \times (1) = 6$ m s^{-1}

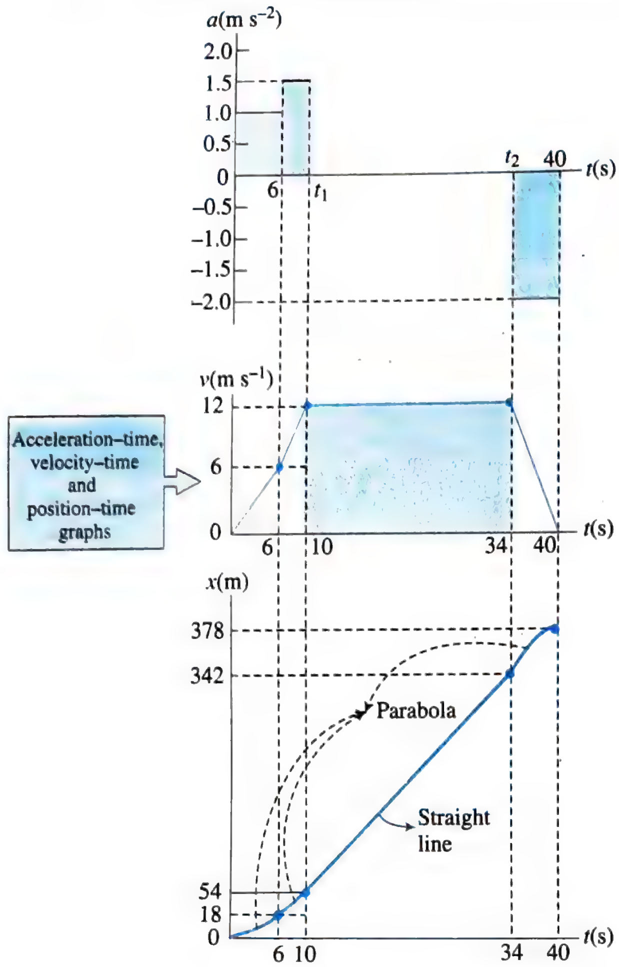
For $6\text{ s} < t < t_2$: $12 - 6 = (t_1 - 6)(1.5) \Rightarrow t_1 = 10$ s

$t_2 = 40 - 6 = 34$ s

For $10\text{ s} < t < 34$ s: Velocity is constant, acceleration is zero.

The train moves with constant velocity 12 m s^{-1} during this interval.

For $34\text{ s} < t < 40$ s: $(0 - 12) = (6)(a_4) \Rightarrow a_4 = -2$ m s^{-2}



The negative acceleration means that a–t curve lies below t-axis; and at represents decrease in velocity.

Analysis from velocity–time curve:

Area under velocity–time curve = Change in position

For $0 < t < 6$ s: $x_6 - 0 = \frac{1}{2}(6)(6) = 18$ m

For 6 s $< t <$ 10 s: $x_{10} - x_6 = \frac{1}{2}(4)(6+12) = 36$ m

For 10 s $< t <$ 34 s: $x_{34} - x_{10} = (24)(12) = 288$ m

For 34 s $< t <$ 40 s: $x_{40} - x_{34} = \frac{1}{2}(6)(12) = 36$ m

On adding the change in positions, we can find the distance between stations A and B

$$d = x_{40} - 0 = 378 \text{ m}$$

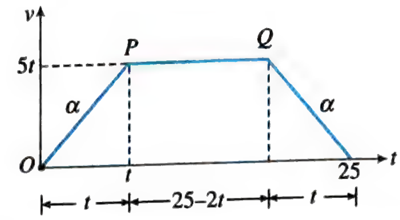
Note: The x–t curve for constant acceleration motion is a parabola. While drawing x–t curve, remember that at any instant slope of the tangent to the curve gives the value of v at that instant.

ILLUSTRATION 4.62

A car starts moving rectilinearly, first with acceleration α = 5 m s^{-2} (the initial velocity is equal to zero), then uniformly, and finally, decelerating at the same rate α comes to a stop. The total time of motion equals t = 25 s. The average velocity during this time is equal to $\langle v \rangle$ = 72 km h^{-1}. How long does the car move uniformly?

Sol.

Let t be the time up to which the car accelerates or decelerates. The maximum velocity attained in this duration is $5t$. The time upto which car moves uniformly = $25 - 2t$. The velocity–time graph of the motion of car is drawn as shown in Figure.



Given the average velocity is whole time of motion

$$v_{av} = \frac{72 \times 5}{18} = 20 \text{ m s}^{-1}$$

The average velocity from the graph can be obtained as

$$v_{av} = \frac{\text{Total displacement}}{\text{Total time}} = \frac{\text{Area of } \vec{v} - t \text{ graph}}{\text{Total time}}$$

$$20 = \frac{\frac{1}{2} \times [25 + (25 - 2t)] \times 5t}{25} = \frac{\frac{1}{2} \times [50 - 2t)] \times 5t}{25}$$

$$200 = 50\,t - 2t^2$$

$\Rightarrow \quad t^2 - 25t + 100 = 0$

$\Rightarrow \quad (t - 20)(t - 5) = 0 \quad \Rightarrow t = 5$ s or 20 s

But t = 20 s is not possible.

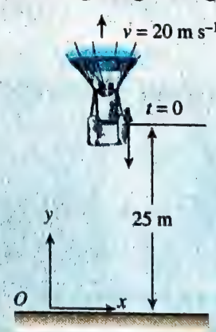
Hence, t = 5 s

The time up to which car moves uniformly

$= 25 - 2t = 25 - 2 \times 5 = 15$ s

ILLUSTRATION 4.63

A hot-air balloonist, rising vertically with a constant velocity of magnitude 20 m s^{-1}, releases a sandbag at an instant when the balloon is 25 m above the ground (Figure). After it is released, the sandbag is in free fall. Sketch a_y–t, v_y–t, and $|v_y|$–t graphs for the motion, taking origin at ground.



Sol. Apply constant acceleration equations to the vertical motion of the sandbag.

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \qquad \text{...(i)}$$

$$v_y = v_{0y} + a_y t \qquad \text{...(ii)}$$

Take +y upward, $a_y = -10$ m s^{-2}. y_0 = 25 m. The initial velocity of the sandbag equals the velocity of the balloon,

so $v_{0y} = +20\ m\,s^{-1}$.

When the balloon reaches the ground, $y = 0$. At its maximum height the sandbag has, $v_y = 0$.

At maximum height, $0 = 20 - 10\,t \quad \Rightarrow t = 2$ s

Position of the sand bag at maximum height using (i)

$$y = 25 + 20 \times 2 - \frac{1}{2} \times 10 \times 2^2 = 45\ \text{m}$$

The maximum height is 45 m above the ground.

For observer at ground, the sand bag will go up and reach to its maximum height $y_{max} = 45$ m and then stop momentary and then start moving and after another 2 s it passes the point of droping.

The sand bag will reach ground where $y = 0$.

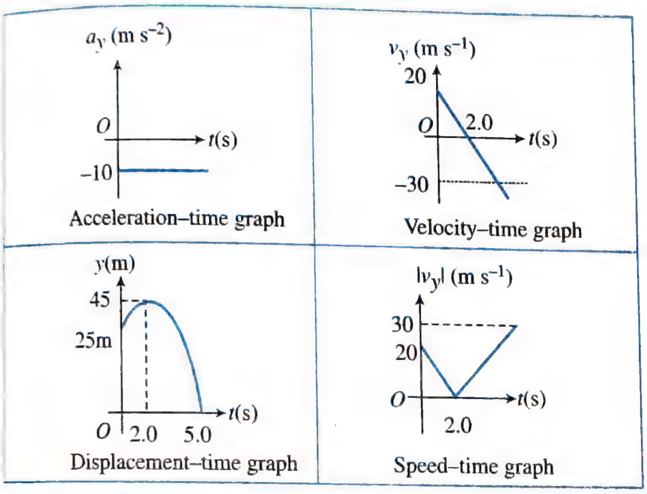
Using (i)

$$0 = 25 + 20t - 5t^2$$

$$t^2 - 4t - 5 = 0$$

or $\quad (t-5)(t+1) = 0 \Rightarrow t = 5$ s

The graphs of a_y, v_y, and y versus t are given in figure. Take $y = 0$ at the ground.



Acceleration–time graph

Velocity–time graph

Displacement–time graph

Speed–time graph

ILLUSTRATION 4.64

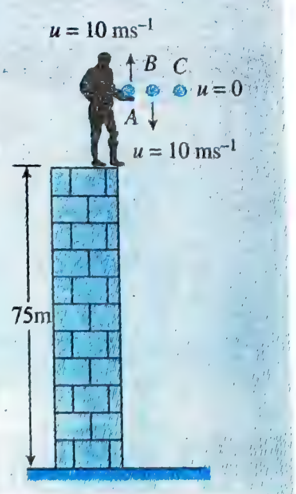
At the height of 75 m, a particle A is thrown up with $u_A = 10\ m\,s^{-1}$ and B particle is thrown down with $u_B = 10\ m\,s^{-1}$ and C particle released with $u_C = 0\ m\,s^{-1}$. Draw graphs of each particle.

(a) Displacement–time

(b) Speed–time

(c) Velocity–time

(d) Acceleration–time



Sol.

Taking origin at the top of building and upward direction positive.

For particle A: $\quad u = +10\ m\,s^{-1} \qquad a = -10\ m\,s^{-2}$

Using $y = ut + \frac{1}{2}at^2$

$$= 10t - \frac{1}{2} \times 10 \times t^2 = 10t - 5t^2 \qquad \text{...(i)}$$

Equation (i) is the equation of parabola open down.

The parabola passes origin when $y = 0$, hence from (i)

$$0 = 10t - 5t^2 \quad \Rightarrow \quad t = 0, 2\ \text{s}$$

For velocity–time relation, $v = u + at$

$$v = 10 - 10t \qquad \text{...(ii)}$$

This relation is straight line relation with negative slope and positive intercept.

At maximum height, velocity of the particle is zero.

$$0 = 10 - 10\,t \Rightarrow t = 1\ \text{s}$$

Maximum height from (i), $y_{max} = 10 \times 1 - 5 \times 1^2 = 5\ m\,s^{-1}$

It will reach ground where $y = -75$ m

Using (i) again, $-75 = 10t - 5t^2 \quad \Rightarrow \quad (t-5)(t+3) = 0$

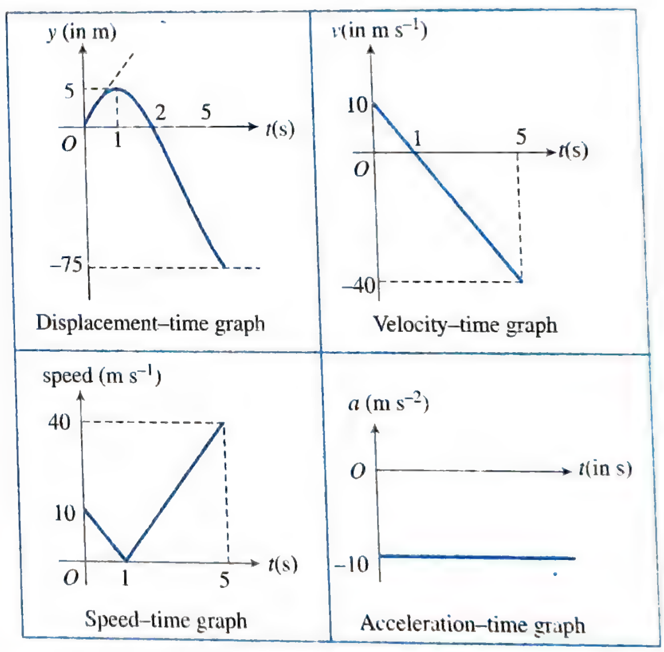
When given $t = 5$ s, i.e., the particle will hit ground at $t = 5$ s.

Velocity at the time of hitting

Using (ii), $v = 10 - 10 \times 5 = 40\ m\,s^{-1}$

For speed–time relation, the speed is always positive the mirror-image of velocity–time relation on positive side.

For drawing acceleration–time graph, we have taken downward direction as negative and acceleration is constant throughout the motion of the particle, i.e., $-10\ m\,s^{-2}$. Hence, acceleration time graph will be straight line parallel with time axis and below the time axis.



Displacement–time graph

Velocity–time graph

Speed–time graph

Acceleration–time graph

For particle B: $u = -10\ m\,s^{-1} \qquad a = -10\ m\,s^{-2}$

Displacement–time relation,

$$y = -10t - \frac{1}{2} \times 10 \times t^2 = -10t - 5t^2 \qquad \text{...(iii)}$$

It is a parabola open down and it passes through origin.

The particle will reach ground when $y = -75$ m

From (iii) $\quad -75 = -10t - 5t^2$

or $\quad t^2 + 2t - 15 = 0 \quad \Rightarrow \quad t = 3$ s

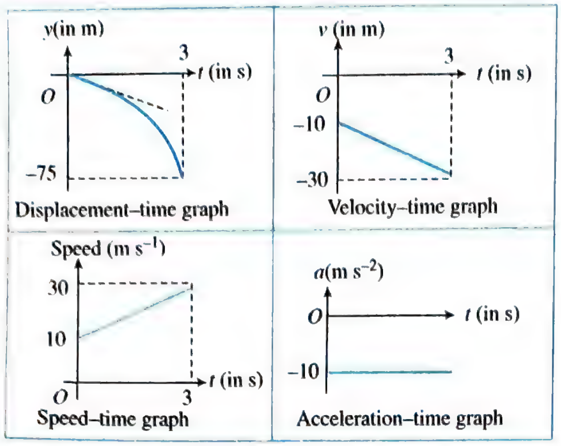
The velocity of the particle in relation with time

As $\quad u = -10$ m/s

$\Rightarrow \quad v = -10 - 10t \qquad \text{...(iv)}$

The particle reaches ground at $t = 3$ s, velocity of the particle when it hits ground is

$$v = -10 - 30 = -40\ m\,s^{-1}$$



For particle *C*: $u = 0, a = -10 \text{ m s}^{-2}$

Using $y = ut + \frac{1}{2}at^2$

$$= 0 - \frac{1}{2}(-10)t^2 = -5t^2 \qquad \text{...(v)}$$

It is a parabola open down and it passes through origin.

For velocity–time relation $\vec{v} = \vec{u} + \vec{a}t$

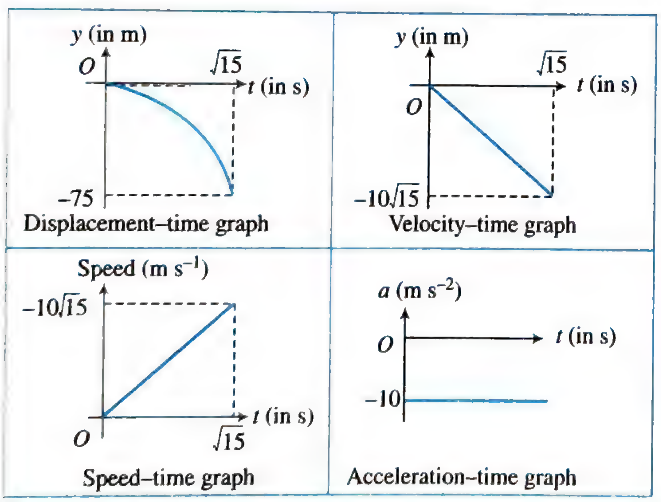
$$v = 0 - 10\,t = -10t \qquad \text{...(vi)}$$

Straight line with negative slope passes through origin.

It will reach round at $y = -75$ m

$$-75 = -5t^2 \Rightarrow t = \sqrt{15} \text{ s}$$

Velocity at the time when it reaches ground $v = -10\sqrt{15}$ s.



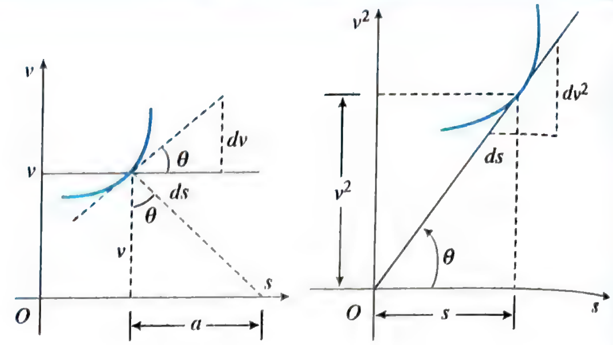
Velocity–Displacement Graph

When v is given as the function of s, we can find the acceleration at any position by using the v–s graph as described below.

The slope of v–s graph at any point gives the ratio of acceleration and speed at that point.

$$\text{Slope} = \frac{dv}{ds} = \frac{a}{v}$$

The slope of v^2–s graph at any point is equal to twice the acceleration that corresponds to that point.



(a) (b)

$$\text{Slope} = \frac{dv^2}{ds} = 2a$$

Since $a = \frac{v\,dv}{ds}$, we have

$$\frac{dv}{ds} = \frac{a}{v}$$

$$\text{Slope of } v\text{–}s \text{ graph} = \frac{\text{Acceleration}}{\text{Velocity}}$$

Since $\frac{v\,dv}{ds} = \frac{1}{2}\frac{dv^2}{ds}$

Slope of v^2–s graph = 2 (acceleration)

Summarizing the above facts, we conclude the following points:

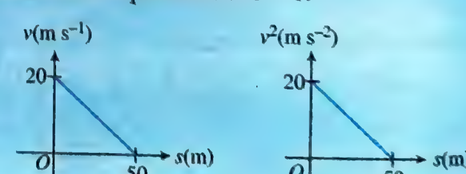
(a) Acceleration = Speed × Slope of v–s graph

$$= \frac{1}{2} \times \text{Slope of } v^2\text{–}s \text{ graph}$$

(b) The slopes can be positive, negative, or zero which signify positive, negative, or zero accelerations.

ILLUSTRATION 4.65

The v–s and v^2–s graph are given for two particles. Find the accelerations of the particles at $s = 0$.



Sol. $a_1 = \frac{v\,dv}{ds}$, where $v = 20 \text{ m s}^{-1}$ (at $s = 0$)

and $\frac{dv}{ds} = -\frac{2}{5} = \frac{a_1}{v} = \frac{a_1}{20}$

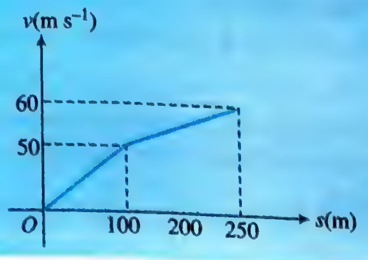
Then, $a_1 = -8 \text{ m s}^{-2}$

$$a_2 = \frac{1}{2}\frac{d(v^2)}{ds}, \text{ where } \frac{d(v^2)}{ds} = -\frac{2}{5}$$

Then, $a_2 = -0.2 \text{ m s}^{-2}$

ILLUSTRATION 4.66

The velocity–displacement for a jet plane on a straight runway is shown in figure. Determine the speed and acceleration of the jet plane at $s = 150$ m.

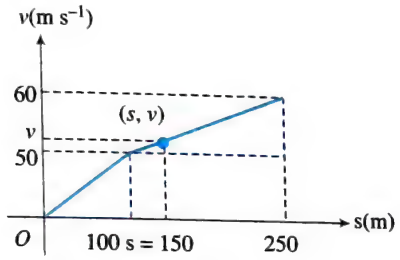


Sol.

Referring to the v–s graph, its slope for $100\text{ m} \le s \le 250\text{ m}$ is equal to $\dfrac{60-50}{250-100} = \dfrac{1}{15}$.

The equation of the above graph is given by $\dfrac{v-50}{s-100} = \dfrac{1}{15}$

This gives $v = \dfrac{650+s}{15}; 100\text{ m} \le s \le 250\text{ m}$



Substituting $s = 150$ m, we have

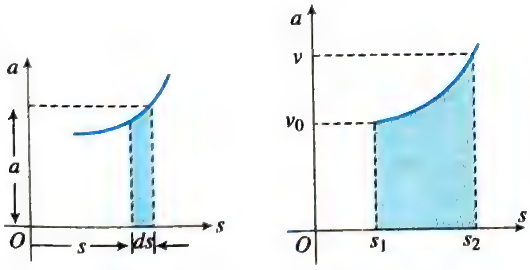
$$v = \frac{160}{3}\text{m s}^{-1}$$

Since $a = \dfrac{v\,dv}{ds}$ and $\dfrac{dv}{ds} = \dfrac{1}{15}$ for $100\text{ m} \le s \le 250\text{ m}$, we have

$$a = \frac{32}{9}\text{m s}^{-2}$$

Acceleration–Displacement Graph

By using the formula $v\,dv = \vec{a}.d\vec{s}$ and integrating both sides, we have $\int_{v_0}^{v} v\,dv = \int_{s_1}^{s_2} \vec{a}.d\vec{s}$. It gives the speed.



The elementary area is equal to $v\,dv$

$$v^2 = v_0^2 + 2\int_{s_1}^{s_2} \vec{a}.d\vec{s}$$

Area A under a–s graph gives the speed as per the formula:

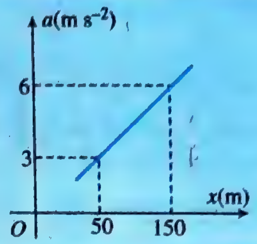
$$v = \sqrt{v_0^2 + 2A} \text{ where } A = \int_{s_1}^{s_2} \vec{a}.d\vec{s}$$

As $\int_{s_1}^{s_2} \vec{a}.d\vec{s}$ represents the area under a–s graph, we have

$$v = \sqrt{v_0^2 + 2(\text{area under } a - s \text{ graph})}$$

ILLUSTRATION 4.67

Referring to a–x graph, find the velocity when the displacement of the particle is 100 m. Assume initial velocity as zero.

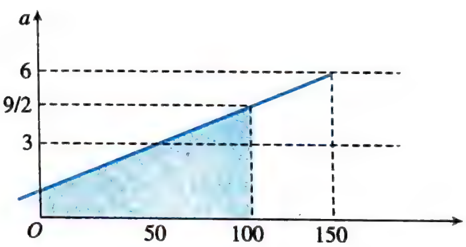


Sol. We can find relation between a and x as:

Using $(y - y_1) = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$$(a-3) = \frac{6-3}{150-50}(x-50)$$

$$\Rightarrow \quad a = \frac{3}{100}x + \frac{3}{2}$$



But $\quad v = \sqrt{v_0^2 + 2(A)}$

As $\quad v_0 = 0 \Rightarrow v^2 = 2\int a\,dx$

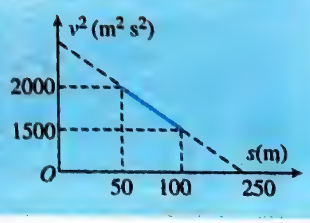
$$\Rightarrow \quad a = \frac{3}{100}x + \frac{3}{2}$$

$\Rightarrow \quad v^2 = 2$[Area of a–x graph upto $x = 100$ m]

$$= 2\left[\frac{1}{2}\left(\frac{3}{2}+\frac{9}{2}\right)100\right] = 600 \quad \Rightarrow v = 10\sqrt{6}\text{ m s}^{-1}$$

ILLUSTRATION 4.68

Referring to the v^2–s diagram of a particle, find the displacement of the particle during the last two seconds.



Sol. Slope of v^2–s graph

$$m = -\frac{(2000-1500)}{100-50} = -10$$

Relation between v^2 and s: $v^2 = -10s + C$

At $s = 50$, $v^2 = 2000$

$\Rightarrow \quad 2000 = -10 \times 50 + C \quad \Rightarrow \quad C = 2500$

$v^2 = -10s + 2500$

Differentiating eq.(i) both sides w.r.t. time

$\Rightarrow \quad \dfrac{2v\,dv}{dt} = -10\dfrac{ds}{dt}$

$\Rightarrow \quad 2va = -10\,v \Rightarrow \quad a = -5\ \mathrm{m\,s^{-2}} \rightarrow$ constant

To find initial velocity:

Put $s = 0$, $v = u$

$\Rightarrow \quad u^2 = 2500 \quad \Rightarrow u = 50\ \mathrm{m\,s^{-1}}$

Apply $v = u + at$

$\Rightarrow \quad 0 = 50 - 5T$

$\Rightarrow \quad T = 10\,\mathrm{s} \rightarrow$ time of motion

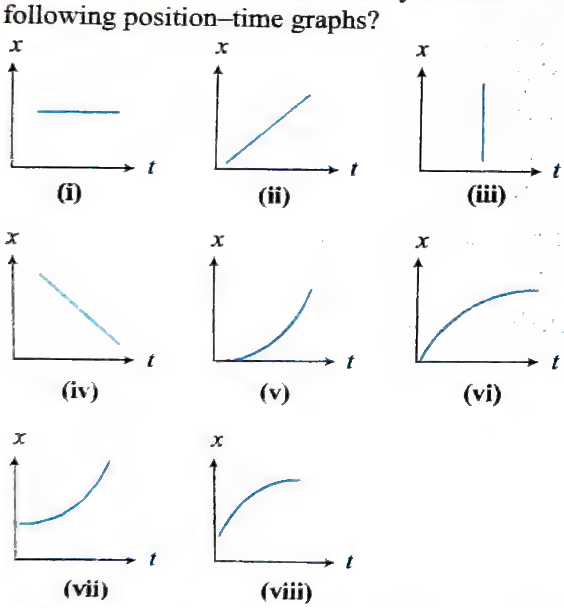
Displacement during last two seconds:

$$S = S_{t=10\,\mathrm{s}} - S_{t=8\,\mathrm{s}}$$

$$= \left[50 \times 10 - \frac{1}{2}5(10)^2\right] - \left[50 \times 8 - \frac{1}{2}5(8)^2\right] = 10\ \mathrm{m}$$

Concept Application Exercise 4.5

1. **(a)** What can you say about velocity in each of the following position–time graphs?



(b) The slope of the velocity–time graph is equal to acceleration. (True/False)

(c) What does the area under acceleration–time graph represents?

(d) Can velocity–time graph be parallel to the velocity axis? (Yes/No). Why?

(e) What is the slope of the v–t graph in uniform motion?

2. **(a)** A ball is thrown vertically upwards. After some time, it returns to the thrower. Draw the velocity–time graph and speed–time graph.

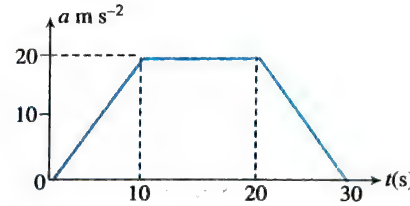
(b) A ball is dropped from some height. After rebounding from the floor, it ascends to the same height. Draw the velocity–time graph and speed–time graph.

3. A body starts at $t = 0$ with velocity u and travels along a straight line. The body has a constant acceleration a. Draw the acceleration–time graph, velocity–time graph, and displacement–time graph from $t = 0$ to $t = 10$ s for the following cases:

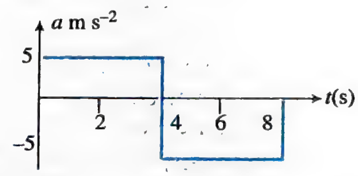
(a) $u = 8\ \mathrm{m\,s^{-1}}$, $a = 2\ \mathrm{m\,s^{-2}}$ **(b)** $u = 8\ \mathrm{m\,s^{-1}}$, $a = -2\ \mathrm{m\,s^{-2}}$

(c) $u = -8\ \mathrm{m\,s^{-1}}$, $a = 2\ \mathrm{m\,s^{-2}}$ **(d)** $u = -8\ \mathrm{m\,s^{-1}}$, $a = -2\ \mathrm{m\,s^{-2}}$

4. In the given figure, find the average acceleration in first 20 s. (Hint: Area under $a - t$ graph is equal to the change in velocity)

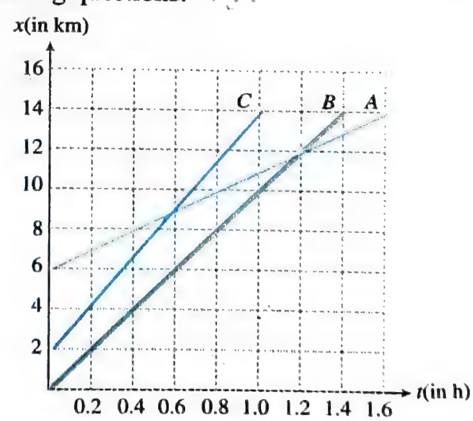


5. At $t = 0$, a particle starts from rest and moves along a straight line, whose acceleration–time graph is shown in figure.



Convert this graph into velocity–time graph. From the velocity–time graph, find the maximum velocity attained by the particle. Also find from v–t graph, the displacement and distance travelled by the particle from 2 to 6 s.

6. Figure shows the position–time graphs of three cars A, B, and C. On the basis of the graphs, answer the following questions:



(a) Which car has the highest speed and which has the lowest?

(b) Are the three cars ever at the same point on the road?

(c) When C passes A, where is B?

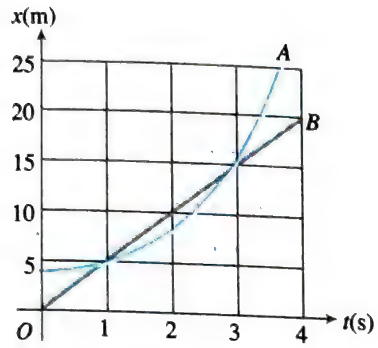
(d) What is the time interval during car A travel between the time it passed cars B and C?

(e) What is the relative velocity of car C with respect to car A?

(f) What is the relative velocity of car B with respect to car C?

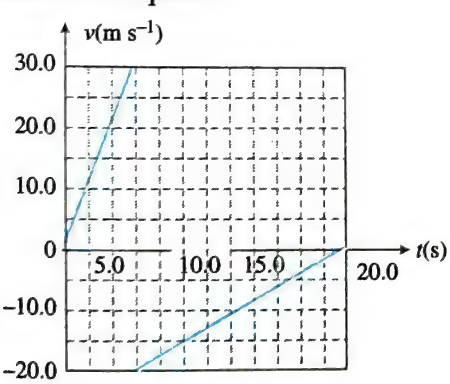
7. A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate of β to come to rest. If the total time elapsed is t second, then calculate
   (a) the maximum velocity attained by the car, and
   (b) the total displacement travelled by the car in terms of α, β and t.

8. Two cars, A and B, move along the x-axis. Car A starts from rest with constant acceleration while car B moves with constant velocity.
   (a) At what time(s), t(s), if any, do A and B have the same position?

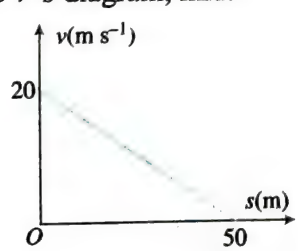


   (b) At what time(s), if any, do A and B have the same velocity? What is the velocity of car B at this time.
   (c) Graph velocity versus time for both A and B.
   (d) At what time(s), if any, does car A pass car B?
   (e) At what time(s), if any, does car B pass car A?

9. A rigid ball traveling in a straight line (the x-axis) hits a solid wall and suddenly rebounds during a brief instant. The v_x–t graph in figure shows this ball's velocity as a function of time. During the first 20 s of its motion, find (a) its displacement, (b) the total distance the ball moves, and (c) sketch a graph of $a_x - t$ for this ball's motion. (d) Is the graph shown really vertical at 5 s? Explain.

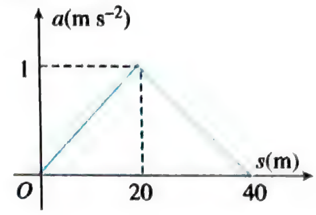


10. Referring to v–s diagram, find:

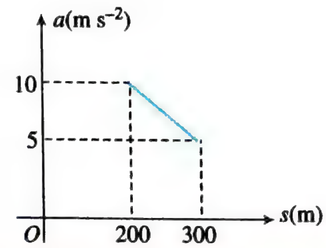


   (a) Acceleration of the particle when its velocity becomes half of the initial velocity.
   (b) Total distance covered by the particle.

11. A racing motor boat speeds up in a straight line in a lake, from rest. Referring to the acceleration–displacement graph for the speeding boat, find its speed when it passes a raft at a distance of 40 m from the starting point.



12. Referring a–s diagram in figure, find the velocity after particle traveles 250 m from starting. Assume $v_0 = 0$.



ANSWERS

1. (a)(i) zero velocity.
   (ii) constant positive velocity.
   (iii) infinite velocity.
   (iv) constant negative velocity.
   (v) positive increasing velocity.
   (vi) positive decreasing velocity.
   (vii) positive increasing velocity.
   (viii) positive decreasing velocity.
   (b) True **(c)** Change in velocity **(d)** No **(e)** Zero

4. 15 m s^{-2} **5.** 20 m s^{-1}, 60 m, 60 m

6. (a) car C has highest speed and car A has lowest speed.
   (b) No
   (c) approximately 6 km from the origin
   (d) 0.6 h **(e)** 7 km h^{-1} **(f)** −2 km h^{-1}

7. (a) $v_{max} = \dfrac{\alpha\beta t}{\alpha + \beta}$ **(b)** $\dfrac{\alpha\beta t^2}{2(\alpha + \beta)}$

8. (a) at $t = 1$ s and $t = 3$ s
   (b) $t = 2$ s, $v_B = 5$ m s^{-1} **(d)** $t = 3$ s **(e)** $t = 1$ s

9. (a) − 75 m **(b)** 225 m

10. (a) − 4 m s^{-2} **(b)** 50 m

11. $\sqrt{40}$ m s^{-1}

12. $25\sqrt{11}$ m s^{-1}

Solved Examples

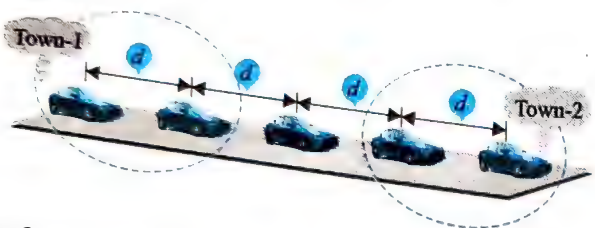
EXAMPLE 4.1

From a town, some cars start at regular intervals of 30 s and run towards another town with constant speed of 60 km/h. At some point of time, all of the cars simultaneously have to reduce their speed to 40 km/h due to bad weather conditions. What will become the time interval between arrivals of the cars at the second town during the bad weather?

Sol.

If two objects moves with constant speed, the distance between the objects remains constant.

If all of the cars simultaneously have to reduce their speed from 60 km/h to 40 km/h, the distance between the cars will remain unchanged.



Speed of cars before bad weather $v_1 = 60(\text{km/h}) \times \frac{5}{18} = \frac{50}{3}\text{m/s}$

Speed of cars after bad weather $v_1 = 40(\text{km/h}) \times \frac{5}{18} = \frac{100}{9}\text{m/s}$

Let I[st] car leaves from town-1 at time $t = 0$, then II[nd] car will leave from town-1 after $t = 30$ sec.

Hence separation between Ist and IInd car $d = \frac{50}{3} \times 30 = 500\,\text{m}$

If all of the cars simultaneously have to reduce their speed from 60 km/h to 40 km/h, the distance between the cars will remain 500 m. If one car reaches town-2, next car will be 500 m behind that car.

Now speed of the car is 40 km/h or $\frac{100}{9}$ m/s.

Hence time to cover 500 m distance $= \frac{500}{\left[100/9\right]} = 45\,\text{s}$

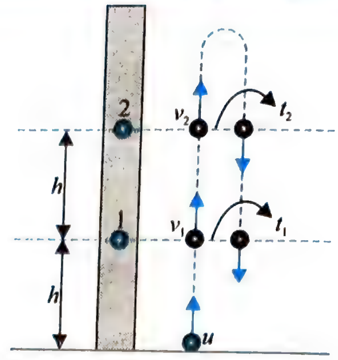
EXAMPLE 4.2

A ball is projected vertically up from ground. Boy A standing at the window of first floor of a nearby building observes that the time interval between the ball crossing him while going up and the ball crossing him while going down is t_1. Another boy B standing on the second-floor notices that time interval between the ball passing him twice (during up motion and down motion) is t_2.

(a) Calculate the height difference (h) between the boy B and A.

(b) Assume that the height of boy A from the point of projection of the ball is also equal to h and calculate the speed with which the ball was projected.

Sol. **(a)** Let the ball is projected with initial velocity u. The ball crosses the first floor and second floor with velocities v_1 and v_2, respectively.



At highest point the velocity of the ball should be zero.

In 1[st] situation:

$$0 = v_1 - g\left(\frac{t_1}{2}\right) \Rightarrow v_1 = \frac{gt_1}{2} \qquad \ldots(\text{i})$$

For 2nd situation:

$$0 = v_2 - g\left(\frac{t_2}{2}\right) \Rightarrow v_2 = \frac{gt_2}{2} \qquad \ldots(\text{ii})$$

Now using: $v_2^2 = v_1^2 - 2gh$

or $$h = \frac{1}{2g}\left(v_2^2 - v_1^2\right) \qquad \ldots(\text{iii})$$

Substituting the values of v_1 and v_2 from (i) and (ii) into equation (iii), we get

$$h = \frac{1}{2g}\left[\frac{g^2t_1^2}{4} - \frac{g^2t_2^2}{4}\right] = \frac{g\left(t_1^2 - t_2^2\right)}{8}$$

(b) Using $v_1^2 = u^2 - 2gh \Rightarrow u^2 = v_1^2 + 2gh \Rightarrow u^2 = v_1^2 + 2gh$ ...(iv)

From equations

$$u^2 = \left(\frac{gt_1}{2}\right)^2 + 2g\cdot\frac{g}{8}\left(t_1^2 - t_2^2\right) = \frac{g^2}{4}\left(2t_1^2 - t_2^2\right)$$

$$\Rightarrow \quad u = \frac{g}{2}\sqrt{2t_1^2 - t_2^2}$$

EXAMPLE 4.3

A platform is moving upwards with a constant acceleration of 2 m/sec². At time $t = 0$, a boy standing on the platform throws a ball upwards with a relative speed of 8 m/sec. At this instant, platform was at the height of 4m from the ground and was moving with a speed of 2 m/sec. Take $g = 10$ m/sec². Find

(a) When and where does the ball strikes the platform?

(b) Maximum height attained by the ball from the ground.

(c) Maximum distance of the ball from the platform.

Sol.

Let us analyze the situation in reference frame of platform.

Taking upward direction as particle direction.

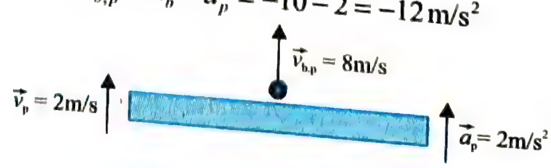
We are given: velocity of the ball w.r.t. platform, $\vec{v}_{b,p} = 8\,\text{m/s}$

Velocity of the platform $\vec{v}_p = 2\,\text{m/s}$

Acceleration of the platform $\vec{a}_p = 2\,\text{m/s}^2$

Acceleration of the ball w.r.t. platform

$$\vec{a}_{b,p} = \vec{a}_b - \vec{a}_p = -10 - 2 = -12\,\text{m/s}^2$$



(a) When the ball returns back to platform, its displacement w.r.t platform will be zero.

Using $\vec{s}_{b,p} = \vec{u}_{b,p}t + \frac{1}{2}\vec{a}_{b,p}t^2$

$$0 = 8\times t - \frac{1}{2}\times 12\times t^2 \Rightarrow t = \frac{4}{3}\text{sec}$$

Position of the platform during this time

$$\vec{y}_p = \vec{y}_0 + \vec{u}_p t + \frac{1}{2}\vec{a}_p t^2$$

$$= 4 + 2\times\frac{4}{3} + \frac{1}{2}\times 2\left(\frac{4}{3}\right)^{20} = \frac{76}{9}\text{m}$$

(b) Velocity of the ball at the time of throwing

$$\vec{v}_b = \vec{v}_{b,p} + \vec{v}_p = 8 + 2 = 10\,\text{m/s}$$

Using $v^2 = u^2 + 2as$, w.r.t earth

$$0 = (10)^2 - 2\times 10\times h \Rightarrow h = 5\text{ m}$$

Hence maximum height reached, $H_{max} = 4 + 5 = 9$ m

(c) For acceleration of maximum distance of the ball from platform, it will be easy to analyze the situation from reference frame of the platform.

Using $v_{b,p}^2 = u_{b,p}^2 + 2a_{b,p}s_{b,p}$

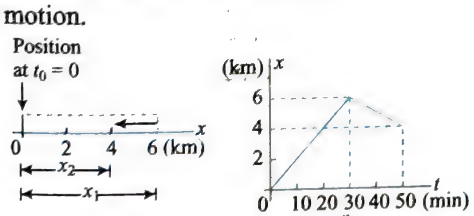
$$0 = 8^2 - 2\times 12\times h_{b,p} \Rightarrow h = \frac{8}{3}\text{m}$$

EXAMPLE 4.4

A runner jogs along a straight road (in the +x direction) for 30 min, travelling a distance of 6 km. She then turns around and walks back towards her starting point for 20 min, travelling 2 km during this time. State true/false:

(a) The final displacement of the runner relative to her starting position is 4 km.
(b) Her average speed for the entire trip is 0.16 km min^{-1}.
(c) The average velocity for the entire trip is 0.16 km min^{-1}.
(d) The runner's average velocity while jogging is 0.4 km min^{-1}.
(e) Her average velocity while walking is 0.1 km min^{-1}.

We first sketch figure (a) to depict the motion and indicate our choice of locating the origin of the coordinate system where the runner starts. We also show significant positions of the runner during the motion.



(a) (b)

Position	Time
$x_0 = 0$	$t_0 = 0$
Jogging $x_1 = 6$ km	$t_1 = 30$ min
Walking $x_2 = 4$ km	$t_2 = 50$ min

(a) True, $\Delta x = (x_2 - x_0) = 3\text{ km} - 0) = 4\text{ km}$

(b) True, average speed $= \dfrac{\text{Total distance}}{\text{Total time}} = \dfrac{8\text{km}}{50\text{min}}$
$= 0.160$ km min^{-1}

(c) False, the average velocity for the entire trip is

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{(x_2 - x_0)}{(t_2 - t_0)} = \frac{4-0}{50-0} = 0.080\text{ km min}^{-1}$$

(d) False, while jogging

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{(x_1 - x_0)}{(t_1 - t_0)} = \frac{6-0}{30-0} = 0.200\text{ km min}^{-1}$$

(e) True, while walking

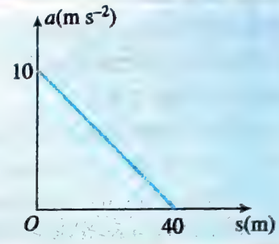
$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{(x_2 - x_1)}{(t_2 - t_1)} = \frac{(4\text{km} - 6\text{km})}{(50\text{min} - 30\text{min})}$$

$$= \frac{-2\text{ km}}{20\text{ min}} = -0.100\text{ km min}^{-1}\text{(walking)}$$

The negative value indicates a velocity in the $-x$ direction. A word of caution: Please note that the runner's average velocity for the entire trip is not simply the average of her jogging and walking velocities, because the times she spends in these two different motions are not equal.

EXAMPLE 4.5

Referring to a–s diagram as shown in figure, find the velocity of the particle when the particle just covers 20 m ($v_0 = \sqrt{50}$ ms^{-1}).



Sol. The acceleration of the particle in relation with position

$$a = -\frac{s}{4} + 10 \quad\Rightarrow\quad \int_{v_0}^{v} v\frac{dv}{ds} = -\frac{s}{4} + 10$$

$$\Rightarrow\quad \int_{v_0}^{v} v\,dv = \int_0^{20}\left(-\frac{s}{4} + 10\right)ds$$

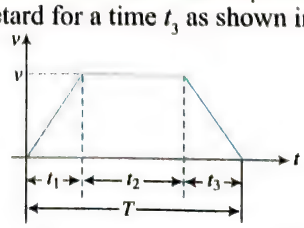
$$\Rightarrow\quad \frac{v^2 - v_0^2}{2} = \left[-\frac{s^2}{8} + 10s\right]_0^{20}$$

$$\Rightarrow\quad \frac{v^2 - 50}{2} = -\frac{(20)^2}{8} + 10\times 20 \quad\Rightarrow v = \sqrt{350}\text{ m s}^{-1}$$

EXAMPLE 4.6

A car moves in a straight line, the car accelerates from rest with a constant acceleration α on a straight road. After gaining a velocity v, the car moves with that velocity for sometime. Then the car decelerates with a retardation β. If the total distance covered by the car is equal to s, find the total time of its motion.

Sol. Let the car accelerate for a time t_1, move uniformly for a time t_2 and then retard for a time t_3 as shown in the $v - t$ graph.



Then the total time (T) of motion of the car is

$$T = t_1 + t_2 + t_3 \quad \text{...(i)}$$

The slope of $v-t$ graph

gives $\frac{v}{t_1} = \alpha$ and $\frac{-v}{t_3} = -\beta$

Then $t_1 + t_3 = \frac{v}{\alpha} + \frac{v}{\beta}$ (ii)

Area under $v-t$ graph gives the total displacement

$$s = \text{Area of the trapezium} = \frac{v}{2}(t_2 + T)$$

This gives $t_2 = \frac{2s}{v} - T$ (iii)

Substituting $t_1 + t_3$ from Eq. (ii) and t_2 from Eq. (iii) in Eq. (i) we have $T = \frac{v}{\alpha} + \frac{v}{\beta} - \frac{2s}{v} - T$

This yields $T = \frac{s}{v} + \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

EXAMPLE 4.7

A particle moving with uniform acceleration along a straight line ABC crosses point A at $t = 0$ with a velocity 12 ms^{-1}. B is 40 m away from A and C is 64 m away from A. The particle passes B at $t = 4$ s.

(a) After what time will the particle be at C?
(b) What is its velocity at C?
(c) When does the particle reach A again?
(d) Locate the point where the particle reverses its direction of motion.
(e) Find the distance covered by the particle in the first 15 s.

Sol. Let the acceleration of the particle be a.

For motion between A and B

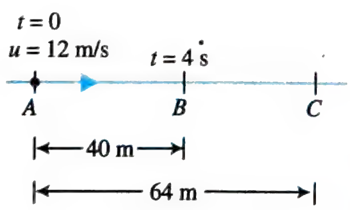
$u = 12\ \text{ms}^{-1}$, $s = 40$ m, $t = 4$ s

$s = ut + 1/2at^2$

$\Rightarrow \quad 40 = 12 \times 4 + \frac{1}{2} \times a \times (4)^2$

$\Rightarrow \quad a = -1\ \text{ms}^{-2}$

For motion between A and C



$64 = 12t + \frac{1}{2}(-1)t^2$

$\Rightarrow \quad t^2 - 24t + 128 = 0$

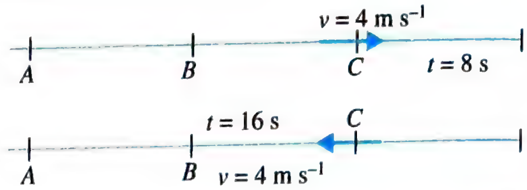
$\Rightarrow \quad (t-8)(t-16) = 0$

$\Rightarrow \quad t = 8$ s, 16 s

(a) The particle will be at C twice, at $t = 8$ s and $t = 16$ s.

(b) Velocity of the particle at C

At $t = 8$ s, velocity of the particle $v = 12 + (-1) \times 8 = 4\ \text{ms}^{-1}$.
At $t = 16$ s, velocity of the particle $v = 12 + (-1) \times 16 = -4\ \text{ms}^{-1}$.
As the acceleration of the particle is negative, it will retard as it moves along ABC. At a point beyond C (i.e., at D), the particle will come to rest momentarily, and then it will move backward with increasing speed. Throughout the motion, the particle decelerates, its velocity decreases continuously, velocity varies as 12, 11, 10, ..., 2, 1, 0, –1, –2, ..., –10, –11, –12, etc. Only from A to D the motion is retarded, during which speed varies as 12, 11,10, ..., 2, 1, 0.



Subsequently, the particle speeds up, speed changing as 1, 2, 3, ..., etc.
Notice the difference between deceleration and retardation.

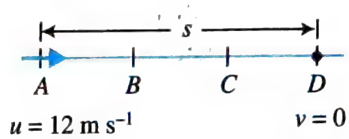
(c) When the particle reaches A again, its displacement = 0.

$\therefore \quad 0 = 12 \times t + \frac{1}{2}(-1)t^2$

$\Rightarrow \quad t = 0,\ 24$ s

At $t = 0$, velocity at $A = 12\ \text{ms}^{-1}$,
At $t = 24$ s, velocity at A is
$12 + (-1) \times 24 = -12\ \text{ms}^{-1}$.

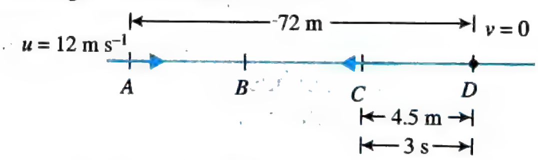
(d) The particle reverses the direction of motion at D. For the motion between A and D



$u = 12\ \text{ms}^{-1}$, $v = 0$, $a = -1\ \text{ms}^{-2}$. If $AD = s$, from the equation
$v^2 = u^2 + 2as$
$(0)^2 = (12)^2 + 2 \times (-1) \times s$

$$s = \frac{12 \times 2}{2} = 72\ \text{m}$$

(e) After 12 s the particle comes to rest momentarily at D after covering a distance of 72 m.



Distance in subsequent 3 s

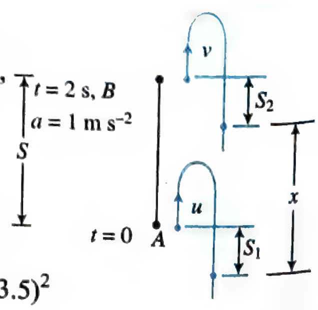
$$= 0 \times 3 + \frac{1}{2} \times (-1) \times (3)^2 = 4.5\ \text{m}$$

$\therefore \quad d = 72\ \text{m} + 4.5\ \text{m} = 76.5$ m

EXAMPLE 4.8

A balloon is ascending vertically with an acceleration of 1 ms^{-2}. Two stones are dropped from it at an interval of 2 s. Find the distance between them 1.5 s after the second stone is released.

Sol. Let at any time $t = 0$, the balloon be at position A, where its velocity is u. At $t = 2$ s, it reaches B, where its velocity becomes v, then



$AB = S = u \times 2 + \frac{1}{2}a(2)^2 = 2u + 2$

Also $v = u + a \times 2 = u + 2$,

First stone: $-S_1 = u \times 3.5 + \frac{1}{2}(-g)(3.5)^2$

Second stone: $-S_2 = v \times 1.5 + \frac{1}{2}(-g)(1.5)^2$

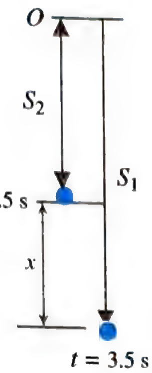
Required distance between the stones

$$x = S_1 + S - S_2$$

Solve to get $x = 55$ m.

Alternatively: (This method can be understood in proper way after studying relative velocity.)

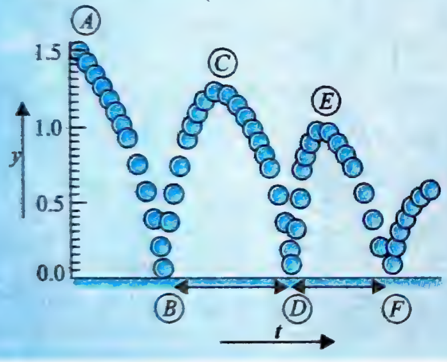
If we work from the frame of balloon, then the acceleration of each stone w.r.t. balloon will be $g + a$ after releasing from it. The initial velocity of each stone will be zero w.r.t. balloon.



$$S_1 = \frac{1}{2}(g + a)(3.5)^2,\ S_2 = \frac{1}{2}(g + a)(1.5)^2;\ x = S_1 - S_2 = 55 \text{ m}$$

EXAMPLE 4.9

A rubber ball is released from a height about 1.5 m. It is caught after three bounces. Sketch graphs of its position, velocity, and acceleration as functions of time. Take positive y-direction as upward direction.



Sol. Check the following points:

(a) Position and time graphs will have the same shape as the given graph.

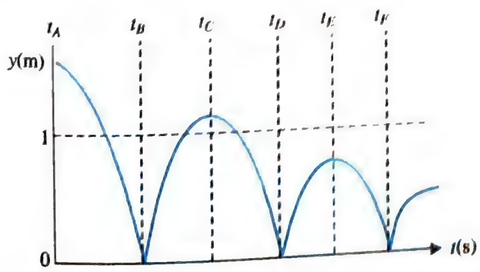
(b) The slope of the position–time graph gives the velocity. Hence, changes in velocity can be observed as per the slope of the position–time graph.

(c) The slope of the velocity–time graph gives the acceleration.

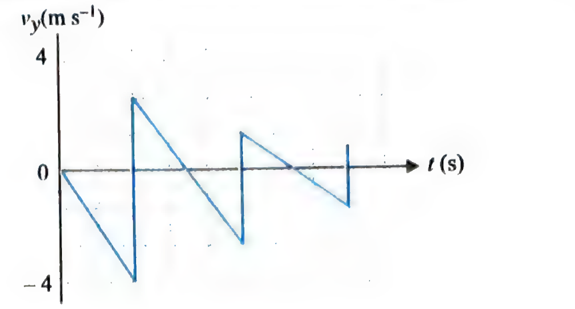
(d) Acceleration in free fall remains constant with time.

Note: Velocity is changing at points B, C, D, E, F. At B, D, and F, velocity changes suddenly from negative to positive and at C and E, velocity changes smoothly from positive to negative. At B, D, and F, velocity changes very quickly so the acceleration must be very large.

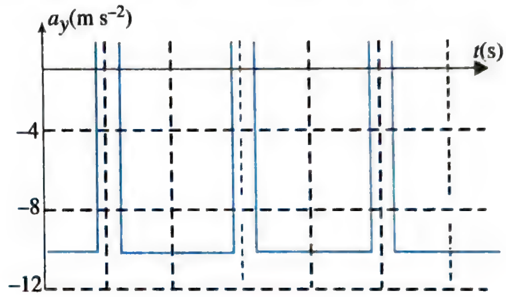
Position–time graph: The diagram given itself conveys position–time graph.



Velocity–time graph: As the slope of v–t graph changes thrice negative to positive during bounces. So v–t graph must observe sharp changes at these points.



Acceleration–time graph: Slope of v–t graph remains same (-10 m s^{-2}), i.e., it has single value during free fall. At the time of contact with floor changes substantially during a very short time interval. So the acceleration will be large and positive, which has to be represented in the form of straight upward lines.



EXAMPLE 4.10

Determined to test the law of gravity for himself, a student walks off a skyscraper 180 m high, stopwatch in hand, and starts his free fall (zero initial velocity). Five seconds later, Superman arrives at the scene and dives off the roof to save the student.

(a) Superman leaves the roof with an initial speed v_0 that he produces by pushing himself downward from the edge of the roof with his legs of steel. He then falls with the same acceleration as any freely falling body. What must the value of v_0 be so that the Superman catches the student just before they reach the ground?

(b) On the same graph, sketch the positions of the student and of the Superman as functions of time. Take Superman's initial speed to have the value calculated in part (a).

(c) If the height of the skyscraper is less than some minimum value, even the Superman cannot reach the student before he hits the ground, what is this minimum height?

Sol.

(a) Suppose that Superman falls for a time t and that the student has been falling for a time t_0 before Superman's leap (in this case, $t_0 = 5$ s). Then the height h of the building is related to t and t_0 in two different ways:

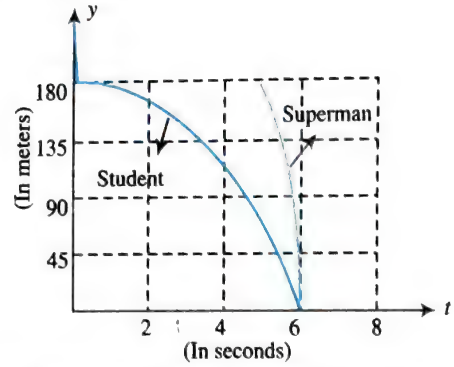
$$-h = v_{0y}t - \frac{1}{2}gt^2 = -\frac{1}{2}g(t + t_0)^2 \quad \text{...(i)}$$

where v_{0y} is Superman's initial velocity. Solving the second t gives $t = \sqrt{\frac{2h}{g}} - t_0 = \sqrt{\frac{2 \times 180}{10}} - 5 = 1$ s

Solving the first for v_{0y} gives $v_{0y} = -\frac{h}{t} + \frac{g}{2}t$ and the

substitution of numerical values gives t = 1.0 s and $v_{0y} = -175$ m s^{-1}, with minus sign indicating a downward initial velocity.

(b)



(c) If the skyscraper is so short that the student is already on the ground, then $h = \frac{1}{2}gt_0^2 = 125$ m.

EXAMPLE 4.11

A student is running at her top speed of 5.0 m s^{-1} to catch a bus, which is stopped at the bus stop. When the student is still 40.0 m from the bus, it starts to pull away, moving with a constant acceleration of 0.2 m s^{-2}.

(a) For how much time and what distance does the student have to run at 5.0 m s^{-1} before she overtakes the bus?

(b) When she reached the bus, how fast was the bus travelling?

(c) Sketch an x–t graph for both the student and the bus.

(d) The equations you used in part (a) to find the time have a second solution, corresponding to a later time for which the student and the bus are again at the same place if they continue their specified motions. Explain the significance of this second solution. How fast is the bus travelling at this point?

(e) If the student's top speed is 3.5 m s^{-1}, will she catch the bus?

(f) What is the minimum speed the student must have to just catch up with the bus? For what time and what distance does she have to run in that case?

Sol.

For convenience, let the student's (constant) speed be v_0 and the bus's initial position be x_0. Note that these quantities are for separate objects, the student and the bus. The initial position of the student is taken to be zero and the initial velocity of the bus is taken to be zero. The positions of student x_1 and bus x_2 as functions of time are then

$$x_1 = v_0 t,\ x_2 = x_0 + (1/2)at^2$$

(a) Setting $x_1 = x_2$ and solving for the time t gives

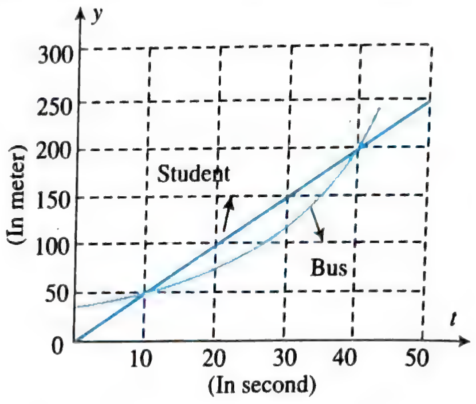
$$t = \frac{1}{a}\left(v_0 \pm \sqrt{v_0^2 - 2ax_0}\right)$$

$$= \frac{1}{(0.2)}\left((5.0) \pm \sqrt{(5.0)^2 - 2(0.2)(40.0)}\right) = 10, 40 \text{ s}$$

The student will be likely to hop on the bus the first time she passes it [see part (d) for a discussion of the later time]. During this time, the student has run a distance $v_0 t$ = (5 m s^{-1})(10 s) = 50 m

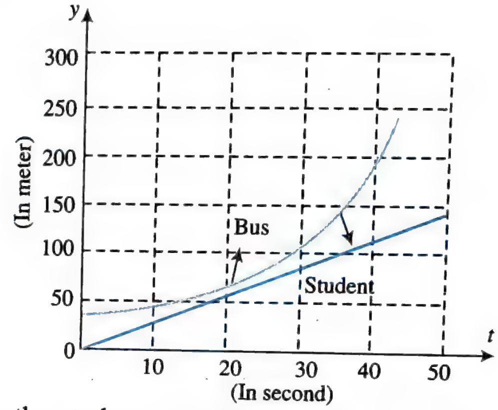
(b) The speed of the bus is $v = at = 0.2 \times 10 = 20$ m s^{-1}

(c) The results can be verified by noting that the x lines for the student and the bus intersect at two points:



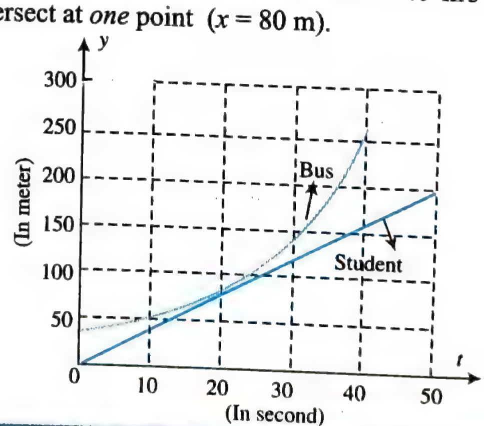
(d) At the later time, the student has passed the bus, maintaining her constant speed, but the accelerating bus then catches up with her. At this later time, the bus's velocity is (0.2 m s^{-2})(40 s) = 8.0 m s^{-1}.

(e) No; $v_0^2 < 2ax_0$, and the roots of the quadratic are imaginary. When the student runs at 3.5 m s^{-1}, the two lines do not intersect:



(f) For the student to catch the bus, $v_0^2 < 2ax_0$ and so the minimum speed is $\sqrt{2(0.2\text{ m s}^{-2})(40\text{ m})} = 4.0\text{ m s}^{-1}$. She would be running for a time $\frac{4.0\text{ m s}^{-1}}{0.2\text{ m s}^{-2}} = 20$ s, and cover a distance of (4.0 m s^{-1})(20 s) = 80.0 m.

However, when the student runs at 4.0 m s^{-1}, the lines intersect at *one* point (x = 80 m).



EXAMPLE 4.12

A particle retards from a velocity v_0 while moving in a straight line. If the magnitude of deceleration is directly proportional to the square root of the speed of the particle, find its average velocity for the total time of its motion.

Sol. The average velocity is given as:

$$v_{av} = \frac{s}{t} = \frac{\text{Total displacment}}{\text{Total time}}$$

We know $a = v\frac{dv}{ds} \Rightarrow v\,dv = a\,ds$

Let us calculate the displacement by substituting $a = -\alpha\sqrt{v}$ (for retardation) in $v\,dv = a\,ds$.

We have $v\,dv = (-\alpha\sqrt{v})ds. \quad \Rightarrow \sqrt{v}\,dv = -\alpha\,ds$

When the particle slows down from v_0 to 0, it covers a distance s.

Hence, $\int_{v_0}^{0}\sqrt{v}\,dv = -\int_0^s \alpha\,ds$

This gives $s = \frac{2v_0^{3/2}}{3\alpha}$

Use of $a = \frac{dv}{dt}$:

Now we will find the time of motion by substituting $a = -\alpha\sqrt{v}$ in $a = \frac{dv}{dt}$, we get

$$-\alpha\sqrt{v} = \frac{dv}{dt} \quad \Rightarrow \quad \frac{dv}{\sqrt{v}} = -\alpha dt$$

If the particle takes time t to stop, we have

$$\int_{v_0}^{0}\frac{dv}{\sqrt{v}} = -\alpha\int_0^t dt$$

This yields $t = \frac{\sqrt{v_0}}{\alpha}$

Finally, substituting s and t in $v_{av} = \frac{s}{t}$, we have $v_{av} = \frac{2v_0}{3}$

EXAMPLE 4.13

A particle moves a long x-axis with acceleration $a = 6(t-1)$, where t is in seconds. If the particle is initially at the origin and moves along positive x-axis with $v_0 = 2\text{ m s}^{-1}$, analyze the motion of the particle.

Sol. Here we are given $a = 6(t-1)$

Here, $s = +x$, $v = +v$.

To find velocity–time relation using $a = \frac{dv}{dt}$, we have

$$\frac{dv}{dt} = 6(t-1) \qquad \text{...(i)}$$

When velocity changes from v_0 to v during time t, integrating both sides with respect to time, we have

$$\int_{v_0}^{v} dv = \int_0^t 6(t-1)dt \Rightarrow v - v_0 = 3t^2 - 6t$$

Substituting $v_0 = 2\text{m s}^{-1}$, we have

$$\left.\begin{aligned} v = \frac{dx}{dt} &= 3t^2 - 6t + 2 \\ dx &= (3t^2 - 6t + 2)dt \end{aligned}\right] \qquad \text{...(ii)}$$

To find position–time relation, again integrating (ii) both sides, we have

$$\int_0^x dx = \int_0^t (3t^2 - 6t + 2)dt$$

This gives $x = t^3 - 3t^2 + 2t = t(t-1)(t-2)$ ...(iii)

The particle passes through the origin where $x = 0$ from (iii), we have $t = 0$, 1 s and 2 s. That means the particle crosses the origin twice at $t = 1$ s and $t = 2$ s. After $t = 2$ s, x is positive. Hence, its displacement points in positive x-direction and goes on increasing its magnitude.

The particle will come at rest when $v = 0$. From (ii), we have

$$t = \left(1 - \frac{1}{\sqrt{3}}\right)s\,(= t_1, \text{say})$$

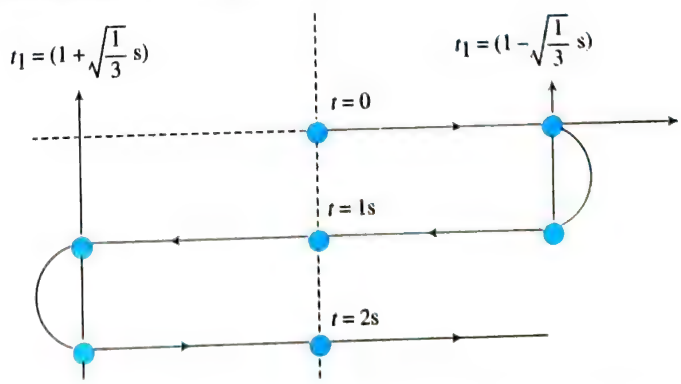
and $\quad t = \left(1 + \frac{1}{\sqrt{3}}\right)s\ (= t_2, \text{say})$.

Hence, the particle at $t = t_1$ and t_2. That signifies to and fro motion of the particle during first two seconds as it starts retracing its path at $t = t_1$ and $t = t_2$.

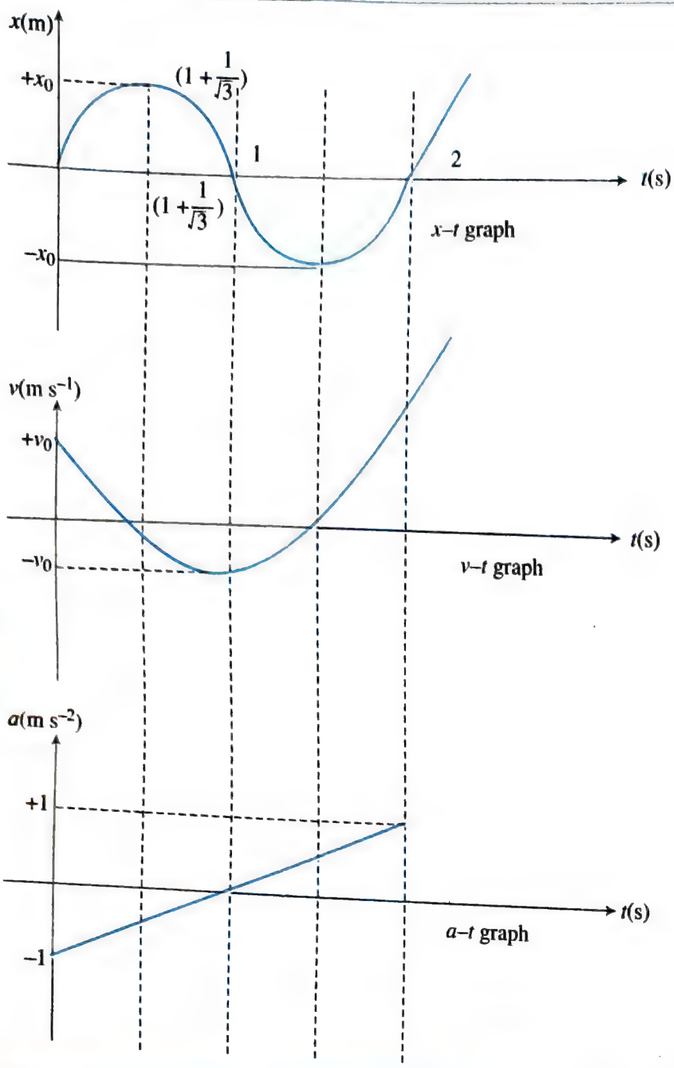
The displacement (x), velocity (v), and acceleration (a) of the particle in different time intervals are given in the following table and shown in the following graphs.

Time interval	x	v	a
$0 < t < \left(1 - \frac{1}{\sqrt{3}}\right)$ s	+ve	+ve	–ve
$\left(1 - \frac{1}{\sqrt{3}}\right)$ s $< t < 1$ s	+ve	–ve	–ve
$1\text{ s} < t < \left(1 + \frac{1}{\sqrt{3}}\right)$ s	–ve	–ve	+ve
$\left(1 + \frac{1}{\sqrt{3}}\right)$ s $< t < 2$ s	–ve	+ve	+ve
$2\text{ s} < t < \infty$	+ve	+ve	+ve

Locus diagram of the particle

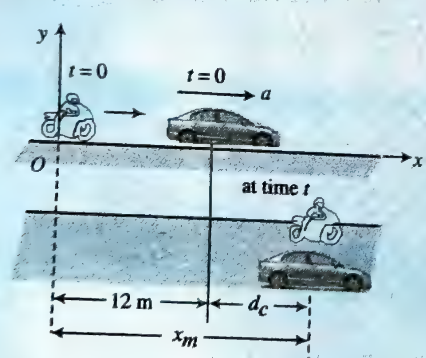


The displacement (x), velocity (v) and acceleration (a) graphs



EXAMPLE 4.14

A motorcyclist situated at origin is located at a distance 12 m behind a car.



At $t = 0$, the motorcyclist starts moving with a constant velocity $v = 8\ \text{m s}^{-1}$ and same time the car starts accelerating from rest with $a = 2\text{m s}^{-2}$. (a) When and where do they meet? (b) Draw the position–time and velocity–time relations of car and motorcycle on the same graph.

Sol.

Let the car and motorcycle meet, after a time t. During the time, the motorcycle covers a distance d_m (say).

Position of motorcycle, $x_m = d_m = vt$ ...(i)

During the time t, the car moves through a distance $d_c = \frac{1}{2}at^2$

Position of car, $x_c = 12 + d_c = 12 + \frac{1}{2}at^2$ ...(ii)

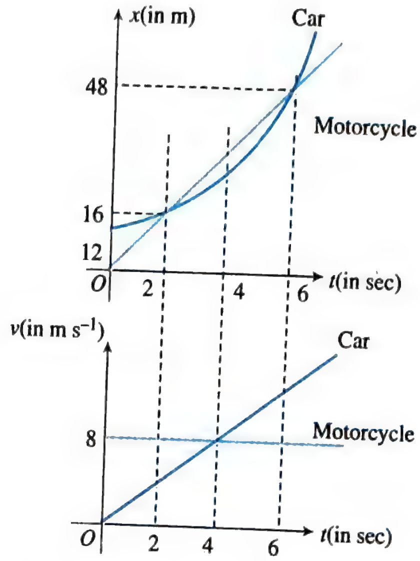
When both motorcycle and car meet their positions will be same.

From (i) and (ii), $x_m = x_c$.

We have $vt = 12 + \frac{1}{2}at^2$

Then substituting $v = 8\ \text{m s}^{-1}$ and $a = 2\ \text{m s}^{-2}$, we have

$$t^2 - 8t + 12 = 0$$



This yields two real values of time, i.e., $t = 2$ s and 6 s. The following conclusion can be made from the above mathematical proceedings.

At $t = 2$ s the motorcycle approaches the car.

The position of motorcyclist $(x_m)_{t=2\,\text{s}} = 8 \times 2 = 16$ m

At $t = 6$ s, the car overtakes the motorcycle.

The position of motorcyclist $(x_m)_{t=6\,\text{s}} = 8 \times 6 = 48$ m

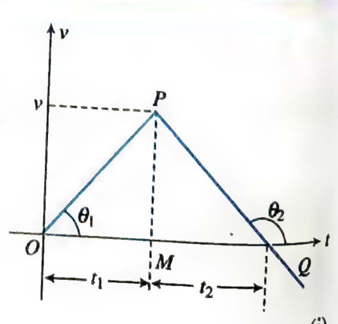
At $t = 2$ s, the motorcycle overtakes the car. Then it diverges (moves away from the car) till the car acquires a velocity equal to that of motorcycle at $t = 4$ s. Then the distance of separation will gradually decreases as the car moves faster than the motorcycle after 4 s. In consequence, at $t = 6$ s, the car overtakes the motorcycle. After that, the car will take a lead leaving the motorcycle behind it.

EXAMPLE 4.15

A diwali rocket moves vertically up with a constant acceleration $a_1 = 20/3\ \text{m s}^{-2}$. After sometimes, its fuel gets exhausted and then it falls freely with an acceleration $a_2 = 10\ \text{m s}^{-2}$. If the maximum height attained by the diwali rocket is h, using graphical method, find its speed when the fuel is just exhausted. Assume $h = 50$ m.

Sol.

Let the rocket acquire a speed v at the time when all its fuel gets exhausted. Referring to the v–t graph in figure the slope of OP gives the acceleration a_1.



The slope of OP gives the acceleration

$$a_1 = \tan\theta_1 = \frac{v}{t_1} \quad \text{...(i)}$$

The slope of PQ gives the acceleration

$$(-a_2) = \tan\theta_2 = \frac{-v}{t_2} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$v = a_1 t_1 = a_2 t_2. \qquad \text{...(iii)}$$

The area under v–t graph = Area of $\Delta OPQ = \frac{1}{2} OQ.PM = s$

Substituting $OQ = t_1 + t_2$ and PM = v,

the total displacement $s = \frac{v}{2}(t_1 + t_2)$ ...(iv)

Substituting $t_1 = \frac{v}{a_1}$ and $t_2 = \frac{v}{a_2}$ from (iii) in (iv), we have

$$s = \frac{v}{2}\left(\frac{v}{a_1} + \frac{v}{a_2}\right)$$

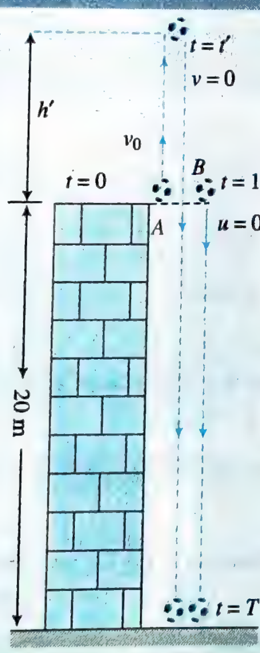
Finally substituting $s = +h$ (as the area is positive), $a_1 = \frac{20}{3}\,\mathrm{m\,s^{-2}}$ and $a_2 = 10\,\mathrm{m\,s^{-2}}$, we have

$$v = \sqrt{\frac{2a_1a_2h}{a_1 + a_2}} = \sqrt{\frac{2\times\frac{20}{3}\times 10\times 50}{\frac{20}{3}+10}}$$

This yields $v = 20\,\mathrm{m\,s^{-1}}$

EXAMPLE 4.16

A ball A is thrown straight up from the edge of the roof of a building. Another ball B is dropped from the roof 1.00 s later. You may ignore air resistance. (a) If the height of the building is 20.0 m, what must the initial speed of ball A be if both are to hit the ground at the same time? (b) On the same graph, sketch the position and velocity of each ball as a function of time, measured from when the first ball is thrown and taken origin at ground.



Sol.

Let ball A is thrown upward at $t = 0$. It takes time T to reach ground. Ball B is dropped from the roof 1.00 s later. If both the balls reach the ground simultaneously, the flying time for ball B will be $(T - 1)$ s.

The displacement of both the balls is (= –20 m).

Using $y = y_0 + ut + \frac{1}{2}at^2$ initially for both balls are at $y_0 = 20$ and when both reaches ground where $y = 0$.

For ball A: $0 = 20 + v_0T + \frac{1}{2}(-10)T^2$ ...(i)

For ball B: $0 = 20 + \frac{1}{2}(-10)(T-1)^2$ ...(ii)

From Eqs. (i) and (ii),

$$v_0T = 5(2T - 1) \qquad \text{...(iii)}$$

From Eqs. (i) and (iii), we get

$$T^2 - 2T - 3 = 0 \Rightarrow (T+1)(T-3) = 0$$
$$T = 3\text{ s}$$

From Eqs. (iii), we get $v_0 = \frac{25}{3}\,\mathrm{m\,s^{-1}}$

Let ball A reaches maximum height at time t' after throwing. For ball A at maximum height, the velocity will be zero.

Using $v = u + at$

$$0 = \frac{25}{3} - 10\times t' \Rightarrow t' = \frac{5}{6}\text{ s}$$

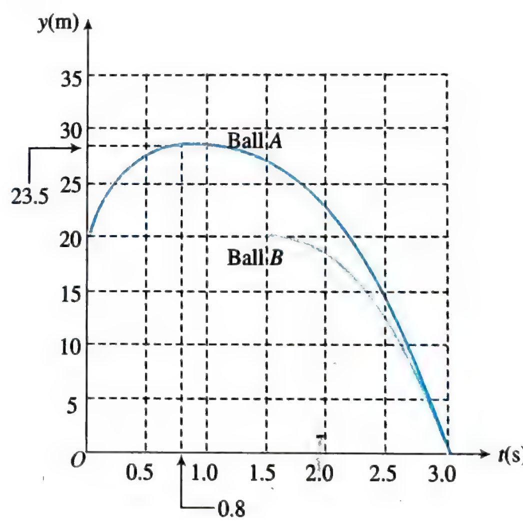
For maximum height reached by ball A,

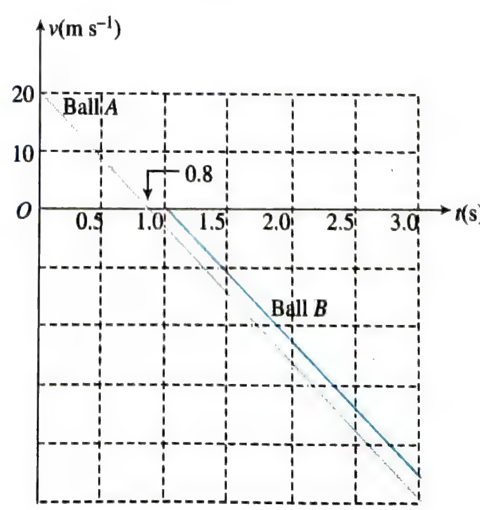
Using $v^2 = u^2 + 2as$

$$0 = \left(\frac{25}{3}\right)^2 - 2\times 10\times h' \Rightarrow h' = \frac{125}{36}\text{ m}$$

Hence, maximum height reach by ball A

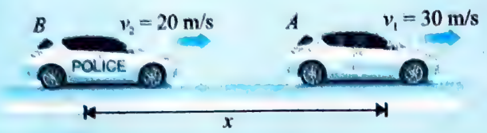
$$H = 20 + \frac{125}{36} = \frac{845}{36}\text{ m}$$





EXAMPLE 4.17

A police car B is chasing a culprit's car A. Cars A and B are moving at constant speed $v_1 = 30$ m/s and $v_2 = 20$ m/s, respectively, along a straight line. The police decides to open fire and a policeman starts firing with his machine gun directly aiming at car A. The bullets have velocity $u = 210$ m/s relative to the gun. The policeman keeps firing for an interval of $T_0 = 40$s. The culprit experiences that the time gap between the first and the last bullet hitting his car is Δt. Find Δt.



Sol. Let x = distance between the car at the instant first bullet is fired (say at time $t = 0$)

Velocity of car A, relative to car B, $v_{AB} = 30 - 20 = 10$ m/s

Speed of bullet, relative to ground, $v_B = u + v_2 = 210 + 20 = 230$ m/s

Velocity of bullet, relative to car A, $v_{BA} = 230 - 30 = 200$ m/s

Time when the first bullet hits the car A,

$$t_1 = \frac{x}{v_{BA}} = \frac{x}{200} \quad \text{...(i)}$$

Distance between the car when the last bullet is fired (at time $t = T_0 = 40$ s),

$$x' = x + v_{AB} \cdot T_0 = x + 10 \times 40 = x + 400$$

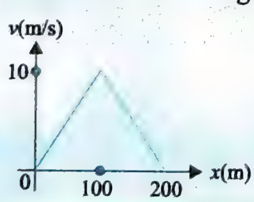
Time when the last bullet hits the car A is

$$t_2 = T_0 + \frac{x'}{v_{BA}} = 40 + \frac{x+400}{200} \quad \text{...(ii)}$$

The interval $\Delta t = t_2 - t_1 = 40 + \dfrac{x+400}{200} - \dfrac{x}{200} = 42\,\text{sec}$

EXAMPLE 4.18

The v-x graph for a car in a race on a straight road is given. Draw a-x graph.



Sol. We know $a = v.\dfrac{dv}{dx}$ ...(i)

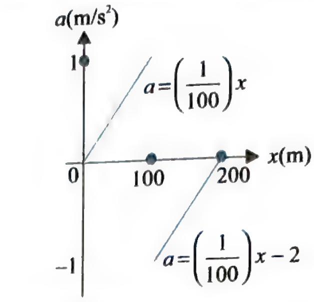
Slope of v-x graph for $0\text{ m} \le x \le 100$ m

$$\frac{dv}{ds} = \frac{10-0}{100-0} = \frac{1}{10}$$

Equation of line: $v = \left(\dfrac{dv}{dx}\right)x \Rightarrow v = \left(\dfrac{1}{10}\right)x$ ...(ii)

From (i) and (ii), a-x relation: $a = \left[\left(\dfrac{1}{10}\right)x\right]\cdot\dfrac{1}{10}$

$$\Rightarrow \quad a = \left(\frac{1}{100}\right)x \quad \text{...(iii)}$$



Acceleration-displacement graph

Slope of v-x graph for $100\text{ m} \le s \le 200$ m

$$\frac{dv}{ds} = \frac{0-10}{200-100} = -\frac{1}{10}$$

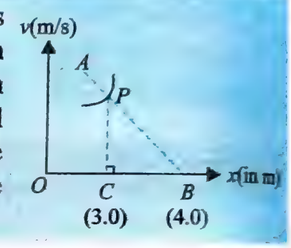
Equation of line: $v - 0 = \left(\dfrac{1}{10}\right)(x - 200)$ ...(iv)

From (i) and (iv) a-x relation: $a = \left[\left(\dfrac{1}{10}\right)(x-200)\right]\left(-\dfrac{1}{10}\right)$

$$\Rightarrow \quad a = \left(\frac{1}{100}\right)x - 2 \quad \text{...(v)}$$

EXAMPLE 4.19

A particle is moving along x axis and its velocity (v) vs position (x) graph is a curve as shown in the figure. Line APB is normal to the curve at point P. Find the instantaneous acceleration of the particle at $x = 3.0$ m.



Sol. Let the velocity of the particle at $x = 3.0$ m be v_0.

The slope of line $APB = -\dfrac{v_0}{1}$

As the line APB is normal to the curve at point P, hence the slope of tangent at $P = \dfrac{1}{v_0}$

It means the slope of v-x graph $\left(\dfrac{dv}{dx}\right)_P = \dfrac{1}{v_0}$ ...(i)

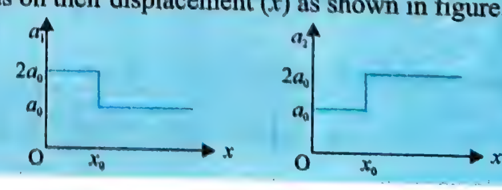
But we define acceleration as $a = v\dfrac{dv}{dx}$ ...(ii)

But at $x = 3.0$m, $v = v_0$ and $\dfrac{dv}{dx} = \left(\dfrac{1}{v_0}\right) a = v_0\left(\dfrac{1}{v_0}\right) = 1\text{ m/s}^2$

Hence $a = v_0\left(\dfrac{1}{v_0}\right) = 1\text{ m/s}^2$

EXAMPLE 4.20

Two particles 1 and 2 start simultaneously from origin and move along the positive x direction. The initial velocity of both particles is zero. The acceleration of the two particles depends on their displacement (x) as shown in figure.



(a) Particles 1 and 2 take t_1 and t_2 time, respectively, for their displacement to become x_0. Find t_1/t_2.

(b) Which particle will cross the point $x = 2x_0$ with greater speed? What is the velocity of the particles at the point $x = 2x_0$?

Sol. (a) For particle 1: $x_0 = \frac{1}{2}(2a_0)t_1^2$ ...(i)

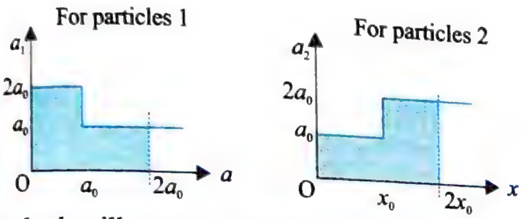
For particle 2: $x_0 = \frac{1}{2}a_0 t_2^2$ ...(ii)

From equation (i) and (ii), $\frac{1}{2}(2a_0)t_1^2 = \frac{1}{2}a_0 t_2^2$

$$\Rightarrow \frac{t_2^2}{t_1^2} = 2 \text{ or } \frac{t_2}{t_1} = \sqrt{2}$$

(b) Speed at any time $v = \sqrt{v_0^2 + 2(\text{area under } a-s \text{ graph})}$

For both particles 1 and 2, this area is same for their displacement $2x_0$

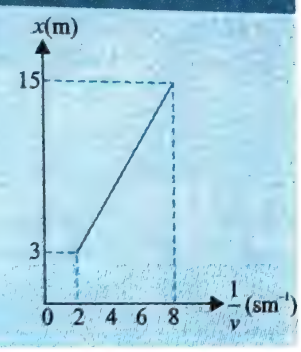


Hence, both will cross $2x_0$ with same speed,

$$v = \sqrt{0 + 2(3a_0x_0)} = \sqrt{6a_0x_0}$$

EXAMPLE 4.21

The graph of position (x) vs inverse of velocity ($1/v$) for a particle moving on a straight line is as shown in the figure. Find the time taken by the particle to move from $x = 3$ m to $x = 15$ m.



Sol. The equation of the straight line shown in the graph is

$$(x-3) = \left(\frac{15-3}{8-2}\right)\left(\frac{1}{v}-2\right)$$

$$(x-3) = 2\left(\frac{1}{v}-2\right) = \frac{2}{v}-4$$

which gives $x = \frac{2}{v} - 1$ ...(i)

But $v = \frac{dx}{dt}$ ...(ii)

From (i) and (ii), we get

$\therefore \quad x = \frac{2dt}{dx} - 1 \Rightarrow x\,dx = 2\,dt - dx$ ...(iii)

Integrating both sides, $\int_3^{15} x\,dx = 2\int_0^t dt - \int_3^{15} dx$

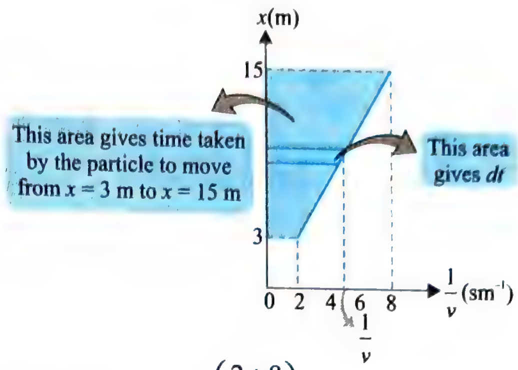
$$\Rightarrow \quad \frac{1}{2}[225-9] = 2t - [15-3] \Rightarrow t = 60\,\text{s}$$

Alternate Approach

We know, $v = \frac{dx}{dt} \Rightarrow dt = \left(\frac{1}{v}\right)dx$ ...(i)

Integrating equation (i) both sides, $\int dt = t = \int\left(\frac{1}{v}\right)dx$

Here $\int\left(\frac{1}{v}\right)dx$ = Area under $\left(\frac{1}{v}\right)$ vs x graph

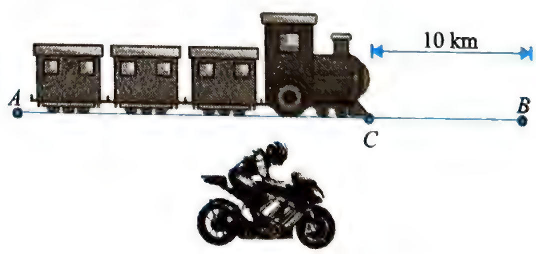


Hence, total time taken, $t = \left(\frac{2+8}{2}\right)\times(15-3) = 60$ s

EXAMPLE 4.22

A railway track runs parallel to a road until a turn brings the road to railway crossing. A cyclist rides along the road every day at a constant speed of 20 km/h. He normally meets a train that travels in same direction at the crossing. One day he was late by 25 minutes and met the train 10 km before the railway crossing. Find the speed of the train.

Sol. Let every day the man and train meet at point B at time t and on particular day when the man is late by 25 min he will reach point B at time $(t + 25)$min. The train meets the cyclist at C which is 10 km from B.



Hence time taken by cyclist to cover the distance CB,

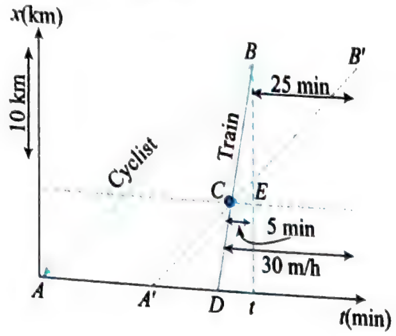
$$t_{CB} = \frac{10}{20} = \frac{1}{2}\text{hr} = 30\,\text{min}$$

It means time taken by train to cover the same distance (CB) will be 5 min.

The speed of the train $v_{\text{train}} = \frac{10}{\left(5/60\right)} = 120\,\text{km/hr}$

Let us draw the position–time graph of the situation. As the speeds of the cyclist and train is constant, hence the position-time graph will be a straight line. The slope of position–time graph gives velocity. The velocity of cyclist is less than the velocity of train. AB line represents the motion of cyclist and line DB represents motion of the train. Both lines meet at point B, at this position train meets cyclist every day.

The line $A'B'$ represents the motion of the cyclist, at the particular day when he is late by 25 min. The line $A'B'$ cuts the line BD at C; C is the position where train crosses the cyclist at the day when the cyclist is late.



Time taken by cyclist to move from C to crossing

$$t_{CB} = \frac{10}{20} = \frac{1}{2}\,\text{m} = 30\,\text{min}$$

From graph it is clear that time taken by train to move from C to crossing should be 5 min.

Hence, velocity of the train $v_{\text{train}} = \dfrac{10}{(5/60)} = 120\,\text{km/hr}$

Exercises

Single Correct Answer Type

1. Between two stations, a train accelerates from rest uniformly at first, then moves with constant velocity, and finally retards uniformly to come to rest. If the ratio of the time taken is 1 : 8 : 1 and the maximum speed attained be 60 km h^{-1}, then what is the average speed over the whole journey?
(1) 48 km h^{-1} (2) 52 km h^{-1}
(3) 54 km h^{-1} (4) 56 km h^{-1}

2. The velocity acquired by a body moving with uniform acceleration is 30 m s^{-1} in 2 s and 60 m s^{-1} in 4 s. The initial velocity is
(1) zero (2) 2 m s^{-1}
(3) 3 m s^{-1} (4) 10 m s^{-1}

3. A particle starts from the origin with a velocity of 10 m s^{-1} and moves with a constant acceleration till the velocity increases to 50 m s^{-1}. At that instant, the acceleration is suddenly reversed. What will be the velocity of the particle, when it returns to the starting point?
(1) Zero (2) 10 m s^{-1}
(3) 50 m s^{-1} (4) 70 m s^{-1}

4. A particle is moving along the x-axis whose instantaneous speed is given by $v^2 = 108 - 9x^2$. The acceleration of the particle is
(1) $-9x$ m s^{-2} (2) $-18x$ m s^{-2}
(3) $\frac{-9x}{2}$ m s^{-2} (4) None of these

5. A ball is released from the top of a tower of height h. It takes time T to reach the ground. What is the position of the ball (from ground) after time $T/3$?
(1) $h/9$ m (2) $7h/9$ m
(3) $8h/9$ m (4) $17h/18$ m

6. Taxies leave station X for station Y every 10 min. Simultaneously, a taxi also leaves station Y for station X every 10 min. The taxies move at the same constant speed and go from X and Y or vice-versa in 2 h. How many taxies coming from the other side will meet each taxi enroute from Y and X?
(1) 24 (2) 23
(3) 12 (4) 11

7. When the speed of a car is u, the minimum distance over which it can be stopped is s. If the speed becomes nu, what will be the minimum distance over which it can be stopped during the same time?
(1) s/n (2) ns
(3) s/n^2 (4) n^2s

8. A thief is running away on a straight road in a jeep moving with a speed of 9 m s^{-1}. A policeman chases him on a motor cycle moving at a speed of 10 m s^{-1}. If the instantaneous separation of the jeep from the motor cycle is 100 m, how long will it take for the policeman to catch the thief?
(1) 1 s (2) 19 s
(3) 90 s (4) 100 s

9. A body is released from the top of a tower of height H m. After 2 s it is stopped and then instantaneously released. What will be its height after next 2 s?
(1) $(H - 5)$ m (2) $(H - 10)$ m
(3) $(H - 20)$ m (4) $(H - 40)$ m

10. A stone is dropped from the top of a tower of height h. After 1 s another stone is dropped from the balcony 20 m below the top. Both reach the bottom simultaneously. What is the value of h? Take $g = 10$ m s^{-2}.
(1) 3125 m (2) 312.5 m
(3) 31.25 m (4) 25.31 m

11. A train 100 m long travelling at 40 m s^{-1} starts overtaking another train 200 m long travelling at 30 m s^{-1}. The time taken by the first train to pass the second train completely is
(1) 30 s (2) 40 s
(3) 50 s (4) 60 s

12. A person is throwing two balls in the air one after the other. He throws the second ball when the first ball is at the highest point. If he is throwing the balls every second, how high do they rise?
(1) 5 m (2) 3.75 m
(3) 2.50 m (4) 1.25 m

13. A stone thrown upwards with speed u attains maximum height h. Another stone thrown upwards from the same point with speed $2u$ attains maximum height H. What is the relation between h and H?
(1) $2h = H$ (2) $3h = H$
(3) $4h = H$ (4) $5h = H$

14. A body dropped from the top of a tower covers a distance $7x$ in the last second of its journey, where x is the distance covered in the first second. How much time does it take to reach the ground?
(1) 3 s (2) 4 s
(3) 5 s (4) 6 s

15. The relation between time t and distance x is $t = \alpha x^2 + \beta x$ where α and β are constants. The retardation is
(1) $2\alpha v^3$ (2) $2\beta v^3$
(3) $2\alpha\beta v^3$ (4) $2b^2v^3$

16. The displacement x of a particle moving in one dimension under the action of a constant force is related to time t by the equation $t = \sqrt{x} + 3$, where x is in meters and t is in seconds. Find the displacement of the particle when its velocity is zero.
(1) Zero (2) 12 m
(3) 6 m (4) 18 m

17. The distances moved by a freely falling body (starting from rest) during 1st, 2nd, 3rd, ..., nth second of its motion are proportional to
(1) Even numbers
(2) Odd numbers
(3) All integral numbers
(4) Squares of integral numbers

18. A drunkard is walking along a straight road. He takes five steps forward and three steps backward and so on. Each step is 1 m long and takes 1 s. There is a pit on the road 11 m away from the starting point. The drunkard will fall into the pit after

(1) 29 s (2) 21 s
(3) 37 s (4) 31 s

19. A stone is dropped from a certain height which can reach the ground in 5 s. It is stopped after 3 s of its fall and then it is again released. The total time taken by the stone to reach the ground will be

(1) 6 s (2) 6.5 s
(3) 7 s (4) 7.5 s

20. A body travels 200 cm in the first 2 s and 220 cm in the next 4 s with deceleration. The velocity of the body at the end of the seventh second is

(1) 5 $cm s^{-1}$ (2) 10 $cm s^{-1}$
(3) 15 $cm s^{-1}$ (4) 20 $cm s^{-1}$

21. A body starts from rest and travels a distance S with uniform acceleration, then moves uniformly a distance $2S$ uniformly, and finally comes to rest after moving further $5S$ under uniform retardation. The ratio of the average velocity to maximum velocity is

(1) 2/5 (2) 3/5
(3) 4/7 (4) 5/7

22. A body sliding on a smooth inclined plane requires 4 s to reach the bottom, starting from rest at the top. How much time does it take to cover one-fourth the distance starting from rest at the top?

(1) 1 s (2) 2 s
(3) 4 s (4) 16 s

23. B_1, B_2, and B_3 are three balloons ascending with velocities v, $2v$, and $3v$, respectively. If a bomb is dropped from each when they are at the same height, then

(1) Bomb from B_1 reaches ground first
(2) Bomb from B_2 reaches ground first
(3) Bomb from B_3 reaches ground first
(4) They reach the ground simultaneously

24. A particle is dropped from rest from a large height. Assume g to be constant throughout the motion. The time taken by it to fall through successive distances of 1 m each will be

(1) All equal, being equal to $\sqrt{2/g}$ second
(2) In the ratio of the square roots of the integers 1, 2, 3, ...
(3) In the ratio of the difference in the square roots of the integers, i.e., $\sqrt{1}, (\sqrt{2}-\sqrt{1}), (\sqrt{3}-\sqrt{2}), (\sqrt{4}-\sqrt{3}), \ldots$
(4) In the ratio of the reciprocals of the square roots of the integers, i.e., $\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots$

25. A ball is dropped into a well in which the water level is at a depth h below the top. If the speed of sound is c, then the time after which the splash is heard will be given by

(1) $h\left[\sqrt{\frac{2}{gh}}+\frac{1}{c}\right]$ (2) $h\left[\sqrt{\frac{2}{gh}}-\frac{1}{c}\right]$

(3) $h\left[\frac{2}{g}+\frac{1}{c}\right]$ (4) $h\left[\frac{2}{g}-\frac{1}{c}\right]$

26. If a particle travels n equal distances with speeds $v_1, v_2, \ldots, v_n$ then the average speed $\bar{V}$ of the particle will be such that

(1) $\bar{V}=\frac{v_1+v_2+\cdots+v_n}{n}$

(2) $\bar{V}=\frac{nv_1v_2+v_n}{v_1+v_2+v_3+\cdots+v_n}$

(3) $\frac{1}{\bar{V}}=\frac{1}{n}\left(\frac{1}{v_1}+\frac{1}{v_2}+\cdots+\frac{1}{v_n}\right)$

(4) $\bar{V}=\sqrt{v_1^2+v_2^2+\cdots+v_n^2}$

27. A ball is thrown from the top of a tower in vertically upward direction. Velocity at a point h meter below the point of projection is twice of the velocity at a point h meter above the point of projection. Find the maximum height reached by the ball above the top of tower.

(1) $2h$ (2) $3h$
(3) $(5/3)h$ (4) $(4/3)h$

28. A juggler keeps on moving four balls in air throwing the balls after regular intervals. When one ball leaves his hand (speed = 20 $m s^{-1}$), the position of other balls (height in meter) will be (take $g = 10\ m s^{-2}$)

(1) 10, 20, 10 (2) 15, 20, 15
(3) 5, 15, 20 (4) 5, 10, 20

29. A particle slides from rest from the topmost point of a vertical circle of radius r along a smooth chord making an angle θ with the vertical. The time of descent is

(1) Least for $\theta = 0$ (2) Maximum for $\theta = 0$
(3) Least for $\theta = 45°$ (4) Independent of θ

30. A body is thrown vertically upwards from A, the top of a tower. It reaches the ground in time t_1. If it is thrown vertically downwards from A with the same speed, it reaches the ground in time t_2. If it is allowed to fall freely from A, then the time it takes to reach the ground is given by

(1) $t=\frac{t_1+t_2}{2}$ (2) $t=\frac{t_1-t_2}{2}$

(3) $t=\sqrt{t_1t_2}$ (4) $t=\sqrt{\frac{t_1}{t_2}}$

31. The deceleration experienced by a moving motor boat, after its engine is cut-off is given by $dv/dt = -kv^3$, where k is constant. If v_0 is the magnitude of the velocity at cut-off, the magnitude of the velocity at a time t after the cut-off is

(1) $v_0/2$ (2) v

(3) v_0e^{-kt} (4) $\frac{v_0}{\sqrt{2v_0^2kt+1}}$

32. For motion of an object along the x-axis, the velocity v depends on the displacement x as $v = 3x^2 - 2x$, then what is the acceleration at $x = 2$ m.

(1) 48 $m s^{-2}$ (2) 80 $m s^{-2}$
(3) 18 $m s^{-2}$ (4) 10 $m s^{-2}$

33. A stone is dropped from the 25th storey of a multistoried building and it reaches the ground in 5 s. In the first second, it passes through how many storeys of the building? ($g = 10\ m s^{-2}$)

(1) 1 (2) 2
(3) 3 (4) none of these

34. A body is projected upwards with a velocity u. It passes through a certain point above the ground after t_1. The time after which the body passes through the same point during the return journey is

(1) $\left(\frac{u}{g}-t_1^2\right)$ (2) $2\left(\frac{u}{g}-t_1\right)$

(3) $3\left(\frac{u^2}{g}-t_1\right)$ (4) $3\left(\frac{u^2}{g^2}-t_1\right)$

35. A parachutist drops first freely from an aeroplane for 10 s and then his parachute opens out. Now he descends with a net retardation of 2.5 m s^{-2}. If he bails out of the plane at a height of 2495 m and $g = 10$ m s^{-2}, his velocity on reaching the ground will be

(1) 5 m s^{-1} (2) 10 m s^{-1}

(3) 15 m s^{-1} (4) 20 m s^{-1}

36. A police party is chasing a dacoit in a jeep which is moving at a constant speed v. The dacoit is on a motorcycle. When he is at a distance x from the jeep, he accelerates from rest at a constant rate. Which of the following relations is true if the police is able to catch the dacoit?

(1) $v^2 \le \alpha x$ (2) $v^2 \le 2\alpha x$

(3) $v^2 \ge 2\alpha x$ (4) $v^2 \ge \alpha x$

37. A train is moving at a constant speed V when its driver observes another train in front of him on the same track and moving in the same direction with constant speed v. If the distance between the trains is x, then what should be the minimum retardation of the train so as to avoid collision?

(1) $\frac{(V+v)^2}{x}$ (2) $\frac{(V-v)^2}{x}$

(3) $\frac{(V+v)^2}{2x}$ (4) $\frac{(V-v)^2}{2x}$

38. A moving car possesses average velocities of 5 m s^{-1}, 10 m s^{-1}, and 15 m s^{-1} in the first, second, and third seconds, respectively. What is the total distance covered by the car in these 3 s?

(1) 15 m (2) 30 m

(3) 55 m (4) None of these

39. The average velocity of a body moving with uniform acceleration after travelling a distance of 3.06 m is 0.34 m s^{-1}. If the change in velocity of the body is 0.18 m s^{-1} during this time, its uniform acceleration is

(1) 0.01 m s^{-2} (2) 0.02 m s^{-2}

(3) 0.03 m s^{-2} (4) 0.04 m s^{-2}

40. Water drops fall from a tap on the floor 5 m below at regular intervals of time, the first drop striking the floor when the fifth drop begins to fall. The height at which the third drop will be from ground (at the instant when the first drop strikes the ground) will be ($g = 10$ m s^{-2})

(1) 1.25 m (2) 2.15 m

(3) 2.75 m (4) 3.75 m

41. Drops of water fall at regular intervals from roof of a building of height $H = 16$ m, the first drop striking the ground at the same moment as the fifth drop detaches itself from the roof. The distances between separate drops in air as the first drop reaches the ground are

(1) 1 m, 5 m, 7 m, 3 m (2) 1 m, 3 m, 5 m, 7 m

(3) 1 m, 3 m, 7 m, 5 m (4) None of the above

42. A point moves in a straight line so that its displacement x metre at time t second is given by $x^2 = 1 + t^2$. Its acceleration in m s^{-2} at time t second is

(1) $\frac{1}{x^3}$ (2) $\frac{-t}{x^3}$

(3) $\frac{1}{x}-\frac{t^2}{x^3}$ (4) $\frac{1}{x}-\frac{1}{x^2}$

43. A point moves with uniform acceleration and v_1, v_2, and v_3 denote the average velocities in the three successive intervals of time t_1, t_2, and t_3. Which of the following relations is correct?

(1) $(v_1-v_2):(v_2-v_3)=(t_1-t_2):(t_2+t_3)$

(2) $(v_1-v_2):(v_2-v_3)=(t_1+t_2):(t_2+t_3)$

(3) $(v_1-v_2):(v_2-v_3)=(t_1-t_2):(t_1-t_3)$

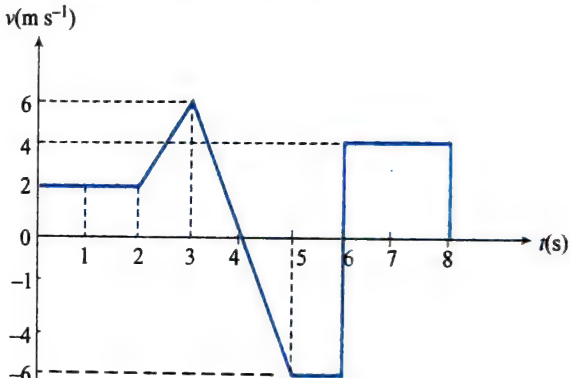
(4) $(v_1-v_2):(v_2-v_3)=(t_1-t_2):(t_2-t_3)$

44. A 2-m wide truck is moving with a uniform speed $v_0 = 8$ m s^{-1} along a straight horizontal road. A pedestrian starts to cross the road with a uniform speed v when the truck is 4 m away from him. The minimum value of v so that he can cross the road safely is

(1) 2.62 m s^{-1} (2) 4.6 m s^{-1}

(3) 3.57 m s^{-1} (4) 1.414 m s^{-1}

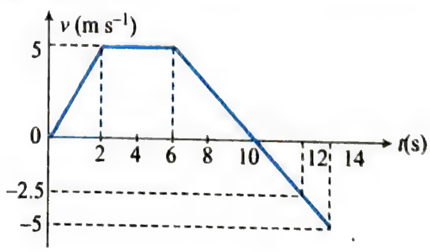
45. The velocity–time graph of a body is shown in figure. The displacement of the body in 8 s is



(1) 9 m (2) 12 m

(3) 10 m (4) 28 m

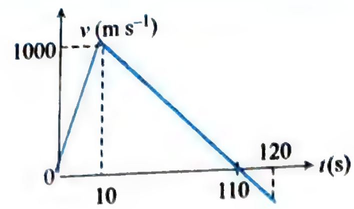
46. The variation of velocity of a particle moving along a straight line is shown in figure. The distance travelled by the particle in 12 s is



(1) 37.5 m (2) 32.5 m

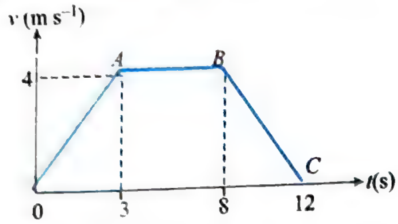
(3) 35.0 m (4) None of these

47. The following graph figure shows the variation of velocity of a rocket with time. Then the maximum height attained by the rocket is



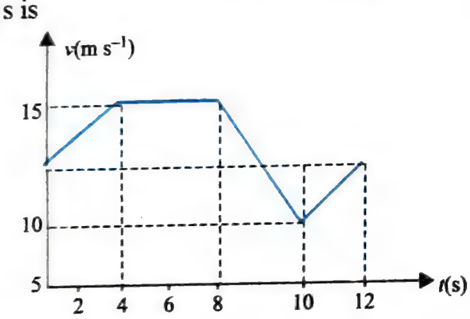
(1) 1.1 km (2) 5 km
(3) 55 km (4) None of these

48. From the velocity–time graph, given in figure of a particle moving in a straight line, one can conclude that



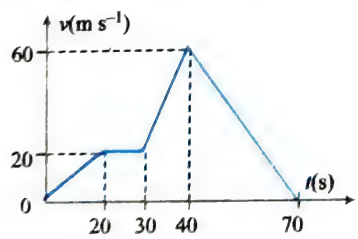
(1) Its average velocity during the 12 s interval is 24/7 m s^{-1}.
(2) Its velocity for the first 3 s is uniform and is equal to 4 m s^{-1}.
(3) The body has a constant acceleration between $t = 3$ s and $t = 8$ s.
(4) The body has a uniform retardation from $t = 8$ s to $t = 12$ s.

49. The velocity–time graph of a particle moving in a straight line is shown in figure. The acceleration of the particle at $t = 9$ s is



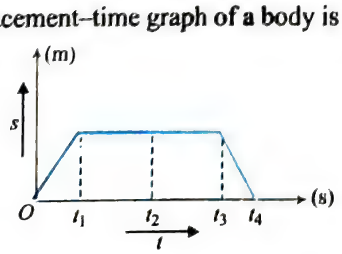
(1) Zero (2) 5 m s^{-2}
(3) –5 m s^{-2} (4) –2 m s^{-2}

50. The velocity–time graph of a body is given in figure. The maximum acceleration in m s^{-2} is

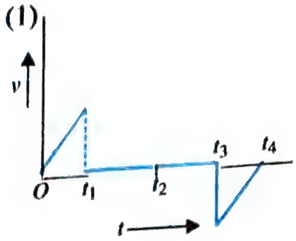


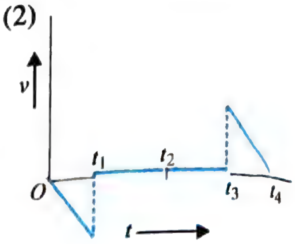
(1) 4 (2) 3
(3) 2 (4) 1

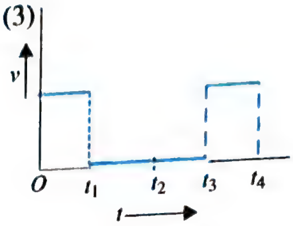
51. The displacement–time graph of a body is shown in figure.

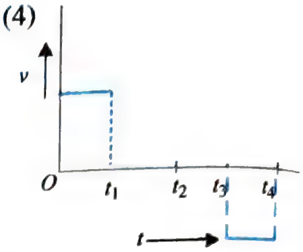


The velocity–time graph of the motion of the body will be

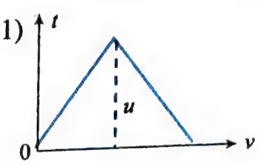
(1)


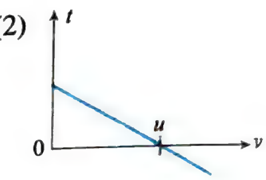
(2)


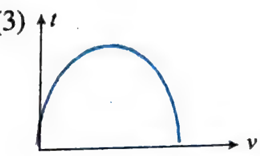
(3)


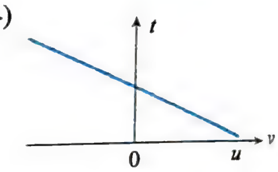
(4)


52. An object is thrown up vertically. The velocity–time graph for the motion of the particle is

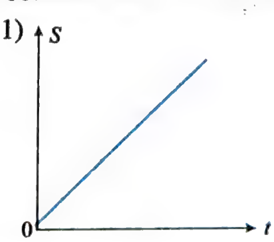
(1)


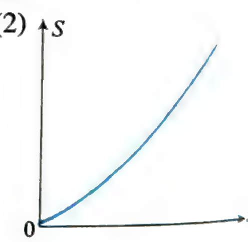
(2)


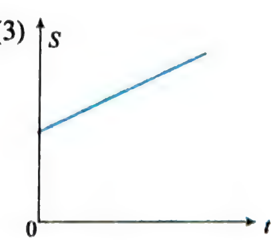
(3)


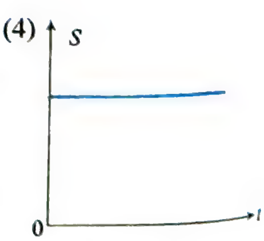
(4)


53. From a high tower, at time $t = 0$, one stone is dropped from rest and simultaneously another stone is projected vertically up with an initial velocity. The graph of distance S between the two stones plotted against time t will be

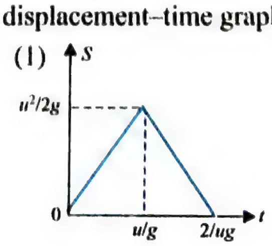
(1)


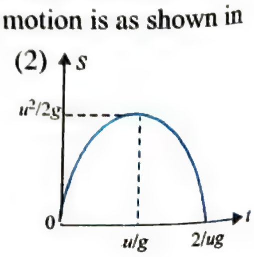
(2)


(3)


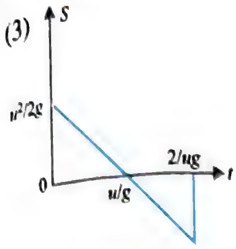
(4)


54. An object is vertically thrown upwards. Then the displacement–time graph for the motion is as shown in

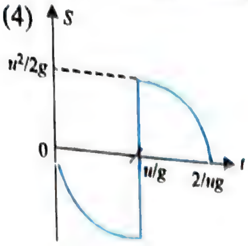
(1)


(2)


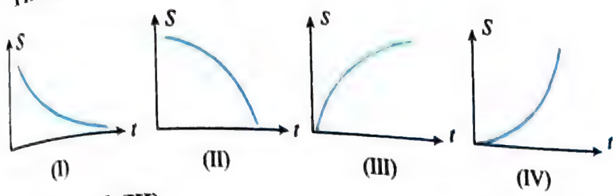
(3)



(4)

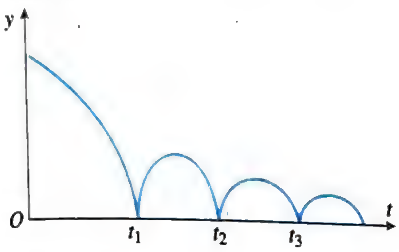


55. The acceleration will be positive in



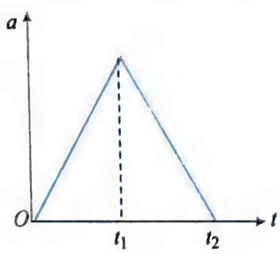
(1) (I) and (III)
(2) (I) and (IV)
(3) (II) and (IV)
(4) None of these

56. The graph as shown in figure below describes the motion of a ball rebounding from a horizontal surface being released from a point above the surface. Assume that the ball collides each time with the floor inelastically. The quantity represented on the y-axis is the ball's (take upward direction as positive)

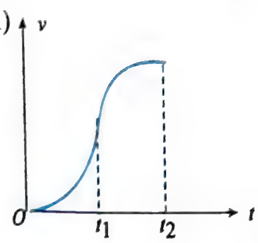


(1) Displacement
(2) Velocity
(3) Acceleration
(4) Momentum

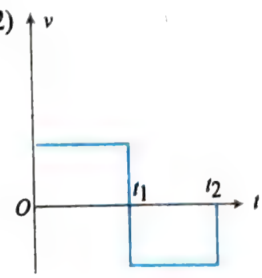
57. The acceleration versus time graph of a particle is shown in figure. The respective v–t graph of the particle is



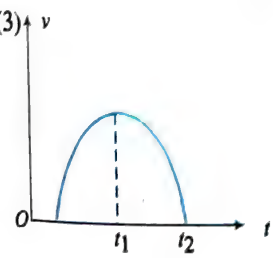
(1)



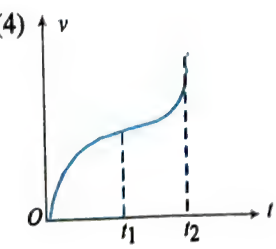
(2)



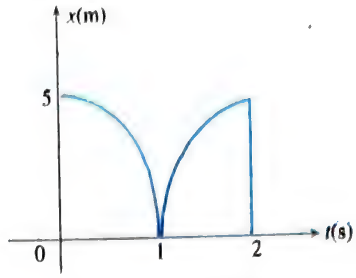
(3)



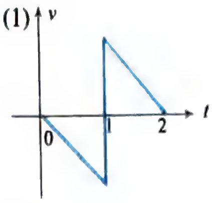
(4)



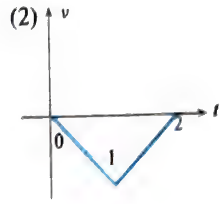
58. The displacement–time graph of a moving particle with constant acceleration is shown in figure. The velocity–time graph is given by



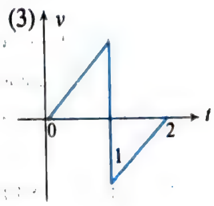
(1)



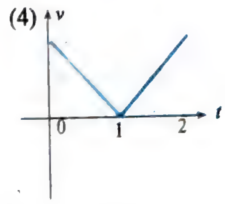
(2)



(3)

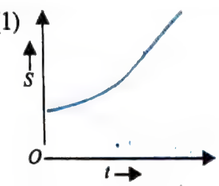


(4)

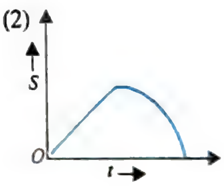


59. Two balls are dropped from the top of a high tower with a time interval of t_0 second, where t_0 is smaller than the time taken by the first ball to reach the floor, which is perfectly inelastic. The distance S between the two balls, plotted against the time lapse t from the instant of dropping the second ball, is best represented by

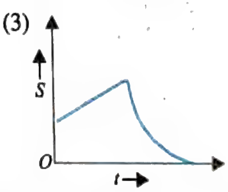
(1)



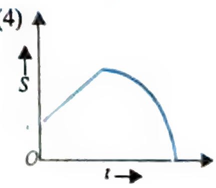
(2)



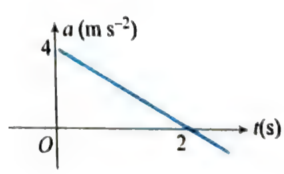
(3)



(4)

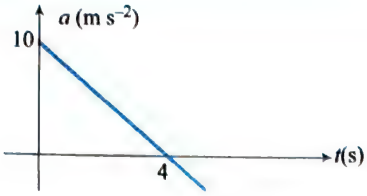


60. The acceleration versus time graph of a particle moving in a straight line is shown in figure. The velocity–time graph of the particle would be



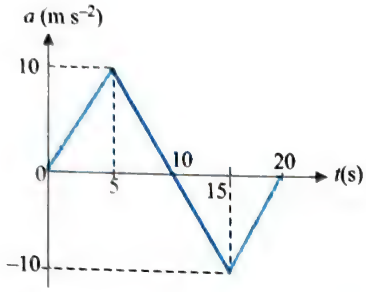
(1) A straight line
(2) A parabola
(3) A circle
(4) An ellipse

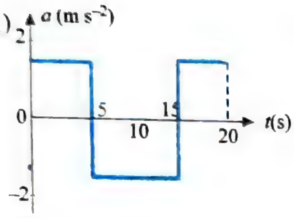
61. The acceleration–time graph of a particle moving along a straight line is as shown in figure. At what time the particle acquires its initial velocity?

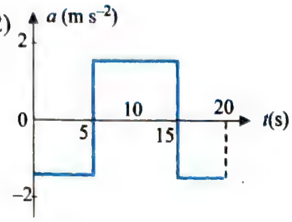


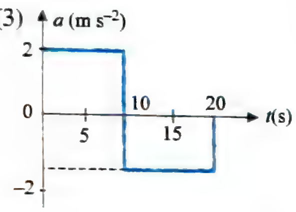
(1) 12 s (2) 5 s
(3) 8 s (4) 16 s

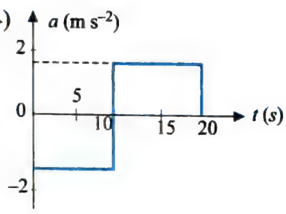
62. Plot the acceleration–time graph of the velocity–time graph given in figure.



(1)


(2)


(3)


(4)


63. A particle is moving along the x-axis whose acceleration is given by $a = 3x - 4$, where x is the location of the particle. At $t = 0$, the particle is at rest at $x = 4/3$ m. The distance travelled by the particle in 5 s is

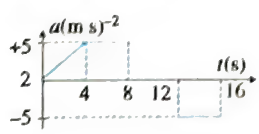
(1) zero (2) 42 m
(3) Infinite (4) None of these

64. A point moves such that its displacement as a function of time is given by $x^3 = t^3 + 1$. Its acceleration as a function of time t will be

(1) $\dfrac{2}{x^5}$ (2) $\dfrac{2t}{x^5}$

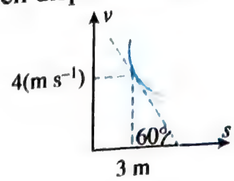
(3) $\dfrac{2t}{x^4}$ (4) $\dfrac{2t^2}{x^5}$

65. The acceleration of a particle starting from rest and travelling along a straight line is shown in figure. The maximum speed of the particle is



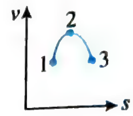
(1) 20 m s^{-1} (2) 30 m s^{-1}
(3) 40 m s^{-1} (4) 60 m s^{-1}

66. A particle is moving along a straight line whose velocity–displacement graph is shown in figure. What is the acceleration when displacement is 3 m?



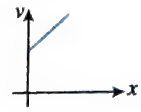
(1) $4\sqrt{3}$ m s^{-2} (2) $3\sqrt{3}$ m s^{-2}
(3) $\sqrt{3}$ m s^{-2} (4) $4/\sqrt{3}$ m s^{-2}

67. Figure below shows the velocity–displacement curve for an object moving along a straight line. At which of the points marked is the object speeding up?



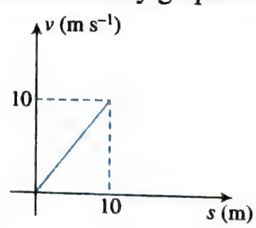
(1) 1 (2) 2
(3) 1 and 3 (4) 1, 2, and 3

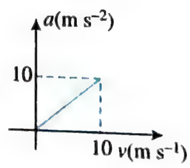
68. Velocity versus displacement graph of a particle moving in a straight line as shown in figure. The acceleration of the particle

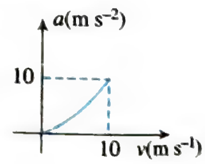


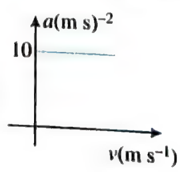
(1) is constant
(2) increases linearly with x
(3) increases parabolically with x
(4) None of these

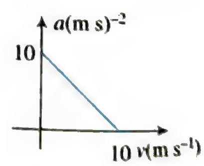
69. Velocity versus displacement graph of a particle moving in a straight line is shown in figure. The corresponding acceleration versus velocity graph will be



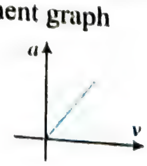
(1)


(2)


(3)


(4)


70. The acceleration–velocity graph of a particle moving in a straight line is shown in figure. Then the slope of the velocity–displacement graph



(1) increases linearly (2) decreases linearly
(3) is constant (4) increases parabolically

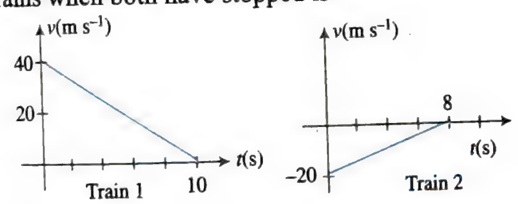
71. A particle starts moving rectilinearly at time $t = 0$ such that its velocity v changes with time t according to the equation $v = t^2 - t$, where t is in seconds and v is in m s^{-1}. The time interval for which the particle retards (i.e., magnitude of velocity decreases) is
(1) $t < 1/2$ (2) $1/2 < t < 1$
(3) $t > 1$ (4) $t < 1/2$ and $t > 1$

72. An object is moving in the x-y plane with the position as a function of time given by $\vec{r} = x(t)\hat{i} + y(t)\hat{j}$. Point O is at $x = 0, y = 0$. The object is definitely moving towards O when
(1) $v_x > 0, v_y > 0$ (2) $v_x < 0, v_y < 0$
(3) $xv_x + yv_y < 0$ (4) $xv_x + yv_y > 0$

73. An object has velocity $\vec{v}_1$ w.r.t. ground. An observer moving with constant velocity $\vec{v}_0$ w.r.t. ground measures the velocity of the object as $\vec{v}_2$. The magnitudes of three velocities are related by
(1) $v_0 \geq v_1 + v_2$ (2) $v_1 \leq v_2 + v_0$
(3) $v_2 \geq v_1 + v_0$ (4) All of the above

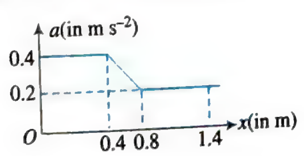
74. A man swimming downstream overcomes a float at a point M. After travelling distance D, he turned back and passed the float at a distance of $D/2$ from the point M. Then the ratio of speed of swimmer with respect to still water to the speed of the river will be
(1) 1 (2) 2
(3) 4 (4) 3

75. Two trains, which are moving along different tracks in opposite directions, are put on the same track due to a mistake. Their drivers, on noticing the mistake, start slowing down the trains when the trains are 300 m apart. Graphs given in figure show their velocities as function of time as the trains slow down. The separation between the trains when both have stopped is



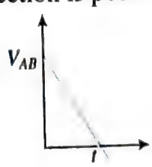
(1) 120 m (2) 280 m
(3) 60 m (4) 20 m

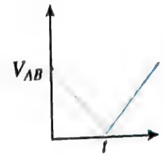
76. The acceleration of a particle which moves along the positive x-axis varies with its position as shown in figure. If the velocity of the particle is 0.8 m s^{-1} at $x = 0$, then velocity of the particle at $x = 1.4$ m is (in m s^{-1})

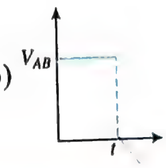


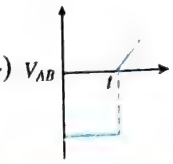
(1) 1.6 (2) 1.2
(3) 1.4 (4) none of these

77. A body A is thrown vertically upwards with such a velocity that it reaches a maximum height of h. Simultaneously, another body B is dropped from height h. It strikes the ground and does not rebound. The velocity of A relative to B versus time graph is best represented by (upward direction is positive)

(1)


(2)


(3)


(4)


Multiple Correct Answers Type

1. Check up the only correct statements in the following:
(1) A body having a constant velocity still can have varying speed.
(2) A body having a constant speed can have varying velocity.
(3) A body having constant speed can have an acceleration.
(4) If velocity and acceleration are in the same direction, then distance is equal to displacement.

2. A block slides down a smooth inclined plane when released from the top, while another falls freely from the same point. Which of the following is/are correct?
(1) Sliding block will reach the ground first
(2) Freely falling block will reach the ground first
(3) Both the blocks will reach the ground with different speeds
(4) Both the blocks will reach the ground with same speed

3. A car accelerates from rest at a constant rate of 2 m s^{-2} for some time. Then it retards at a constant rate of 4 m s^{-2} and comes to rest. It remains in motion for 6 s.
(1) Its maximum speed is 8 m s^{-1}
(2) Its maximum speed is 6 m s^{-1}
(3) It travelled a total distance of 24 m.
(4) It travelled a total distance of 18 m.

4. At $t = 0$, an arrow is fired vertically upwards with a speed of 100 m s^{-1}. A second arrow is fired vertically upwards with the same speed at $t = 5$ s. Then
(1) The two arrows will be at the same height above the ground at $t = 12.5$ s.
(2) The two arrows will reach back their starting points at $t = 20$ s and at $t = 25$ s.
(3) The ratio of the speeds of the first and second arrows at $t = 20$ s will be 2 : 1.
(4) The maximum height attained by either arrow will be 1000 m.

5. Two bodies of masses m_1 and m_2 are dropped from heights h_1 and h_2, respectively. They reach the ground after time t_1 and t_2 and strike the ground with v_1 and v_2, respectively. Choose the correct relations from the following.

(1) $\frac{t_1}{t_2}=\sqrt{\frac{h_1}{h_2}}$ (2) $\frac{t_1}{t_2}=\sqrt{\frac{h_2}{h_1}}$

(3) $\frac{v_1}{v_2}=\sqrt{\frac{h_1}{h_2}}$ (4) $\frac{v_1}{v_2}=\frac{h_2}{h_1}$

6. From the top of a tower of height 200 m, a ball A is projected up with 10 m s^{-1}, and 2 s later another ball B is projected vertically down with the same speed. Then
   (1) Both A and B will reach the ground simultaneously.
   (2) Ball A will hit the ground 2 s later than B hitting the ground.
   (3) Both the balls will hit the ground with the same velocity.
   (4) Both the balls will hit the ground with different velocity.

7. A body starts from rest and then moves with uniform acceleration. Then
   (1) Its displacement is directly proportional to the square of the time.
   (2) Its displacement is inversely proportional to the square of the time.
   (3) It may move along a circle.
   (4) It always moves in a straight line.

8. Which of the following statements is/are correct?
   (1) If the velocity of a body changes, it must have some acceleration.
   (2) If the speed of a body changes, it must have some acceleration.
   (3) If the body has acceleration, its speed must change.
   (4) If the body has acceleration, its speed may change.

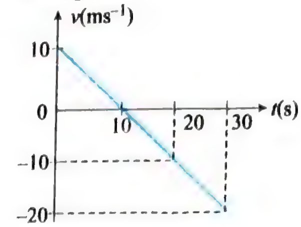
9. The body will speed up if
   (1) Velocity and acceleration are in the same direction.
   (2) Velocity and acceleration are in opposite directions.
   (3) Velocity and acceleration are in perpendicular direction.
   (4) Velocity and acceleration are acting at acute angle w.r.t. each other.

10. Average acceleration is in the direction of
   (1) Initial velocity (2) Final velocity
   (3) Change in velocity
   (4) Final velocity if initial velocity is zero.

11. A particle is projected vertically upward with velocity u from a point A, when it returns to the point of projection
   (1) Its average speed is $u/2$.
   (2) Its average velocity is zero.
   (3) Its displacement is zero.
   (4) Its average speed is u.

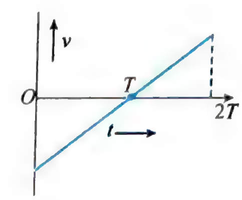
12. A particle moves along a straight line and its velocity depends on time as $v = 4t - t^2$. Then for first 5 s:
   (1) Average velocity is 25/3 m s^{-1}
   (2) Average speed is 10 m s^{-1}
   (3) Average velocity is 5/3 m s^{-1}
   (4) Acceleration is 4 m s^{-2} at $t = 0$

13. The velocity–time plot for a particle moving on a straight line is shown in figure.



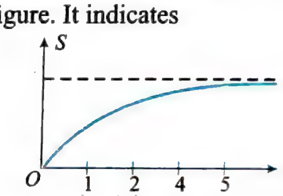
   (1) The particle has a constant acceleration.
   (2) The particle has never turned around.
   (3) The particle has zero displacement.
   (4) The average speed in the interval 0 to 10 s is the same as the average speed in the interval 10 s to 20 s.

14. Figure shows the velocity (v) of a particle plotted against time (t).



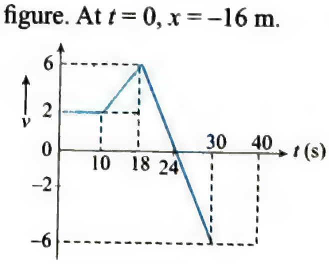
   (1) The particle changes its direction of motion at some point
   (2) The acceleration of the particle remains constant.
   (3) The displacement of the particle is zero.
   (4) The initial and final speeds of the particle are the same

15. The displacement of a particle as a function of time is shown in figure. It indicates



   (1) The particle starts with a certain velocity, but the motion is retarded and finally the particle stops.
   (2) The velocity of the particle decreases.
   (3) The acceleration of the particle is in opposite direction to the velocity.
   (4) The particle starts with a constant velocity, the motion is accelerated and finally the particle moves with another constant velocity.

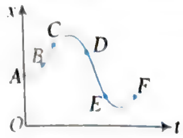
16. A particle moves in a straight line with the velocity as shown in figure. At $t = 0$, $x = -16$ m.



   (1) The maximum value of the position coordinate of the particle is 54 m.
   (2) The maximum value of the position coordinate of the particle is 36 m.
   (3) The particle is at the position of 36 m at t = 18 s.
   (4) The particle is at the position of 36 m at t = 30 s.

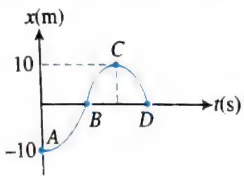
17. For a particle moving along the x-axis, mark the correct statement(s).
(1) If x is positive and is increasing with the time, then average velocity of the particle is positive.
(2) If x is negative and becoming positive after some time, then the velocity of the particle is always positive.
(3) If x is negative and becoming less negative as time passes, then the average velocity of the particle is positive.
(4) If x is positive and is increasing with time, then the velocity of the particle is always positive.

18. For a particle moving along the x-axis, x–t graph is as given in figure. Mark the correct statement(s).



(1) Initial velocity of the particle is zero.
(2) For BC, acceleration is positive and for DE, acceleration is negative.
(3) For EF, the acceleration is positive.
(4) Velocity becomes zero three times in the motion

19. For a particle moving along the x-axis, a scaled x–t graph is shown in figure. Mark the correct statement(s).



(1) Speed of the particle is greatest at C.
(2) Speed of the particle is greatest at B.
(3) Particle is speeding up in region marked CD.
(4) Particle is speeding up in the region marked AB.

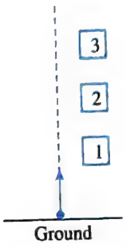
20. Mark the correct statement(s).
(1) A particle can have zero displacement and non-zero average velocity.
(2) A particle can have zero displacement and non-zero velocity.
(3) A particle can have zero acceleration and non-zero velocity.
(4) A particle can have zero velocity and non-zero acceleration.

21. At time $t = 0$, a car moving along a straight line has a velocity of $16\ \mathrm{m\,s^{-1}}$. It slows down with an acceleration of $-0.5t\ \mathrm{m\,s^{-2}}$, where t is in seconds. Mark the correct statement(s).
(1) The direction of velocity changes at $t = 8$ s.
(2) The distance travelled in 4 s is approximately 59 m.
(3) The distance travelled by the particle in 10 s is 94 m.
(4) The velocity at $t = 10$ s is $9\ \mathrm{m\,s^{-1}}$

22. A ball is thrown upwards into air with a speed greater than its terminal speed. It lands at the same place from where it was thrown. Mark the correct statement(s).
(1) It acquires terminal speed before it gets to the highest point of the trajectory.
(2) Before reaching the highest point of trajectory, it speed is continuously decreasing.
(3) During the entire flight, the force of air resistance i greatest just after it is thrown.
(4) The magnitude of net force experienced by the ball i maximum just after it is thrown.

23. A particle is moving along the x-axis whose position i given by $x = 4 - 9t + \dfrac{t^3}{3}$. Mark the correct statement(s) in relation to its motion.
(1) The direction of motion is not changing at any of th instants.
(2) The direction of the motion is changing at $t = 3$ s.
(3) For $0 < t < 3$ s, the particle is slowing down.
(4) For $0 < t < 3$ s, the particle is speeding up.

24. A particle is thrown in vertically in upward direction and passes three equally spaced windows of equal height Then



(1) The average speed of the particle while passing th windows satisfy the relation $v_{av_1} > v_{av_2} > v_{av_3}$.
(2) The time taken by the particle to cross the window satisfies the relation $t_1 < t_2 < t_3$.
(3) The magnitude of the acceleration of the particle whi crossing the windows, satisfies the relation $a_1 = a_2 \neq a_3$.
(4) The change in the speed of the particle, while crossin the windows, would satisfy the relation $\Delta v_1 < \Delta v_2 < \Delta v$

25. An object moves with constant acceleration $\vec{a}$. Which o the following expressions is/are also constant?
(1) $\dfrac{d|\vec{v}|}{dt}$ (2) $\left|\dfrac{d\vec{v}}{dt}\right|$
(3) $\dfrac{d(v^2)}{dt}$ (4) $\dfrac{d(\vec{v}/|\vec{v}|)}{dt}$

26. A ball is dropped from a height of 49 m, the wind blow horizontally and imparts a constant acceleration of 4.9 $\mathrm{m\,s^{-2}}$ to the ball. Choose the correct statement(s).
(1) Path of the ball is a straight line.
(2) Path of the ball is a curved one.
(3) The time taken by the ball to reach the ground is 3.16 s
(4) The angle made by the line joining initial and fina positions (on ground after 1st strike) of the ball wit horizontal is greater than 45°.

27. An object may have
(1) varying speed without having varying velocity.
(2) varying velocity without having varying speed.
(3) non-zero acceleration without having varying velocity
(4) non-zero acceleration without having varying speed

28. From the top of a tower of height 200 m, a ball A is projected up with speed 10 m s^{-1} and 2 s later, another ball B is projected vertically down with the same speed. Then

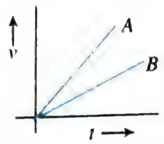
(1) Both A and B will reach the ground simultaneously

(2) Ball A will hit the ground 2 s later than B hitting the ground

(3) Both the balls will hit the ground with the same velocity

(4) Both will rebound to the same height from the ground, if both have same coefficient of restitution.

29. The velocity–time graph of two bodies A and B is shown in figure. Choose correct statement.



(1) acceleration of B > acceleration of A

(2) acceleration of A > acceleration of B

(3) both are starting from same point

(4) A covers greater distance than B in the same time.

30. A particle starts moving along a straight line path with a velocity 10 m s^{-1}. After 5 s, the distance of the particle from the starting point is 50 m. Which of the following statement about the nature of motion of the particle are correct?

(1) The body may be speeding up with constant positive acceleration.

(2) The motion may be moving with constant velocity.

(3) The body may have constant negative acceleration.

(4) The motion may be first accelerated and then retarded.

Linked Comprehension Type

For Problems 1–3

A body is allowed to fall from a height of 100 m. If the time taken for the first 50 m is t_1 and for the remaining 50 m is t_2.

1. Which is correct?

(1) $t_1 = t_2$ (2) $t_1 > t_2$

(3) $t_1 < t_2$ (4) Depends upon the mass

2. The ratio of t_1 and t_2 is nearly

(1) 5 : 2 (2) 3 : 1

(3) 3 : 2 (4) 5 : 3

3. The ratio of times to reach the ground and to reach first half of the distance is

(1) $\sqrt{3}:1$ (2) $\sqrt{2}:1$

(3) 5 : 2 (4) $1:\sqrt{3}$

For Problems 4 and 5

Two particles A and B are initially 40 m apart, A is behind B. Particle A is moving with uniform velocity of 10 m s^{-1} toward B. Particle B starts moving away from A with constant acceleration of 2 m s^{-2}.

4. The time at which there is a minimum distance between the two is

(1) 2 s (2) 4 s

(3) 5 s (4) 6 s

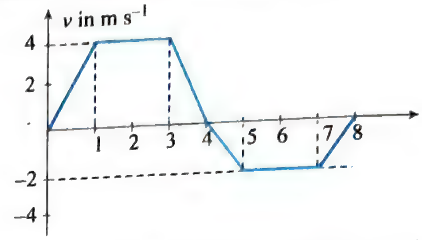
5. The minimum distance between the two is

(1) 20 m (2) 15 m

(3) 25 m (4) 30 m

For Problems 6–8

The velocity–time graph of a particle in straight line motion is shown in figure. The particle starts its motion from origin.



6. The distance travelled by the particle in 8 s is

(1) 18 m (2) 16 m

(3) 8 m (4) 6 m

7. The distance of the particle from the origin after 8 s is

(1) 18 m (2) 16 m

(3) 8 m (4) 6 m

8. Find the average acceleration from 2 s to 6 s.

(1) – 2 m s^{-2} (2) – 3/2 m s^{-2}

(3) 2 m s^{-2} (4) 3/2 m s^{-2}

For Problems 9–11

The velocity–time graph of a particle moving along a straight line is shown in figure The rate of acceleration and deceleration is constant and it is equal to 5 m s^{-2}. If the average velocity during the motion is 20 m s^{-1}, then

9. The value of t is

(1) 5 s (2) 10 s

(3) 20 s (4) $5\sqrt{2}$ s

10. The maximum velocity of the particle is

(1) 20 m s^{-1} (2) 25 m s^{-1}

(3) 30 m s^{-1} (4) 40 m s^{-1}

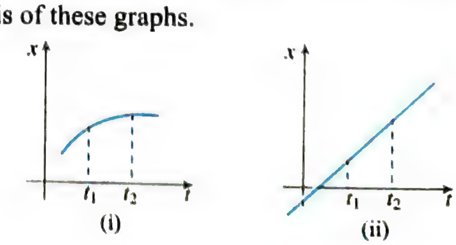
11. The distance travelled with uniform velocity is

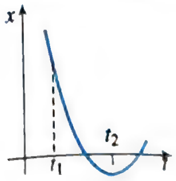
(1) 375 m (2) 125 m

(3) 300 m (4) 450 m

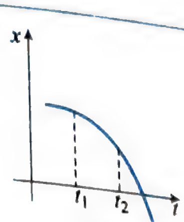
For Problems 12 and 13

Study the four graphs given below. Answer the following questions on the basis of these graphs.





(iii)



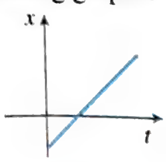
(iv)

12. In which of the graphs, the particle has more magnitude of velocity at t_1 than at t_2.
(1) (i), (iii), and (iv) (2) (i) and (iii)
(3) (ii) and (iii) (4) None of the above

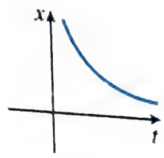
13. Acceleration of the particle is positive
(1) In graph (i) (2) In graph (ii)
(3) In graph (iii) (4) In graph (iv)

For Problems 14 and 15

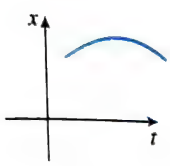
Study the following graphs:



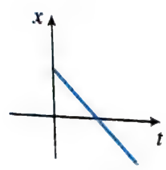
(i)



(ii)



(iii)



(iv)

14. The particle is moving with constant speed
(1) In graphs (i) and (iii) (2) In graphs (i) and (iv)
(3) In graphs (i) and (ii) (4) In graphs (i)

15. The particle has negative acceleration
(1) In graph (i) (2) In graph (ii)
(3) In graph (iii) (4) In graph (iv)

For Problems 16–20

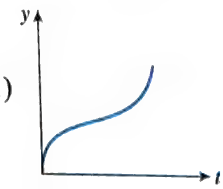
An inquisitive student, determined to test the law of gravity for himself, walks to the top of a building of 145 floors, with every floor of height 4 m, having a stopwatch in his hand (the first floor is at a height of 4 m from the ground level). From there he jumps off with negligible speed and hence starts rolling freely. A rocketeer arrives at the scene 5 s later and dives off from the top of the building to save the student. The rocketeer leaves the roof with an initial downward speed v_0. In order to catch the student and to prevent injury to him, the rocketeer should catch the student at a sufficiently great height above ground so that the rocketeer and the student slow down and arrive at the ground with zero velocity. The upward acceleration that accomplishes this is provided by rocketeer's jet pack, which he turns on just as he catches the student, before the rocketeer is in free fall. To prevent any discomfort to the student, the magnitude of the acceleration of the rocketeer and the student as they move downward together should not exceed 5 g.

16. Just as the student starts his free fall, he presses the button of the stopwatch. When he reaches at the top of 100th floor, he has observed the reading of stopwatch as 00: 00: 06 : 00 (hh: mm: ss: 100th part of the second). Find the value of g. (Correct up to two decimal places)
(1) 10.00 $m s^{-2}$ (2) 9.25 $m s^{-2}$
(3) 9.75 $m s^{-2}$ (4) 9.50 $m s^{-2}$

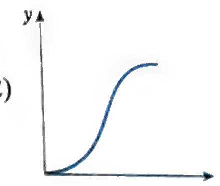
17. What should be the initial downward speed of the rocketeer so that he catches the student at the top of 100th floor for safe landing?
(1) It can have many values
(2) 180 $m s^{-1}$
(3) 175 $m s^{-1}$
(4) Cannot be determined

18. The position–time graph for rocketeer would be (take the top of building as origin, and vertical downward direction as positive y-axis)

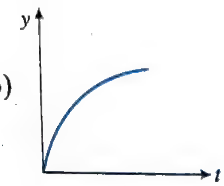
(1)



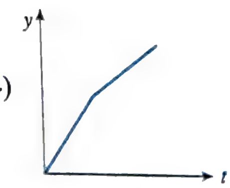
(2)



(3)



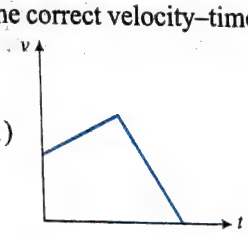
(4)



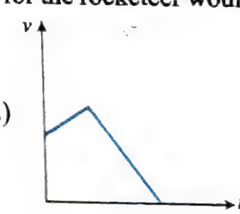
19. In Q.16, what would be the approximate retardation to be given by jet pack along for safe landing?
(1) $5g$ $m s^{-2}$ (2) $2g$ $m s^{-2}$
(3) $4g$ $m s^{-2}$ (4) Cannot be determined

20. The correct velocity–time graph for the rocketeer would be

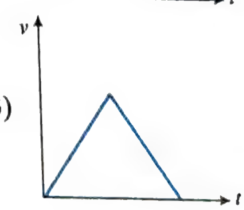
(1)



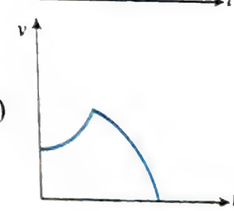
(2)



(3)



(4)



For Problems 21–25

An elevator without a ceiling is ascending up with an acceleration of 5 $m s^{-2}$. A boy on the elevator shoots a ball in vertically upward direction from a height of 2 m above the floor of elevator. At this instant, the elevator is moving up with a velocity of 10 $m s^{-1}$ and floor of the elevator is at a height of 50 m from the ground. The initial speed of the ball is 15 $m s^{-1}$ w.r.t. the elevator. Consider the duration for which the ball strikes the floor of the elevator in answering following questions:

21. The time in which the ball strikes the floor of elevator is given by
(1) 2.13 s (2) 4.26 s
(3) 1.0 s (4) 2.0 s

22. The maximum height reached by the ball as measured from the ground would be

(1) 52 m (2) 31.25 m
(3) 83.25 m (4) 63.25 m

23. The displacement of ball w.r.t. ground during its flight is

(1) 32.64 m (2) 2 m
(3) 52 m (4) 30.64 m

24. The distance travelled by the ball during its flight is

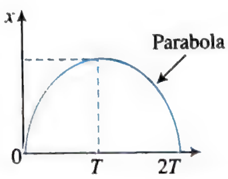
(1) 32.64 m (2) 31.86 m
(3) 52 m (4) 30.64 m

25. The maximum separation between the floor of elevator and the ball during its flight would be

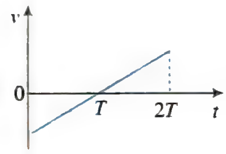
(1) 30 m (2) 15 m
(3) 7.5 m (4) 9.5 m

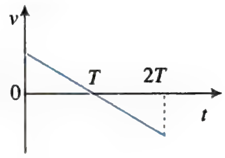
For Problems 26–28

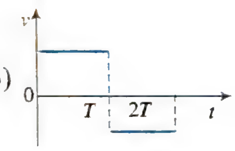
The x–t graph of a particle moving along a straight line is shown in figure.

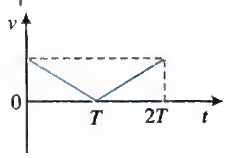


26. The v-t graph of the particle is correctly shown by

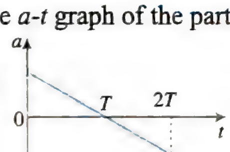
(1)


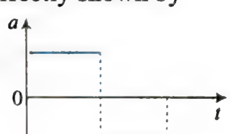
(2)


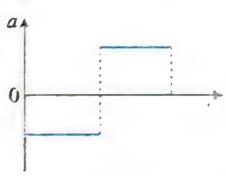
(3)


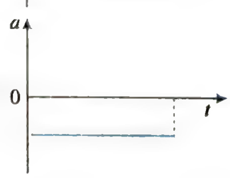
(4)


27. The a-t graph of the particle is correctly shown by

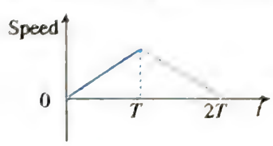
(1)


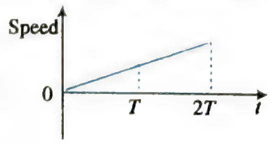
(2)


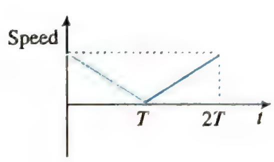
(3)


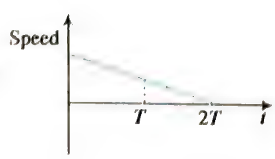
(4)


28. The speed–time graph of the particle is correctly shown by

(1)


(2)


(3)


(4)


Matrix Match Type

1. A particle moves along a straight line such that its displacement S varies with time t as $S = a + bt + gt^2$.

Column I	Column II
i. Acceleration at $t = 2$ s	**a.** $\beta + 5\gamma$
ii. Average velocity during third second	**b.** 2γ
iii. Velocity at $t = 1$ s	**c.** α
iv. Initial displacement	**d.** $\beta = 2\gamma$

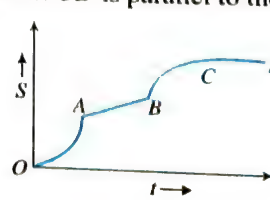
2. For a body projected vertically up with a velocity $\vec{v}_0$ from the ground, match the following.

Column I	Column II
i. $\vec{v}_{av}$ (Average velocity)	**a.** Zero for round trip
ii. u_{av} (average speed)	**b.** $\dfrac{\vec{v}_1 + \vec{v}_2}{2}$ over any time interval
iii. T_{ascent}	**c.** $\dfrac{v_0}{2}$ over the total time of its flight
iv. $T_{descent}$	**d.** $\dfrac{v_0}{g}$

3. A ball is thrown vertically upwards from the top of a cliff. Take starting position of motion as origin and upward direction as positive. Column I specifies the position, velocity, and/or acceleration of the particle at any instant. Column II gives their sign, (+) or (–), at that moment. Match the columns.

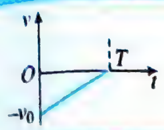
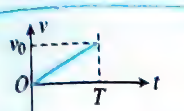
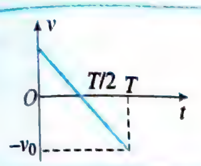
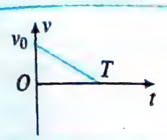
Column I	Column II
i. When the ball is above the point of projection, its displacement is	**a.** Always positive
ii. When the ball is above the point of projection, its velocity is	**b.** Always negative
iii. When the ball is above the point of projection, its acceleration is or may be negative	**c.** May be positive
iv. When the ball is below the point of projection, its acceleration is	**d.** May be zero

4. The displacement versus time curve is given. Sections OA and BC are parabolic. CD is parallel to the time axis.



Column I	Column II
i. OA	a. Velocity increases with time linearly
ii. AB	b. Velocity decreases with time
iii. BC	c. Velocity is independent of time
iv. CD	d. Velocity is zero

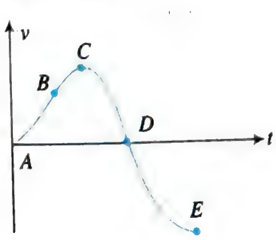
5. Study the following v–t graphs in Column I carefully and match appropriately with the statements given in Column II. Assume that motion takes place from time 0 to T.

Column I	Column II
i. 	a. Net displacement is positive, but not zero
ii. 	b. Net displacement is negative, but not zero
iii. 	c. Particle returns to its initial position again
iv. 	d. Acceleration is positive

6. For a particle moving along the x-axis, if acceleration (constant) is acting along negative x-axis, then match the entries of Column I with entries of Column II.

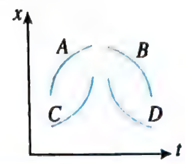
Column I	Column II
i. Initial velocity > 0	a. Particle may move in positive x-direction with increasing speed.
ii. Initial velocity < 0	b. Particle may move in positive x-direction with decreasing speed.
iii. $x > 0$	c. Particle may move in negative x-direction with increasing speed.
iv. $x < 0$	d. Particle may move in negative x-direction with decreasing speed.

7. The velocity–time graph of a particle moving along the x-axis is shown in figure. Match the entries of Column I with entries of Colcumn II.



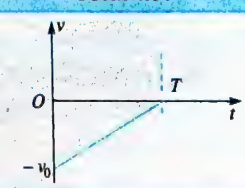
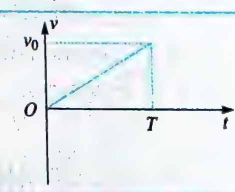
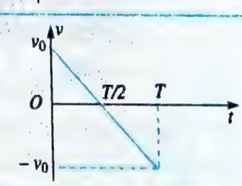
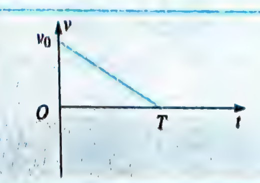
Column I	Column II
i. For AB, the particle is	a. moving in positive x-direction with increasing speed
ii. For BC, the particle is	b. moving in positive x-direction with decreasing speed.
iii. For CD, the particle is	c. moving in negative x-direction with increasing speed.
iv. For DE, the particle is	d. moving in negative x-direction with decreasing speed.

8. Figure shows the position–time graph of particle moving along a straight line. Match the entries of Column I with the entries of Column II.



Column I	Column II
i. The particle A is	a. accelerating
ii. The particle B is	b. decelerating
iii. The particle C is	c. speeding up
iv. The particle D is	d. slowing down

9. Study the following v–t graphs in Column I carefully and match appropriately with the statements given in Column II. Assume that motion takes place from time 0 to T.

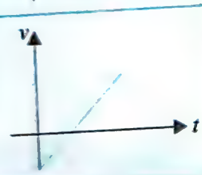
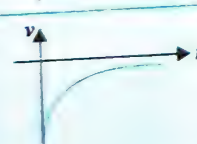
Column I	Column II
i. 	a. Net displacement is positive, but not zero.
ii. 	b. Net displacement is negative, but not zero.
iii. 	c. Particle returns to its initial position again.
iv. 	d. Acceleration is positive.

10. A particle is performing rectilinear motion on x axis, such that its x-coordinate varies with time as $x = \frac{t^3}{3} - 2t^2 + 3t + 5(\text{m})$ where t is in second. In column I, time instant is given and column II describes the motion of particle at particular instant. Match the proper entry from column II to column I.

Column I	Column II
i. $t=\frac{1}{2}$ sec	**a.** Particle is moving in positive x-direction and speeding up.
ii. $t=\frac{3}{2}$ sec	**b.** Particle is moving in negative x-direction and slowing down.
iii. $t=\frac{5}{2}$ sec	**c.** Particle is moving in negative x-direction and speeding up.
iv. $t=4$ sec	**d.** Particle is moving in positive x-direction and slowing down

(1) i → d; ii → c; iii → b; iv → a
(2) i → a; ii → c; iii → b; iv → d
(3) i → a; ii → b; iii → c; iv → d
(4) i → d; ii → b; iii → c; iv → a

11. The velocity–time graph for a particle moving along a straight line is given in each situation of column I. In the time interval $v > t > 0$, match the graph in column I with corresponding statements in column II.

Column I	Column II
i. 	**a.** Speed of particle is continuously decreasing
ii. 	**b.** Magnitude of acceleration of particle is decreasing with time.
iii. 	**c.** Direction of acceleration of particle does not change.
iv 	**d.** Magnitude of acceleration of particle does not change.
	e. Particle will never come back to its initial position.

(1) i → b,d; ii → a,d; iii → a,b,c,e; iv → a,b,c,e
(2) i → c,d; ii → c,d; iii → a,b,c,e; iv → a,b,c,e
(3) i → a,b; ii → c,e; iii → a,b,c,e; iv → a,b,c
(4) i → b,d; ii → a,c; iii → a,b,c; iv → a,b,c,e

Numerical Value Type

1. From a lift moving upwards with a uniform acceleration $a = 2\ ms^{-2}$, a man throws a ball vertically upwards with a velocity $v = 12\ ms^{-1}$ relative to the lift. The ball comes back to the man after a time t. Find the value of t in seconds.

2. A train starts from station A with uniform acceleration a_1 for some distance and then goes with uniform retardation a_2 for some more distance to come to rest at station B. The distance between stations A and B is 4 km and the train takes 1/15 h to complete this journey. If accelerations are in km per minute unit, then show that $\frac{1}{a_1}+\frac{1}{a_2}=x$. Find the value of x.

3. In a car race, car A takes 4 s less than car B at the finish and passes the finishing point with a velocity v more than the car B. Assuming that the cars start form rest and travel with constant accelerations $a_1 = 4\ ms^{-2}$ and $a_2 = 1\ ms^{-2}$ respectively, find the velocity of v in ms^{-1}.

4. A cat, on seeing a rat at a distance $d = 5$ m, starts with velocity $u = 5\ ms^{-1}$ and moves with acceleration $\alpha = 2.5\ ms^{-2}$ in order to catch it, while the rate with acceleration β starts from rest. For what value of β will the cat overtake the rat? (in ms^{-2})

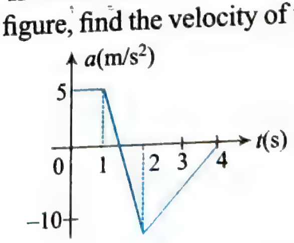
5. A body is thrown up with a velocity $100\ ms^{-1}$. It travels 5 m in the last second of its journey. If the same body is thrown up with a velocity $200\ ms^{-1}$, how much distance (in metre) will it travel in the last second ($g = 10\ ms^{-2}$)?

6. In quick succession, a large number of balls are thrown up vertically in such a way that the next ball is thrown up when the previous ball is at the maximum height. If the maximum height is 5 m, then find the number of the thrown up per second ($g = 10\ ms^{-2}$).

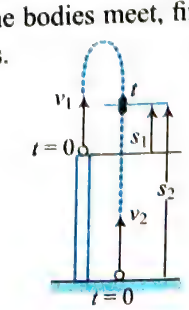
7. A police jeep is chasing a culprit going on a motorbike. The motorbike crosses a turning at a speed of 72 kmh^{-1}. The jeep follows it at a speed of 90 kmh^{-1}, crossing the turning 10 s later than the bike.
Assuming that they travel at constant speeds, how far from the turning will the jeep catch up with the bike? (In km)

8. In a square cut, the speed of the cricket ball changes from 30 ms^{-1} to 40 ms^{-1} during the time of its contact $\Delta t = 0.01$ s with the bat. If the ball is deflected by the bat through an angle of $\theta = 90°$, find the magnitude of the average acceleration (in × $10^2\ ms^{-2}$) of the ball during the square cut.

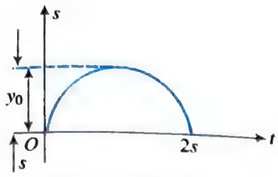
9. A particle moves vertically with an upward initial speed $v_0 = 10.5\ ms^{-1}$. If its acceleration varies with time as shown in a–t graph in figure, find the velocity of the particle at $t = 4$ s.



10. Two bodies 1 and 2 are projected simultaneously with velocities $v_1 = 2\ ms^{-1}$ and $v_2 = 4\ ms^{-1}$ respectively. The body 1 is projected vertically up from the top of a cliff of height $h = 10$ m and the body 2 is projected vertically up from the bottom of the cliff. If the bodies meet, find the time (in s) of meeting of the bodies.



11. A particle moves rectilinearly possessing a parabolic s–t graph. Find the average velocity of the particle over a time interval from $t = \frac{1}{2}$ s to $t = 1.5$ s.



12. A particle moves in a straight line. Its position (in m) as function of time is given by

$$x = (at^2 + b)$$

What is the average velocity in time interval $t = 3$ s to $t = 5$ s in ms^{-1}. (Where a and b are constants and $a = 1\ ms^{-2}$, $b = 1$ m)

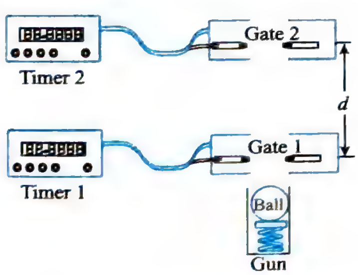
13. A particle can move only along x-axis. Three pairs of initial and final positions of particle at two successive times are given

Pair	Initial position	Final position
1	–3 m	+5 m
2	–3 m	–7 m
3	+7 m	–3 m

Find the sum of magnitudes of displacement in the pairs which give negative displacement in m.

14. **Measuring *g***

The figure shows a method for measuring the acceleration due to gravity. The ball is projected upward by a "gun." The ball passes electronic "gates" 1 and 2 as it rises and again as it falls. Each gate is connected to a separate timer. The first passage of the ball through each gate starts the corresponding timer, and the second passage through the same gate stops the timer. The time intervals Δt_1 and Δt_2 are thus measured. The vertical distance between the two gates is d. If $d = 5$ m, $\Delta t_1 = 3$ s, $\Delta t_2 = 2$ s, find the measured value of acceleration due to gravity (in $m s^{-2}$).



15. Acceleration of particle moving rectilinearly is $a = 4 - 2x$ (where x is position in metre and a in $m s^{-2}$). It is at instantaneous rest at $x = 0$. At what position x (in meter) will the particle again come to instantaneous rest?

16. A big Diwali rocket is projected vertically upward so as to attain a maximum height of 160 m. The rocket explodes just as it reaches the top of its trajectory sending out luminous particles in all possible directions all with same speed v. The display, consisting of the luminous particles, spreads out as an expanding, brilliant sphere. The bottom of this sphere just touches the ground when its radius is 80 m. With what speed (in m/s) are the luminous particles ejected by the explosion?

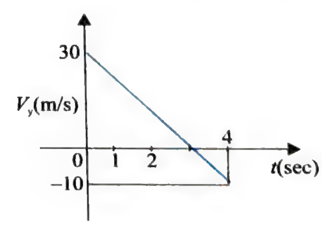
17. A steel ball is dropped from the roof of a building. A man standing in front of a 1-m high window in the building notes that the ball takes 0.1 s to the fall from the top to the bottom of the window. The ball continues to fall and strikes the ground. On striking the ground, the ball gets rebounded with the same speed with which it hits the ground. If the ball reappears at the bottom of the window 2 s after passing the bottom of the window on the way down, find the height of the building.

18. A passenger reaches the platform and finds that the second least boggy of the train is passing him. The second last boggy takes 3 s to pass the passenger, and the last boggy takes 2 s to pass him. Find the time by which the passenger late for the departure of the train? Assume that the train accelerates at constant rate and all the boggies are of equal length.

19. A particle is projected vertically from the ground takes time $t_1 = 1$ s upto point A, $t_2 = 3$ s from point A to B, and time $t_3 = 4$ s from point B to highest point. Find the height of the middle point of A and B from the ground.

20. The loaded bucket of a crane achieves a maximum velocity 5 m/s in some time at a uniform rate and then takes half of this time to stop at a uniform rate after the application of brake. The time difference between the instants when half of the maximum velocity is achieved is t (sec). Find the displacement of the bucket.

21. The velocity time graph for the vertical component of the velocity of an object thrown upward from the ground which reaches at the roof of a building is shown in the figure. If the height of the building is h meter. Find $h/5$.



22. Two balls are dropped from the top of a cliff at a time interval $\Delta t = 2$ s. The first ball hits the ground, rebounds elastically (essentially reversing direction instantly without losing speed) and collides with the second ball at height $h = 55$ m above the ground. How high is the top of the cliff (in m)?

23. A passenger is standing on the platform at the beginning of the n^{th} (= 3rd) coach of a train. The train starts moving with constant acceleration. The third coach passes by the passenger in $\Delta t_1 = 5.0$ s and rest of the train (including the 3rd coach) in $\Delta t_2 = 20$ s. In what time interval Δt (in sec) did the last coach passed by the passenger?

JEE Main

Single Correct Answer Type

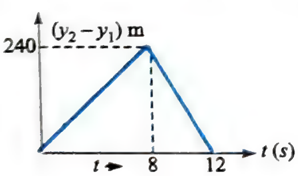
1. A particle has an initial velocity $3\hat{i}+4\hat{j}$ and an acceleration of $0.4\hat{i}+0.3\hat{j}$. Its speed after 10 s is
   (1) 10 units (2) $7\sqrt{2}$ units
   (3) 7 units (4) 8.5 units **(AIEEE 2009)**

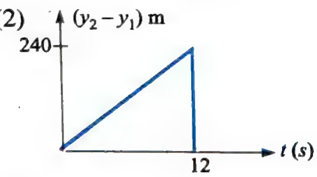
2. A particle is moving with velocity $\vec{v}=K(y\hat{i}+x\hat{j})$, where K is a constant. The general equation for its path is
   (1) $y=x^2$ + constant (2) $y^2=x$ + constant
   (3) xy = constant (4) $y^2=x^2$ + constant **(AIEEE 2010)**

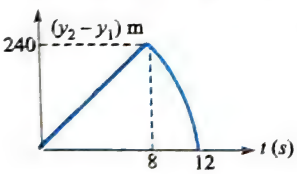
3. An object moving with a speed of 6.25 m/s, is decelerated at a rate given by:
$$\frac{dv}{dt}=-2.5\sqrt{v}$$
   where v is instantaneous speed. The time taken by the object, to come to rest, would be:
   (1) 1 s (2) 2 s
   (3) 4 s (4) 8 s **(AIEEE 2011)**

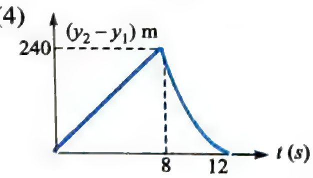
4. A boy can throw a stone up to a maximum height of 10 m. The maximum horizontal distance that the boy can throw the same stone up to will be
   (1) $20\sqrt{2}$ m (2) 10 m
   (3) $10\sqrt{2}$ m (4) 20 m **(AIEEE 2012)**

5. Two stones are thrown up simultaneously from the edge of a cliff 240 m high with initial speed of 10 m/s and 40 m/s respectively. Which of the following graphs best represents the time variation of relative position of the second stone with respect to the first?
   (Assume stones do not rebound after hitting the ground and neglect air resistance, take $g=10$ m/s^2)
   (The figures are schematic and not drawn to scale.)

(1)


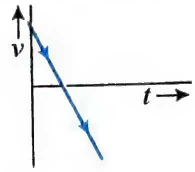
(2)


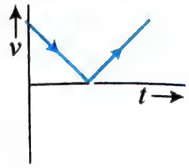
(3)


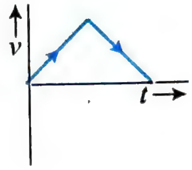
(4)


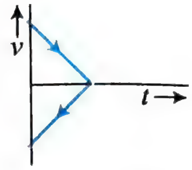
(JEE Main 2015)

6. A body is thrown vertically upwards. Which one of the following graphs correctly represent the velocity vs. time?

(1)


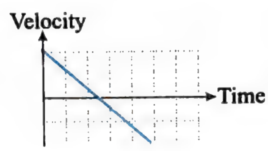
(2)


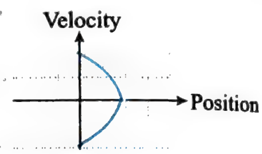
(3)


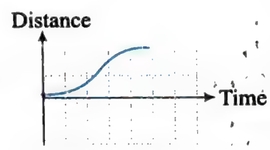
(4)


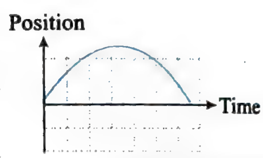
(JEE Main 2017)

7. All the graphs below are intended to represent the same motion. One of them does it incorrectly. Pick it up.

(1)


(2)


(3)


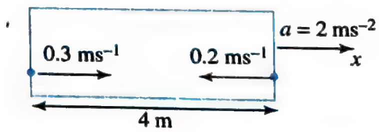
(4)


(JEE Main 2018)

JEE Advanced

Numerical Value Type

1. A rocket is moving in a gravity free space with a constant acceleration of 2 ms^{-2} along $+x$ direction (see figure). The length of a chamber inside the rocket is 4 m. A ball is thrown from the left end of the chamber in $+x$ direction with a speed of 0.3 ms^{-1} relative to the rocket. At the same time, another ball is thrown in $-x$ direction with a speed of 0.2 ms^{-1} from its right end relative to the rocket. The time in seconds when the two balls hit each other is



(JEE Advanced 2014)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3) 2. (1) 3. (4) 4. (1) 5. (3)
6. (2) 7. (4) 8. (4) 9. (4) 10. (3)
11. (1) 12. (1) 13. (3) 14. (2) 15. (1)
16. (1) 17. (2) 18. (1) 19. (3) 20. (2)
21. (3) 22. (2) 23. (1) 24. (3) 25. (1)
26. (3) 27. (3) 28. (2) 29. (4) 30. (3)
31. (4) 32. (2) 33. (1) 34. (2) 35. (1)
36. (3) 37. (4) 38. (2) 39. (2) 40. (4)
41. (2) 42. (3) 43. (2) 44. (3) 45. (3)
46. (1) 47. (3) 48. (4) 49. (3) 50. (1)
51. (4) 52. (4) 53. (1) 54. (1) 55. (2)
56. (1) 57. (1) 58. (1) 59. (4) 60. (2)
61. (3) 62. (1) 63. (1) 64. (2) 65. (2)
66. (1) 67. (1) 68. (2) 69. (1) 70. (3)
71. (2) 72. (3) 73. (2) 74. (4) 75. (4)
76. (2) 77. (3)

Multiple Correct Answers Type

1. (2),(3),(4) 2. (2),(4) 3. (1),(3)
4. (1),(2),(3) 5. (1),(3) 6. (1),(3)
7. (1),(4) 8. (1),(2),(4) 9. (1),(4)
10. (3),(4) 11. (1),(2),(3) 12. (3),(4)
13. (1),(4) 14. (1),(2),(3),(4) 15. (1),(2),(3)
16. (1),(3),(4) 17. (1),(3),(4) 18. (1),(3),(4)
19. (2),(3),(4) 20. (2),(3),(4) 21. (1),(2),(3)
22. (2),(3),(4) 23. (2),(3) 24. (1),(2),(4)
25. (2) 26. (1),(3),(4) 27. (2),(4)
28. (1),(3),(4) 29. (2),(4) 30. (2),(3),(4)

Linked Comprehension Type

1. (2) 2. (1) 3. (2) 4. (3) 5. (2)
6. (1) 7. (4) 8. (2) 9. (1) 10. (2)
11. (1) 12. (2) 13. (3) 14. (2) 15. (3)
16. (1) 17. (3) 18. (2) 19. (1) 20. (2)
21. (1) 22. (3) 23. (4) 24. (2) 25. (4)
26. (2) 27. (4) 28. (3)

Matrix Match Type

1. i → b; ii → a; iii → d; iv → c
2. i → a, b; ii → c; iii → d; iv → d
3. i → a; ii → c,d; iii → b; iv → b
4. i → a; ii → c; iii → b; iv → d
5. i → b, d; ii → a,d; iii → c; iv → a
6. i → b; ii → c; iii → b, c; iv → b, c
7. i → a; ii → a; iii → b; iv → c
8. i → b, d; ii → a, c; iii → a, c; iv → b, d
9. i → b, d; ii → a, d; iii → c; iv → a
10. (1) 11. (2)

Numerical Value Type

1. (2) 2. (2) 3. (8) 4. (5) 5. (5)
6. (1) 7. (1) 8. (5) 9. (3) 10. (5)
11. (0) 12. (8) 13. (14) 14. (8) 15. (4)
16. (2) 17. (21) 18. (3.5) 19. (60) 20. (5)
21. (8) 22. (180) 23. (0.64)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2) 2. (1) 3. (2) 4. (4) 5. (3)
6. (1) 7. (1)

JEE Advanced

Numerical Value Type

1. (2)

5 Kinematics II

INTRODUCTION

The motion of a particle along a straight line such as the x-axis is completely known if its position is known as a function of time. The concepts of displacement, velocity, and acceleration are used to describe an object moving in one dimension. We also analyzed one-dimensional motion of a particle under constant acceleration and developed the particle under constant acceleration model. Let us now extend this idea to two-dimensional motion of a particle in the xy-plane motion during which the acceleration of a particle remains constant in both magnitude and direction. As we shall see, this approach is useful for analyzing some common types of motion.

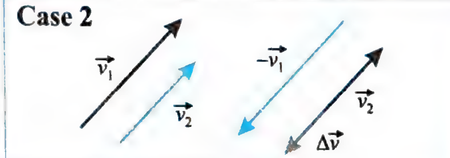
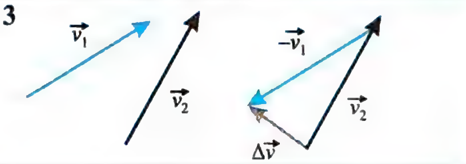
VELOCITY AND ACCELERATION IN TWO-DIMENSIONAL MOTION

The most noticeable difference between velocity along one dimension and velocity in two or more dimensions is that the latter can change direction as well as magnitude. Because acceleration is defined as a change in velocity—any change in velocity—divided by a time interval, there can be acceleration even when the magnitude of the velocity does not change.

Let us consider a particle is moving in two dimensions (that is, in a plane). At time t_1, the particle has velocity $\vec{v}_1$, and at a later time t_2, the particle has velocity $\vec{v}_2$. The change in velocity of the particle is $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ and the average acceleration, $\vec{a}_{av}$, for the time interval $\Delta t = t_2 - t_1$ is defined by

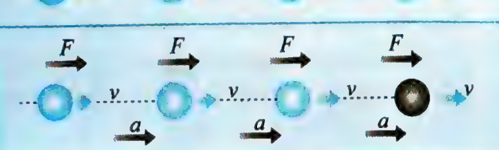
$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}. \quad \text{...(i)}$$

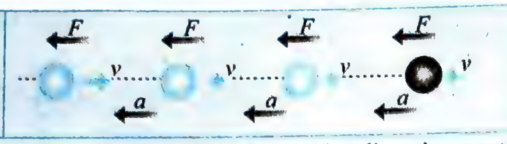
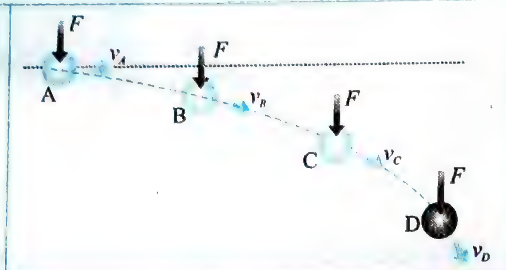
Now consider three cases:

Case 1	Case 2	Case 3
		
In this case, we can observe the initial and final velocities of the particle having the same direction, but the magnitude of the final velocity is greater than the magnitude of the initial velocity. The resulting change in velocity and the average acceleration are in the same direction as the velocities.	Here again, the initial and final velocities are pointing in the same direction, but the magnitude of the final velocity is less than the magnitude of the initial velocity. The resulting change in velocity and the average acceleration are in the opposite direction from the velocities.	If $\lvert\vec{v}_1\rvert = \lvert\vec{v}_2\rvert$, then in this case, the initial and final velocities have the same magnitude, but the direction of the final velocity vector is different from the direction of the initial velocity vector. In this case, the change in velocity and the average acceleration are not zero and can be in a direction not obviously related to the initial or final velocity directions.

Thus, various changes can occur when a particle accelerates.

- First, the magnitude of the velocity vector (the speed) may change with time as in straight line (one-dimensional) motion.
- Second, the direction of the velocity vector may change with time even if its magnitude (speed) remains constant. In this situation, the particle moves along a curved path.
- Finally, both the magnitude and the direction of the velocity vector may change simultaneously. In this situation also, the particle moves along a curved path.

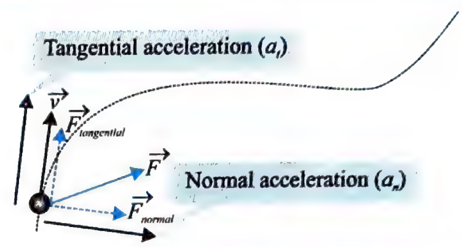
Situation 1		If a disc moves in straight line with constant speed, no external force is required.
Situation 2		If we want to increase the speed of the disc, we need to apply a force along the direction of its motion.

Situation 3		If we want to decrease the speed of the disc, we need to apply a force opposite to the direction of its motion.
In the situations listed above ,we cannot change the direction motion of the disc. If we apply a force in the direction of velocity, it will change the magnitude of velocity.		
Situation 4		Now consider if a force is applied perpendicular to velocity. The disc starts moving in curve path. Now we cannot describe the motion of disc by a single coordinate. For complete description of its motion, we need both coordinates *x* and *y*, and the motion of the disc is called 'motion in two dimensions'.

It means for changing the direction of a moving particle we need to apply a force or a component of force normal to its velocity.The component of force along the velocity change the magnitude of speed and the component of force perpendicular to the velocity change the direction of motion.

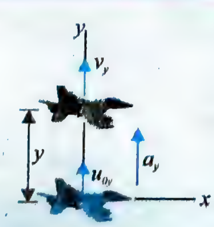
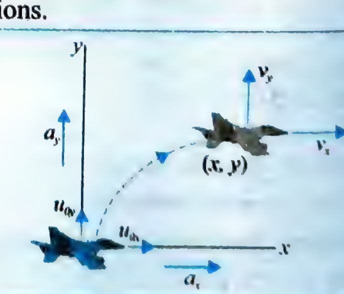
A particle moves to the right along a curved path, and its velocity changes both in direction and in magnitude as described in figure. In this situation, the velocity vector is always tangent to the path; the acceleration vector $\vec{a}$, however, is at some angle to the path.

As the particle moves along the curved path in the figure, the direction of the total acceleration vector $\vec{a}$ changes from point to point. At any instant, this vector can be resolved into two components, a radial or normal component a_r along the direction perpendicular to velocity and a tangential component a_t along the velocity. The total acceleration vector $\vec{a}$ can be written as the vector sum of the component vectors: $\vec{a} = \vec{a}_r + \vec{a}_t$.



PRINCIPLES OF PHYSICAL INDEPENDENCE OF MOTIONS

To understand how displacement, velocity, and acceleration are applicable to two-dimensional motion, consider a toy aircraft equipped with two engines that are fitted perpendicular to each other. These engines produce the only forces that the craft experiences, and the aircraft is assumed to be at the coordinate origin when $t_0 = 0$ s, so that $\vec{r}_0 = 0$ m. At a later time *t*, the aircraft's displacement has vector components of $\vec{x}$ and $\vec{y}$.

Only the engine responsible for horizontal motion is active and the engine responsible for vertical motion is turned off. The toy aircraft is moving with a constant acceleration a_x parallel to the *x* axis. There is no motion in the *y* direction.	Only the engine responsible for vertical motion is active and the engine responsible for horizontal motion is turned off. The toy aircraft is moving with a constant acceleration a_y parallel to the *y* axis. There is no motion in the *x* direction.	If both engines of the spacecraft are active at the same time, the resulting motion takes place in part along the *x* axis and in part along the *y* axis. The two-dimensional motion of the spacecraft can be viewed as the combination of the separate *x* and *y* motions.
		

The engine responsible for horizontal motion accelerates the aircraft in the *x* direction and causes a change in the *x* component of the velocity. Similarly, the engine responsible for vertical motion accelerates the aircraft in the *y* direction and causes a change in the *y* component of the velocity. *It is to be noted that the x part of the motion occurs exactly as it would if the y part did not occur at all. Similarly, the y part of the motion occurs exactly as it would if the x part of the motion did not exist.* In other words, the *x* and *y* motions are independent of each other.

ILLUSTRATION 5.1

In the figure below, the directions to the right and upward are the positive directions. In the *x* direction, the toy aircraft has an initial velocity component of $u_x = +20$ m/s and an acceleration component of $a_x = +20$ m/s^2. In the *y* direction, the analogous quantities are $u_y = +15$ m/s and $a_y = +10$ m/s^2. At a time of $t = 5.0$ s, find

(a) the x and y components of the toy aircraft's displacement.

(b) the toy aircraft's final velocity (magnitude and direction).

Sol. We can treat the motion in the x direction and the motion in the y direction separately, each as a one-dimensional motion subject to the equations of kinematics for constant.

The given data for the motion in the x direction and y direction are given in the following table:

x-Direction			y-Direction			
x component of acceleration (a_x)	x component of initial velocity (u_x)	x component of displacement (x)	y component of acceleration (a_y)	y component of initial velocity (u_y)	y component of displacement (y)	
+20 m/s²	+20 m/s	?	+10 m/s²	+15 m/s	?	5.0 s

(a) Let us calculate the x and y components of the toy aircraft's displacements.

The x component of the aircraft's displacement:

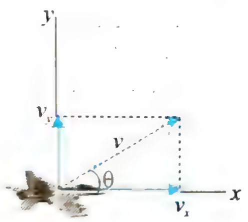
$$x = u_x t + \frac{1}{2}a_x t^2 = (20\,\text{m/s})(5.0\,\text{s}) + \frac{1}{2}(20\,\text{m/s}^2)(5.0\,\text{s})^2 = +350\,\text{m}$$

The y component of the aircraft's displacement:

$$y = u_y t + \frac{1}{2}a_y t^2 = (15\,\text{m/s})(5.0\,\text{s}) + \frac{1}{2}(10\,\text{m/s}^2)(5.0\,\text{s})^2 = +200\,\text{m}$$

After 5.0 s, the aircraft is 350 m to the right and 200 m above the origin.

(b) We can make the components of velocity at time $t = 5.0$ s. In the figure, we can observe the final velocity vector, which has components v_x and v_y and a magnitude v.



The final velocity is directed at an angle θ above the +x axis. The vector and its components form a right triangle, the hypotenuse being the magnitude of the velocity and the components being the other two sides. Thus, we can use the Pythagorean theorem to determine the magnitude v from values for the components v_x and v_y. We can also use trigonometry to determine the directional angle θ.

Let us calculate the x and y components of the toy aircraft's velocity at time $t = 5.0$ s.

The x component of the aircraft's velocity:

$$v_x = u_x + a_x t = 20\,\text{m/s} + (20\,\text{m/s}^2)(5.0\,\text{s}) = +120\,\text{m/s}$$

The y component of the aircraft's velocity:

$$v_y = u_y + a_y t = 15\,\text{m/s} + (10\,\text{m/s}^2)(5.0\,\text{s}) = +65\,\text{m/s}$$

Hence magnitude of final velocity:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(120)^2 + (65)^2} = 5\sqrt{745}\ \text{m/s}$$

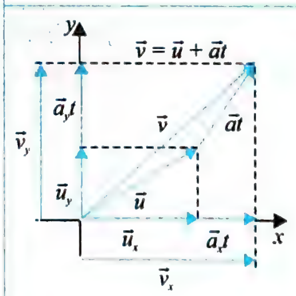
The angle made by final velocity with x-axis,

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{65}{120}\right) = \tan^{-1}\left(\frac{13}{24}\right)$$

TWO-DIMENSIONAL MOTION WITH CONSTANT ACCELERATION

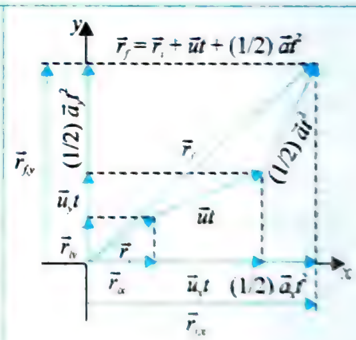
The independence of x and y motions is a very supporting tool in simplifying the cases of two-dimensional kinematics. It allows us to treat two-dimensional motion as two distinct one-dimensional motions, one for the x direction and one for the y direction. In doing so, we will be able to describe the x and y variables separately and then bring these descriptions together to understand the two-dimensional picture. The motion in the x direction and the motion in the y direction can be treated separately, each as a one-dimensional motion subject to the equations of kinematics for constant acceleration.

The two-dimensional motion with constant acceleration can be analyzed as a combination of two independent motions in the x and y directions, with accelerations a_x and a_y.



$\vec{v}_x = \vec{u}_x + \vec{a}_x t$

and $\vec{v}_y = \vec{u}_y + \vec{a}_y t$



$\vec{r}_{f,x} = \vec{r}_{i,x} + \vec{u}_x t + \frac{1}{2}\vec{a}_x t^2$

and $\vec{r}_{f,x} = \vec{r}_{i,x} + \vec{u}_x t + \frac{1}{2}\vec{a}_x t^2$

Table: Equations of Kinematics for Two-Dimensional Motion

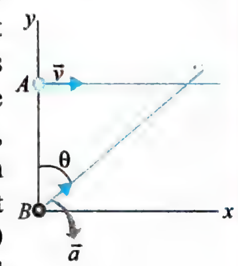
Variable	x Component	y Component
Final velocity relation with time	$v_x = u_x + a_x t$	$v_y = u_y + a_y t$
Displacement relation with time	$x = \frac{1}{2}(u_x + v_x)t$ and $x = u_x t + \frac{1}{2}a_x t^2$	$y = \frac{1}{2}(u_y + v_y)t$ and $y = u_y t + \frac{1}{2}a_y t^2$
Final velocity relation with displacement	$v_x^2 = u_x^2 + 2a_x x$	$v_y^2 = u_y^2 + 2a_y y$

Concept Application Exercise 5.1

1. A particle leaves the origin with $\vec{v} = (3.0\hat{i})$ m/s as initial velocity and $\vec{a} = (-1.0\,\hat{i} - 0.50\,\hat{j})$ m/s² as constant acceleration. When it reaches its maximum x coordinate, what are its (a) velocity and (b) position vector?

2. A moderate wind accelerates a pebble over a horizontal xy plane with a constant acceleration $\vec{a} = (4.0 \text{ m/s}^2) + (8.0 \text{ m/s}^2)\hat{j}$. At time $t = 0$, the velocity is $(4.00 \text{ m/s})\hat{i}$. What are the (a) magnitude and (b) angle of its velocity when it has been displaced by 16.0 m parallel to the x axis?

3. In the given figure, particle A moves along the line $y = 30$ m with a constant velocity $\vec{v}$ of magnitude 3.0 m/s and parallel to the x axis. At the instant particle A passes the y-axis, particle B leaves the origin with a zero initial speed and a constant acceleration $\vec{a}$ of magnitude 0.40 m/s². What angle θ between $\vec{a}$ and the positive direction of the y axis would result in a collision?

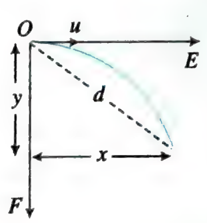


4. A particle moves so that its coordinates vary with time as $x = a \sin \omega t$, $y = a \cos \omega t$ and $z = bt^2$. Find the initial: (a) position, (b) velocity and (c) acceleration of the particle.

5. A ball is projected horizontally in air with velocity v_0 such that it moves with a constant horizontal acceleration a due to air flow. Taking the gravitational acceleration into account, find the (a) velocity and (b) displacement as the function of time till it strikes the ground.

6. A body starts from the origin with an acceleration of 6 m s⁻² along the x-axis and 8 m s⁻² along the y-axis. Find its distance from the origin after 4 s.

7. A body of mass 2 kg has an initial velocity of 3 m s⁻¹ along OE and it is subjected to a force of 4 N in OF direction perpendicular to OE. Find the distance of the body from O after 4 s.



Answers

1. (a) $(-1.5 \text{ m/s})\hat{j}$ (b) $(4.50\text{m})\hat{i} - (2.25\text{m})\hat{j}$

2. (a) 20.0 m/s

(b) $\tan^{-1}\left(\dfrac{4}{3}\right)$ measured from $+x$ axis

3. 60°

4. (a) $a\hat{j}$ (b) $a\omega$ (c) $-a\omega^2\hat{j} + 2b\hat{k}$

5. (a) $(v_0 + at)\hat{i} - gt\,\hat{j}$ (b) $\left(v_0 t + \dfrac{1}{2}at^2\right)\hat{i} - \dfrac{1}{2}gt^2$

6. 80 m

7. 20 m

PROJECTILE MOTION

The motion of bodies under gravity is generally known as projectile motion. It is really spectacular to watch the motion of cricket, volley, footballs in air. Neglecting the air resistance and swing of the balls, ideally the balls follow parabolic paths when projected obliquely.

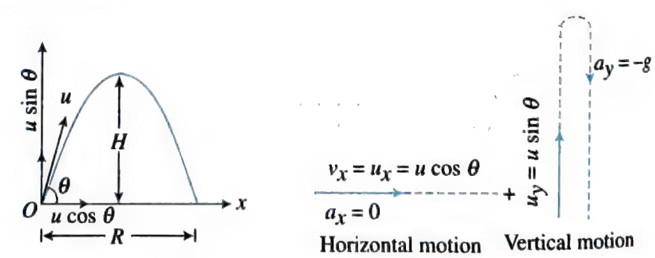
Generally "projectile motion" refers to the motion of a point object thrown in earth's gravitational field. However, the above definition of a projectile is still valid when you throw a point object in gravitational field of any other planet which may not be practically possible for everyone. This tells us that the motion of rockets, crackers, etc., driven by some other external forces other than gravity, cannot be termed as projectile motion in general sense.

The assumptions of projectile motion are as follows:

- There is no resistance due to air.
- The effect due to the curvature of earth is negligible.
- The effect due to the rotation of earth is negligible.
- For all points of the trajectory, the acceleration due to gravity g is constant in magnitude and direction.

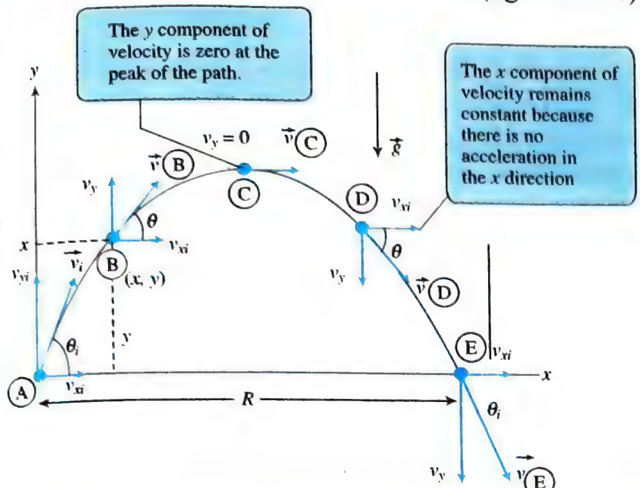
Principles of Physical Independence of Motions in Projectile Cases

In this section, we stated that the two-dimensional motion with constant acceleration can be analyzed as a combination of two independent motions in the x- and y-directions, with accelerations a_x and a_y. Projectile motion can also be handled in this way, with zero acceleration in the x-direction and a constant acceleration in the y-direction, $a_y = -g$ (figure below). Therefore, when analyzing projectile motion, model it to be the superposition of two motions:



- The motion of a projectile is a two-dimensional motion. So, it can be discussed in two parts: horizontal motion and vertical motion. These two motions take place independent of each other. This is called the principle of physical independence of motions.
- The velocity of the particle can be resolved into two mutually perpendicular components. Horizontal component and vertical component. The vertical component of velocity first decreases and becomes zero at the topmost point (C) and again increases while downward motions.
- The force of gravity continuously affects the vertical component.
- The horizontal motion is a uniform motion and the vertical motion is a uniformly accelerated retarded motion at all positions (A), (B), (C), (D) and (E). The horizontal

component of velocity remains constant (figure below).



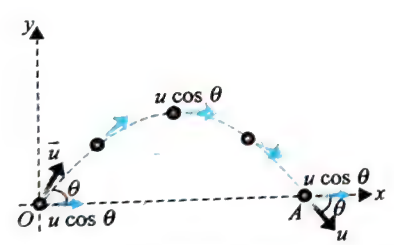
Parabolic path = Vertical motion + Horizontal motion

	Horizontal direction	Vertical direction
(a)	Initial velocity $u_x = u\cos\theta$	Initial velocity $u_y = u\sin\theta$
(b)	Acceleration $a_x = 0$	Acceleration $a_y = g$
(c)	Velocity after time t, $v_x = u\cos\theta$	Velocity after time t, $v_y = u\sin\theta - gt$

ILLUSTRATION 5.2

What is the average velocity when the projectile moves from O to A.

Sol. From O to A, the displacement in vertical direction is zero.



It means the component of average velocity in vertical direction should be zero.

The horizontal component of initial velocity remains constant.

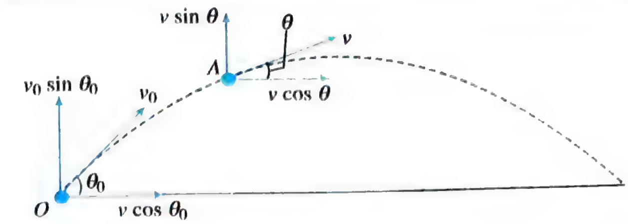
Hence, the average velocity when the projectile moves from O to A is equal to the horizontal component of initial velocity.

$$v_{av} = u_x = u\cos\theta$$

ILLUSTRATION 5.3

At what point on a projectile's trajectory, its speed is minimum? If a stone is thrown with a speed v_0 at an angle θ_0 with horizontal, find the velocity of the stone when its line of motion makes an angle θ with horizontal.

Sol. The horizontal component of the velocity will be constant during the projectile motion.



Horizontal component of the initial velocity, $u_x = v_0\cos\theta_0$.

Horizontal component of the velocity when the particle makes an angle θ with horizontal, $v_x = v\cos\theta$.

Hence, $v\cos\theta = v_0\cos\theta_0 \Rightarrow v = \dfrac{v_0\cos\theta_0}{\cos\theta}$

ILLUSTRATION 5.4

A particle is projected with velocity u = 10 m/s at an angle $\theta = 37°$ with the horizontal. Find

(a) Velocity of the particle after 1 s

(b) Angle between initial velocity and the velocity after 1s

Sol. We can write initial velocity of projection in unit as

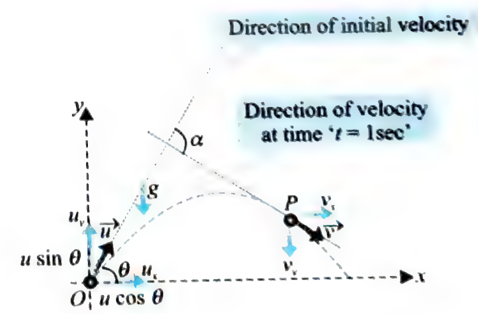
$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$
$$= 10\cos 37°\,\hat{i} + 10\,\sin 37°\,\hat{j} = 8\hat{i} + 6\hat{j} \text{ m/s}$$

In horizontal direction, acceleration of the particle is zero i.e., horizontal component of the velocity is constant. In vertical direction, the acceleration is 'acceleration due to gravity'. It means

$$\vec{a}_y = -g\,\hat{j}$$

Hence, net acceleration of the particle,

$$\vec{a} = \vec{a}_x + \vec{a}_y = 0 + \left(-g\,\hat{j}\right) = -g\,\hat{j}$$



(a) Net velocity of the particle at any instant is $\vec{v} = \vec{v}_0 + \vec{a}t$

Hence, the velocity of the particle after 1 s is

$$\vec{v} = \left(8\hat{i} + 6\hat{j}\right) + \left(-10\hat{j}\right) \times 1$$
$$= 8\hat{i} + (6 - 10)\,\hat{j} = 8\hat{i} - 4\hat{j} \text{ m/s}$$
$$\Rightarrow \quad |\vec{v}| = \sqrt{(8)^2 + (4)^2} = 4\sqrt{5} \text{ m/s}$$

(b) For finding angle between two vectors we use 'dot product of two vectors'. We can define dot product of $\vec{u}$ and $\vec{v}$ as:

$\vec{u}.\vec{v} = |\vec{u}|.|\vec{v}|.\cos\alpha$. Here α is the angle between the vectors.

Then, $\cos\alpha = \dfrac{\vec{u}.\vec{v}}{|\vec{u}| \times |\vec{v}|}$

We have $\vec{u} = 8\hat{i} + 6\hat{j}$ m/s and $\vec{v} = 8\hat{i} - 4\hat{j}$ m/s

Hence, $\vec{u}.\vec{v} = 8\times8 - 6\times4 = 64 - 24 = 40$

and $|\vec{u}| = 10$ m/s and $|\vec{v}| = 4\sqrt{5}$ m/s

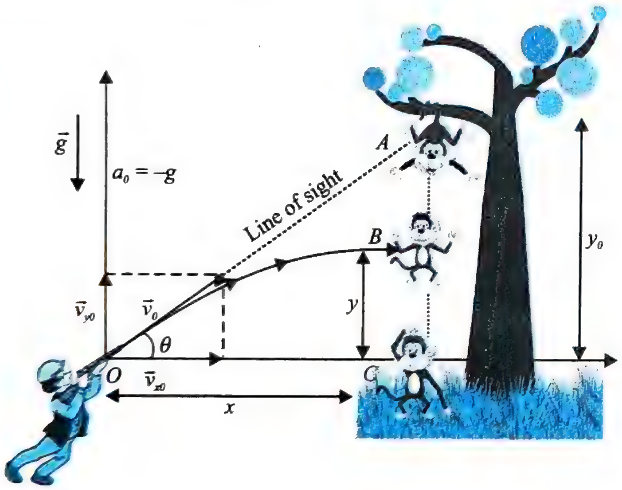
Then $\cos\alpha = \dfrac{40}{10\times4\sqrt{5}} = \dfrac{1}{\sqrt{5}} \Rightarrow \alpha = \cos^{-1}\left[\dfrac{1}{\sqrt{5}}\right]$

ILLUSTRATION 5.5

A hunter aims his gun and fires a bullet directly at a monkey in a tree. At the instant the bullet leaves the barrel of the gun, the monkey drops. Will the bullet hit the monkey? Support your answer with proper reasoning.

Sol. Let the monkey, stationed at A in the tree, be fired with a gun with a velocity $\vec{v}_0$ at angle θ with the horizontal along the line of sight as shown in figure.

If g (the magnitude of the free-fall acceleration) were zero, the bullet would follow the straight-line path OA as shown in figure and the monkey would float in place of dropping. The bullet would certainly hit the monkey. However, g is *not zero*, but the bullet *still* hits the monkey.



Motion of the bullet

Let $\vec{v}_{x0}$ and $\vec{v}_{y0}$ be the two rectangular components of $\vec{v}_0$.

Clearly, $v_{x0} = v_0\cos\theta$ and $v_{y0} = v_0\sin\theta$

Let the bullet take a time t to reach a point $B(x, y)$ directly below A.

As $OC = x$ and $BC = y$,

$$x = v_{x0}t \text{ and } y = v_{y0}t + \frac{1}{2}a_y t^2$$

or $$y = v_{y0}t + \frac{1}{2}(-g)t^2 = v_{y0}\times\frac{x}{v_{x0}} - \frac{1}{2}gt^2$$

or $$y = x\tan\theta - \frac{1}{2}gt^2 \quad \text{...(i)} \left[\text{as } \frac{v_{y0}}{v_{x0}} = \tan\theta\right]$$

Motion of the monkey

Since monkey is initially not at the origin, let $AC = y_0$

If after time t, monkey's y-coordinate is y', then

$$y' = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$$

or $$y' = x\tan\theta - \frac{1}{2}gt^2 \quad \text{...(ii)}$$

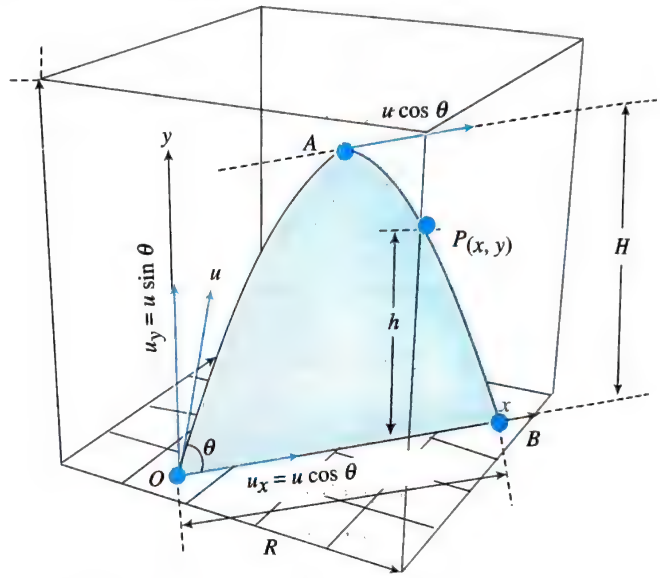
$$\left(\text{as } \tan\theta = \frac{AC}{OC} = \frac{y_0}{x}, y_0 = x\tan\theta \text{ and } v_{y0} = 0 \text{ and } a_y = g\right)$$

From equations (i) and (ii), $a_y = g$. Therefore, the bullet will hit the monkey when it reaches B.

The figure shows that during the time of flight of the bullet, both bullet and monkey fall the same distance from their origin ($g = 0$).

CALCULATION OF VARIOUS PARAMETERS IN PROJECTILE MOTION

Now we will use the above concept to calculate some important parameters of projectile motion.



Let a particle is projected with velocity u at angle θ with the horizontal from point O as shown in figure. We can write the x and y components of the initial velocity as $u_x = u\cos\theta$, $u_y = u\sin\theta$.

The acceleration of the particle in x- and y-direction is $a_x = 0$ and $a_y = -g$.

Time of Flight

The total time taken by the projectile to go up (O to A) and come down (A to B) to the same level from which it was projected is called time of flight.

For vertical upward motion, using $v_y = u_y + a_y t$, we get $u_y = u\sin\theta$ and $a_y = -g$

Hence, $0 = u_y + a_y t = u\sin\theta - gt$

$$\Rightarrow \quad t = \frac{u_y}{g} = \frac{u\sin\theta}{g}$$

Now as the time taken to go up is equal to the time taken to come down, so time of flight, L

$$T = 2t = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$$

Alternate method: The displacement along the vertical direction is zero for the complete flight (i.e., from O to B). Hence, along vertical direction, the net displacement = 0. Using $y = u_y T + \frac{1}{2}a_y T^2$,

$$0 = (u\sin\theta)T - \frac{1}{2}gT^2 \Rightarrow T = \frac{2u\sin\theta}{g}$$

Some important points regarding time of flight

- For complementary angles of projection θ and $90° - \theta$,

$$\text{Ratio of time of flight} = \frac{T_1}{T_2} = \frac{2u\sin\theta/g}{2u\sin(90-\theta)/g}$$

$$= \tan\theta \Rightarrow \frac{T_1}{T_2} = \tan\theta$$

- If t_1 is the time taken by the projectile from point O to P and t_2 is the time taken in moving from point P to B at ground level, then $t_1 + t_2 = \frac{2u\sin\theta}{g}$ = time of flight. $u\sin\theta = \frac{g(t_1+t_2)}{2}$

  and height of the point P is given by

$$h = u\sin\theta\, t_1 - \frac{1}{2}gt_1^2 = g\frac{(t_1+t_2)}{2}t_1 - \frac{1}{2}gt_1^2$$

By solving, we get $h = \frac{g t_1 t_2}{2}$.

Maximum Height

It is the maximum height from the point of projection, a projectile can reach.

Using $v_y^2 = u_y^2 + 2a_y s \Rightarrow 0 = (u\sin\theta)^2 - 2gH$, we get

$$H = \frac{u^2\sin^2\theta}{2g}$$

Some important points regarding maximum height

- Maximum height can also be expressed as:

$$H = \frac{u_y^2}{2g}$$

  where u_y is the vertical component of initial velocity.

- $H_{max} = \frac{u^2}{2g}$ (when $\sin^2\theta = \max = 1$ i.e., $\theta = 90°$)

  i.e., for maximum height, the body should be projected vertically upward. So it falls back to the point of projection after reaching the maximum height.

- For complementary angles of projection θ and $90° - \theta$

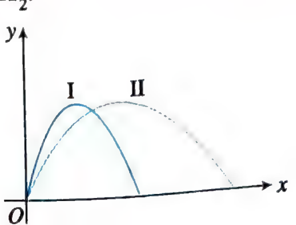
  Ratio of maximum height

$$= \frac{H_1}{H_2} = \frac{u^2\sin^2\theta/2g}{u^2\sin^2(90^o-\theta)2g} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

ILLUSTRATION 5.6

Which of the path (I) or (II) of a projectile has more time of flight? Use necessary assumptions.

Sol. Given $H_1 = H_2$.



$$\frac{(u_y)_1^2}{2g} = \frac{(u_y)_2^2}{2g} \Rightarrow \frac{u_1^2\sin^2\theta_1}{2g} = \frac{u_2^2\sin^2\theta_2}{2g}$$

$$\Rightarrow \quad u_1\sin\theta_1 = u_2\sin\theta_2 \text{ Or } (u_y)_1 = (u_y)_2$$

We know $T_1 = \frac{2(u_y)_1}{g} = \frac{2(u_y)_2}{g} = T_2$

Hence, both paths take same time.

Horizontal Range

It is the horizontal distance travelled by a body during the time of flight.

In horizontal direction, the acceleration of the particle is zero, i.e., horizontal component of the velocity is constant. Hence, displacement in the horizontal direction can be written as $x = R = u_x \times T$.

$$\Rightarrow \quad R = u\cos\theta \times T = u\cos\theta \times \left(\frac{2u\sin\theta}{g}\right)$$

$$= \frac{u^2(2\sin\theta\cdot\cos\theta)}{g} = \frac{u^2\sin 2\theta}{g}$$

Some important points regarding range

- The range of projectile can also be expressed as:

$$R = u\cos\theta \times T = u\cos\theta\frac{2u\sin\theta}{g}$$

$$= \frac{2u\cos\theta\, u\sin\theta}{g} = \frac{2u_x u_y}{g}$$

$\therefore \quad R = \frac{2u_x u_y}{g}$ (where u_x and u_y are the horizontal and vertical components of initial velocity)

Multiplication of time of flight

$$T_1T_2 = \frac{2u\sin\theta}{g}\frac{2u\cos\theta}{g} \Rightarrow T_1T_2 = \frac{2R}{g}$$

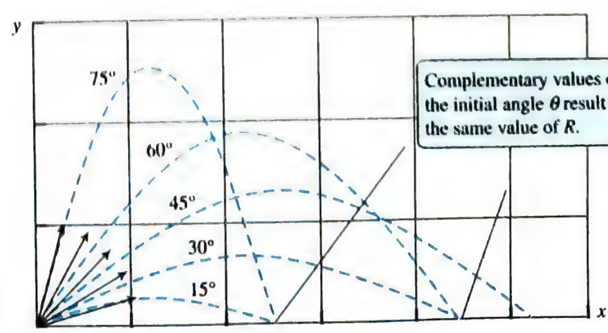
- If the angle of projection is changed from θ to $\theta' = (90 - \theta)$, then the range remains unchanged.

$$R' = \frac{u^2\sin 2\theta'}{g} = \frac{u^2\sin[2(90° - \theta)]}{g} = \frac{u^2\sin 2\theta}{g} = R$$

  So a projectile has same range at angles of projection θ and $(90° - \theta)$, though time of flight, maximum height and trajectories are different.

  These angles θ and $90° - \theta$ are called complementary angles of projection, and for complementary angles of projection, ratio of range

$$\frac{R_1}{R_2} = \frac{u^2\sin 2\theta/g}{u^2\sin[2(90^o - \theta)]/g} = 1 \Rightarrow \frac{R_1}{R_2} = 1$$



- **Maximum range:** For range to be maximum

$$\frac{dR}{d\theta}=0 \Rightarrow \frac{d}{d\theta}\left[\frac{u^2\sin 2\theta}{g}\right]=0$$

$\Rightarrow$ $\cos 2\theta = 0$, i.e., $2\theta = 90°$

$\Rightarrow$ $\theta = 45°$ and $R_{max} = \left(\frac{u^2}{g}\right)$

i.e., a projectile will have maximum range when it is projected at an angle of 45° to the horizontal and the maximum range will be (u^2/g).

When the range is maximum, the height H reached by the projectile is

$$H=\frac{u^2\sin^2\theta}{2g}=\frac{u^2\sin^2 45}{2g}=\frac{u^2}{4g}=\frac{R_{max}}{4}$$

i.e., if a person can throw a projectile to a maximum distance R_{max}, The maximum height to which it will rise is $\left(\frac{R_{max}}{4}\right)$.

- Relation between horizontal range and maximum height:

$$R=\frac{u^2\sin 2\theta}{g} \text{ and } H=\frac{u^2\sin^2\theta}{2g}$$

$$\therefore \quad \frac{R}{H}=\frac{u^2\sin 2\theta/g}{u^2\sin^2\theta/2g}=4\cot\theta$$

$\Rightarrow$ $R = 4H\cot\theta$

- If in case of projectile motion range R is n times the maximum height H

i.e., $R = nH \Rightarrow \frac{u^2\sin 2\theta}{g}=n\frac{u^2\sin^2\theta}{2g}$

$\Rightarrow$ $\tan\theta = [4/n]$ or $\theta = \tan^{-1}[4/n]$

The angle of projection is given by $\theta = \tan^{-1}[4/n]$

Note: If $R = H$, then $\theta = \tan^{-1}(4)$.
If $R = 4H$, then $\theta = \tan^{-1}(1)$.

ILLUSTRATION 5.7

A grasshopper can jump upto a height h. Find the maximum distance through which it can jump along the horizontal ground.

Sol. The grasshopper can jump upto a height h. It means the maximum possible height it can achieve, i.e., when it jumps vertically.

Since $h=\frac{v^2\sin^2\theta}{2g}$, when $\theta = 90°$,

$$h=\frac{v^2}{2g} \Rightarrow v^2 = 2gh \qquad \text{...(i)}$$

Now, maximum range, $\frac{v^2}{g}=\frac{2gh}{g}=2h \Rightarrow R=2h$

ILLUSTRATION 5.8

At what angle should a projectile be thrown such that the horizontal range of the projectile will be equal to half of its maximum value?

Sol. We know: $R=\frac{u^2\sin 2\theta}{g}$ and $R_{max}=\frac{u^2}{g}$.

Given $R=\frac{1}{2}R_{max}$

or $R=\frac{u^2\sin 2\theta}{g}=\frac{1}{2}R_{max}=\frac{u^2}{2g}$

$\Rightarrow$ $\sin 2\theta = \frac{1}{2}$ or $2\theta = 30° \Rightarrow \theta = 15°$

Also $R=\frac{u^2\sin(\pi-2\theta)}{g}$

Now, $\sin(\pi-2\theta)=\frac{1}{2} \Rightarrow 180°-2\theta = 30° \Rightarrow \theta = 75°$

ILLUSTRATION 5.9

A batsman hits a ball at an angle of 30° with an initial speed of 30 m s^{-1}. Assuming that the ball travels in a vertical plane, calculate

(a) The time at which the ball reaches the highest point
(b) The maximum height reached
(c) The horizontal range of the ball
(d) The time for which the ball is in the air

Sol. Here $\theta = 30°$, $u = 30$ m s^{-1}.

(a) The time taken by the ball to reach the highest point is half the total time of flight. As the time of ascending and descending is same for a projectile without air resistance, the time to reach the highest point

$$t_H=\frac{T}{2}=\frac{u\sin\theta}{g}=\frac{30}{10}\times\sin 30° = 1.5 \text{ s}$$

(b) The maximum height reached is

$$\frac{u^2\sin^2\theta}{2g}=\frac{(30)^2\times(\sin 30°)^2}{2g}=\frac{900}{2\times10\times4}=11.25 \text{ m}$$

(c) Horizontal range

$$=\frac{u^2\sin 2\theta}{g}=\frac{(30)^2\sin 2(30°)}{10}=\frac{900\sqrt{3}}{20}=45\sqrt{3} \text{ m}$$

(d) The time for which the ball is in air is same as its time of flight, i.e.,

$$\frac{2u\sin\theta}{g}=\frac{2\times30\times\sin 30°}{10}=3 \text{ s}$$

ILLUSTRATION 5.10

The horizontal range of a projectile is $2\sqrt{3}$ times its maximum height. Find the angle of projection.

Sol. If u and α are the initial velocity of projection and angle of projection, respectively, then

Maximum height attained $= \dfrac{u^2 \sin^2 \alpha}{2g}$

Horizontal range $= \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

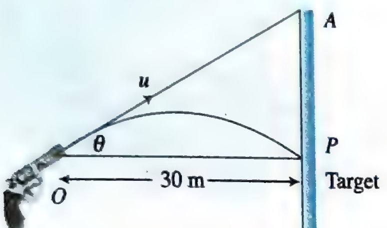
According to the problem,

$$\frac{2u^2 \sin\alpha\cos\alpha}{g} = 2\sqrt{3}\left(\frac{u^2\sin^2\alpha}{2g}\right) \Rightarrow \tan\alpha = \left(\frac{2}{\sqrt{3}}\right)$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

ILLUSTRATION 5.11

A bullet with muzzle velocity 100 m s^{-1} is to be shot at a target 30 m away in the same horizontal line. How high above the target must the rifle be aimed so that the bullet will hit the target?



Sol. Horizontal range of bullet is 30 m.

Using range formula, $R = \dfrac{u^2 \sin 2\theta}{g} = 30$

or $\quad \sin 2\theta = \dfrac{30\times 10}{(100)^2} \quad$ or $\quad \sin 2\theta = 0.03$

For small θ, $\sin\theta \approx \theta = \tan\theta$, i.e., $2\theta = 0.03$

Therefore, $\theta = 0.015$ rad

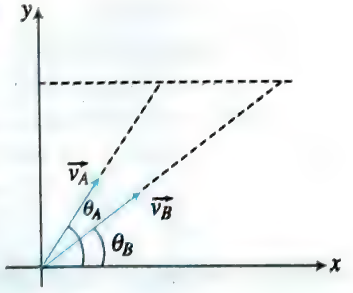
In ΔOAP, $\tan\theta = \dfrac{AP}{OP} \quad \Rightarrow AP = OP\tan\theta$

The rifle must be aimed at an angle $\theta = 0.015$ above horizontal.

Height to be aimed $= 30\tan\theta \approx 30(\theta) = 30\times 0.015 = 45$ cm

ILLUSTRATION 5.12

Two particles A and B are projected from the same point in different directions in such a manner that vertical components of their initial velocities are same (Figure). Find the ratio of range.



Sol. We know that the range of projectile,

$$R = \frac{u^2\sin 2\theta}{g} = \frac{(2u\sin\theta)}{g}\times u\cos\theta = \frac{2u_x\cdot u_y}{g}$$

As $(u_y)_1 = (u_y)_2 \Rightarrow u_A\sin\theta_A = u_B\sin\theta_B$

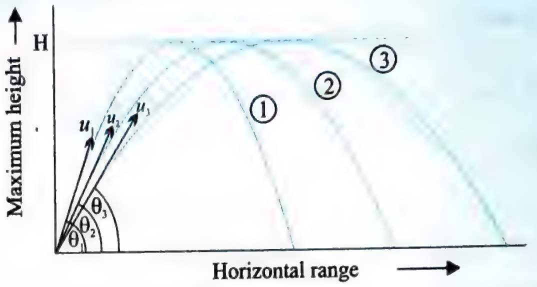
or $\quad \dfrac{u_A}{u_B} = \dfrac{\sin\theta_B}{\sin\theta_A}$

$$\frac{R_A}{R_B} = \frac{2(u_A\sin\theta_A)(u_A\cos\theta_A)/g}{2(u_B\sin\theta_B)(u_B\cos\theta_B)/g}$$

$$= \left[\frac{u_A\cos\theta_A}{u_B\cos\theta_B}\right] = \frac{\sin\theta_B\cos\theta_A}{\sin\theta_A\cos\theta_B} = \frac{\tan\theta_B}{\tan\theta_A}$$

ILLUSTRATION 5.13

Three projectiles are fired with velocities u_1, u_2 and u_3 at inclinations θ_1, θ_2 and θ_3, respectively, with the horizontal such that the maximum heights attained by all of them are same.



(a) Which projectile will take maximum time to reach the ground?

(b) Which projectile will possess the maximum speed on reaching the ground?

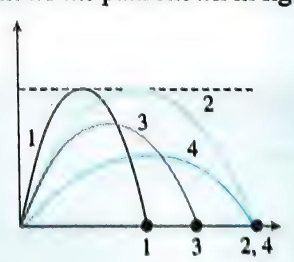
Sol.

(a) Since the maximum heights attained by all the three projectiles are same, the vertical components of their velocities must be equal i.e., $u_1\sin\theta_1 = u_2\sin\theta_2 = u_3\sin\theta_3$. Obviously, for all the three projectiles, the time of ascent and descent will be same and have they will reach the ground at the same time.

(b) The horizontal range of the three projectiles are different. The one, which has the maximum horizontal range will also have the maximum horizontal component of the velocity i.e., $u_3\cos\theta_3 > u_2\cos\theta_2 > u_1\cos\theta_1$. Obviously, the third projectile will reach the ground with maximum velocity i.e. u_3 will be maximum.

ILLUSTRATION 5.14

Four cannon balls, (1), (2), (3), and (4) are fired from level ground. Cannon ball (1) is fired at an angle of 60° above the horizontal and follows the path shown in figure.



Cannon balls (2) and (3) are fired at an angle of 45° and (4) is fired at an angle of 30° above the horizontal. Which cannon ball has the largest initial speed?

Sol. Expressions for range and maximum height are given by

$$R = \frac{u^2 \sin 2\theta}{g}; \quad H = \frac{u^2 \sin^2 \theta}{2g}$$

From the figure, cannon balls (2) and (4) have same range.

$$u_2^2 \sin 2 \times 45° = u_4^2 \sin 2 \times 30°$$

$$u_2^2 \sin 90° = u_4^2 \sin 60°$$

Thus, $u_2^2(1) = u_4^2\left(\frac{\sqrt{3}}{2}\right)$ or $u_4 > u_2$

Now we compare (2) and (3), $(\text{Range})_3 < (\text{Range})_2$

As the projection angle is same, $u_3 < u_2$.

Now we compare (1) and (2), the maximum height achieved by balls are same, $H_1 = H_2$.

or $\quad \dfrac{u_1^2 \sin^2 60°}{2g} = \dfrac{u_2^2 \sin^2 45°}{2g} \quad$ or $\quad u_1 = u_2 \times \sqrt{\dfrac{2}{3}}$.

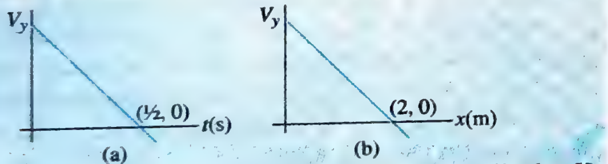
or $\quad u_2 > u_1$

As $u_4 > u_2$, $u_2 > u_3$ and $u_2 > u_1$, therefore, u_4 is maximum.

Hence, the initial velocity of ball (4) is maximum.

ILLUSTRATION 5.15

Two graphs of the same projectile motion (in the x- y-plane) projected from origin are shown in figure. X-axis is along horizontal direction and Y-axis is vertically upwards. Take $g = 10$ m s^2.



Find (i) The Y component of initial velocity and (ii) the X component of initial velocity.

Sol. From graph (i): $v_y = 0$ at $t = \frac{1}{2}$ s, i.e., time taken to reach maximum height H is

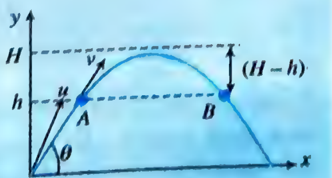
$$t = \frac{u_y}{g} = \frac{1}{2} \Rightarrow u_y = 5 \text{ m s}^{-1}$$

From graph (2); $v_y = 0$ at $x = 2$ m, i.e., when the particle is at maximum height, its displacement along horizontal, $x = 2$ m.

$$x = u_x \times t \Rightarrow 2 = u_x \times \frac{1}{2} \Rightarrow u_x = 4 \text{ m s}^{-1}$$

ILLUSTRATION 5.16

The figure shows two positions A and B at the same height h above the ground. If the maximum height of the projectile is H, then determine the time t elapsed between the positions A and B in terms of H and h.

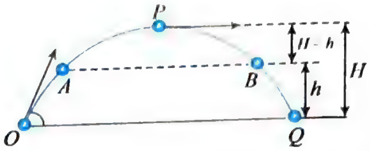


Sol. We know the time of flight

$$T = \frac{2u_y}{g} \text{ or } u_y = \frac{Tg}{2} \quad \text{...(i)}$$

Maximum height reached by projectile, $H = \dfrac{u_y^2}{2g}$ ...(ii)

From (i) and (ii), $H = \dfrac{1}{2g}\left(\dfrac{Tg}{2}\right)^2 = \dfrac{T^2 g}{8}$ or $T^2 = \dfrac{8H}{g}$ ...(iii)



Let the time taken to move from A to B is t. We can write in similar manner

$$t^2 = \frac{8(H-h)}{g} \text{ or } t = \sqrt{\frac{8}{g}(H-h)}$$

Alternate: If the time taken by the projectile in moving from A to B is t, then the time taken for moving from P to B should be $t/2$.

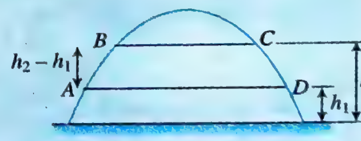
Using $y = u_y t + \frac{1}{2}a_y t^2$ in vertical direction. Here as the top position $u_y = 0$, hence

$$(H-h) = \frac{1}{2}g\left(\frac{t}{2}\right)^2$$

$$\Rightarrow \quad t^2 = \frac{8(H-h)}{g} \text{ or } t = \sqrt{\frac{8}{g}(H-h)}$$

Important Observation:

If t_{AD} is the time interval to travel from A to D and t_{BC} is the time interval to travel from B to C, then



$$(t_{AD})^2 - (t_{BC})^2 = \frac{8(h_2 - h_1)}{g}$$

Equation of Trajectory

A projectile thrown with velocity u at an angle θ with the horizontal. The velocity u can be resolved into two rectangular components: $u \cos\theta$ component along X-axis and $u \sin\theta$ component along Y-axis.

For horizontal motion, $x = u\cos\theta \times t$

$$\Rightarrow \quad t = \frac{x}{u\cos\theta} \quad \text{...(i)}$$

For vertical motion, $y = (u\sin\theta)t - \frac{1}{2}gt^2$ ...(ii)

From equation (i) and (ii),

$$y = u\sin\theta\left(\frac{x}{u\cos\theta}\right) - \frac{1}{2}g\left(\frac{x^2}{u^2\cos^2\theta}\right)$$

$$= x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta} \quad \text{...(iii)}$$

This equation shows that the trajectory of projectile is parabolic because it is similar to the equation of parabola.

$$y = ax - bx^2$$

It is known as the equation of trajectory. It is an equation of parabola. Hence, path of a projectile is parabolic.

Equation (iii) can also be written as:

$$y = x\tan\theta - \frac{x^2}{\dfrac{2u^2\cos^2\theta}{g}} = x\tan\theta - \frac{x^2}{\left(\dfrac{2u^2\cos\theta\cdot\sin\theta}{g}\right)}\frac{\sin\theta}{\cos\theta}$$

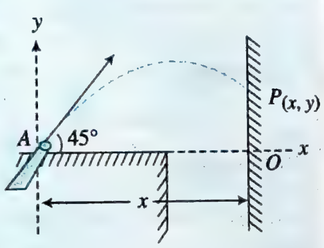
$$= x\tan\theta - \frac{x^2\tan\theta}{R} \qquad y = x\tan\theta\left(1-\frac{x}{R}\right)$$

Note: The equation of oblique projectile also can be written as $y = x\tan\theta\left[1-\frac{x}{R}\right]$ $\left(\text{where } R = \text{horizontal range} = \frac{u^2\sin 2\theta}{g}\right)$

Since $R = \frac{u^2\sin 2\theta}{g}$, substituting $\frac{g}{u^2} = \frac{\sin 2\theta}{R}$ in the locus equation, we have $\frac{y}{x} = \tan\theta\left(1-\frac{x}{R}\right)$, whereas $\frac{dy}{dx} = \tan\theta\left(1-\frac{2x}{R}\right)$ The shape of the curve above x-axis is symmetrical about a vertical line passing through the highest point.

ILLUSTRATION 5.17

A jet of water is projected at an angle $\theta = 45°$ with horizontal from point A which is situated at a distance $x = OA$ = (a) 1/2 m, (b) 2 m from a vertical wall. If the speed of projection is $v_0 = \sqrt{10}$ m s^{-1}, find point P of striking of the water jet with the vertical wall.



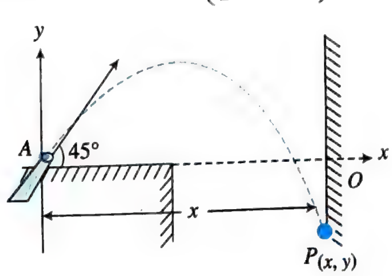
Sol. Point P lies at the trajectory of jet of water. Hence, the coordinate of point P (x,y) should satisfy the trajectory equation.

$$y = x\tan\theta\left(1-\frac{x}{R}\right) \qquad \text{...(i)}$$

$$R = \frac{u^2\sin 2\theta}{g} = \frac{(\sqrt{10})^2\cdot\sin 2\times 45°}{10} = 1 \text{ m}$$

(a) If $x = 1/2$ m from (i) $y = \frac{1}{2}\tan 45°\left(1-\frac{1/2}{1}\right) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$ m

Hence, the coordinate of $P = \left(\frac{1}{2}\text{m}, \frac{1}{4}\text{m}\right)$



(b) If $x = 2$ m, from (i),

$$y = 2\cdot\tan 45°\left(1-\frac{2}{1}\right) = -2 \text{ m}$$

Hence, water jet will strike below the horizontal dotted line (x-axis) at coordinate (2 m, –1 m).

ILLUSTRATION 5.18

A particle is projected from the ground at $t = 0$ so that on its way it just clears two vertical walls of equal height on the ground. The particle was projected with initial velocity u and at angle θ with the horizontal. If the particle passes just grazing top of the wall at time $t = t_1$ and $t = t_2$, then calculate

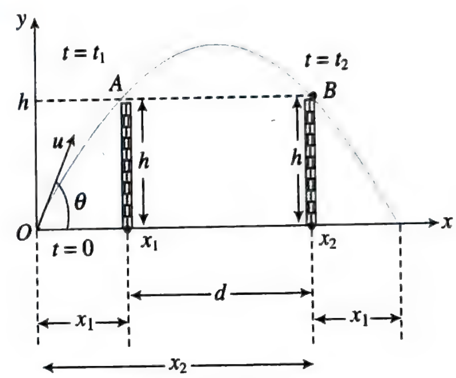
(a) the height of the wall.

(b) the time t_1 and t_2 in terms of height of the wall.

(c) Write the expression for calculating the range of this projectile and separation between the walls.

Sol.

(a)



$$h = (u\sin\theta)t_1 - \frac{1}{2}gt_1^2 \qquad \text{...(i)}$$

and $$h = (u\sin\theta)t_2 - \frac{1}{2}gt_2^2 \qquad \text{...(ii)}$$

Comparing equation (i) with equation (ii), we get

$$u\sin\theta = \frac{g(t_1+t_2)}{2}$$

Substituting this value in equation (i), we get

$$h = g\left(\frac{t_1+t_2}{2}\right)t_1 - \frac{1}{2}gt_1^2 \Rightarrow h = \frac{gt_1t_2}{2}$$

(b) **Time (t_1 and t_2):** $h = u\sin\theta\, t - \frac{1}{2}gt^2$

$$t^2 - \frac{2u\sin\theta}{g}t + \frac{2h}{g} = 0$$

Solving for t, we get

$$t_1 = \frac{u\sin\theta}{g}\left[1+\sqrt{1-\left(\frac{\sqrt{2gh}}{u\sin\theta}\right)^2}\right]$$

and $$t_2 = \frac{u\sin\theta}{g}\left[1-\sqrt{1-\left(\frac{\sqrt{2gh}}{u\sin\theta}\right)^2}\right]$$

(c) Finding Range and Separation Between Walls

With the help of the equation of trajectory, we can determine the range of projectile.

The path of the projectile grazes the top of the walls, so coordinates of the top point of the wall must satisfy the equation of trajectory.

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2$$

$$h = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2$$

If x_1 and x_2 are roots of this quadratic equation,

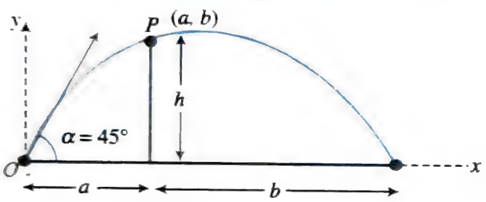
$x_1 - x_2 = d$ (difference of two roots gives desired relation)

$x_1 + x_2 =$ (sum of the roots) gives horizontal range.

ILLUSTRATION 5.19

From a point on the ground at a distance a from the foot of a pole, a ball is thrown at an angle of 45°, which just touches the top of the pole and strikes the ground at a distance of b, on the other side of it. Find the height of the pole.

Sol. Let h be the height of the pole.



We have $y = x\tan\alpha\left(1 - \frac{x}{R}\right)$

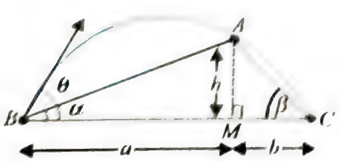
Since the top of pole lies on curve (1),

$$h = a\tan 45°\left(1 - \frac{a}{a+b}\right) = a\left[\frac{a+b-a}{a+b}\right] = \frac{ab}{a+b}$$

ILLUSTRATION 5.20

A particle is projected over a triangle from one extremity of its horizontal base. Grazing over the vertex, it falls on the other extremity of the base. If a and b are the base angles of the triangle and θ the angle of projection, prove that $\tan\theta = \tan\alpha + \tan\beta$.

Sol. Let ABC be the triangle with base BC. Let h be the height of the vertex A above BC. If AM is the perpendicular drawn on base BC from vertex A, then $\tan\alpha = h/a$ and $\tan\beta = h/b$, where $BM = a$ and $CM = b$.



Since A (a, h) lies on the trajectory of the projectile,

$$y = x\tan\theta\left(1 - \frac{x}{R}\right) \quad ...(i)$$

Therefore, it should satisfy (i) i.e.,

$$h = a\tan\theta\left(1 - \frac{a}{a+b}\right) \qquad [\because \text{ Range, } R = a+b]$$

$$\frac{h}{a} = \tan\theta\left[\frac{b}{a+b}\right]$$

$$\Rightarrow \quad \tan\theta = \frac{h}{a} + \frac{h}{b} = \tan\alpha + \tan\beta$$

Hence, proved.

ILLUSTRATION 5.21

A particle projected at a definite angle α to the horizontal passes through points (a, b) and (b, a), referred to horizontal and vertical axes through the point of projection. Show that:

(a) The horizontal range $R = \dfrac{a^2+ab+b^2}{a+b}$

(b) The angle of projection α is given by $\tan^{-1}\left[\dfrac{a^2+ab+b^2}{ab}\right]$

Sol.

(a) The equation of the trajectory of the particle

$$y = x\tan\alpha\left(1 - \frac{x}{R}\right) \quad ...(i)$$

Since the particle passes through the points with coordinates (a, b) and (b, a), they should satisfy the equation of the curve.

$$b = a\tan\alpha\left(1 - \frac{a}{R}\right) \quad ...(ii)$$

and $$a = b\tan\alpha\left(1 - \frac{b}{R}\right) \quad ...(iii)$$

Dividing (ii) and (iii), $$\frac{b^2}{a^2} = \frac{\left(1 - \frac{a}{R}\right)}{\left(1 - \frac{b}{R}\right)}$$

$$\Rightarrow \quad b^2 - \frac{b^3}{R} = a^2 - \frac{a^3}{R}$$

$$\Rightarrow \quad \frac{1}{R}[a^3 - b^3] = a^2 - b^2$$

$$\Rightarrow \quad R = \frac{a^3 - b^3}{a^2 - b^2} = \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)} = \frac{a^2+ab+b^2}{a+b}$$

Hence, proved. $[\because a \neq b]$

(b) Substituting the expression for R in (ii),

$$\frac{b}{a} = \tan\alpha\left[1 - \frac{a(a+b)}{a^2+ab+b^2}\right]$$

$$= \tan\alpha\left[\frac{a^2+ab+b^2-a^2-ab}{a^2+ab+b^2}\right]$$

$$\tan\alpha = \frac{a^2+ab+b^2}{ab} \Rightarrow \alpha = \tan^{-1}\left[\frac{a^2+ab+b^2}{ab}\right]$$

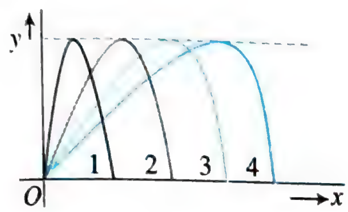
Hence proved.

CONCEPT APPLICATION EXERCISE 5.2

1. The equation of projectile is $y = 16x - \frac{5x^2}{4}$. Find the horizontal range.

2. A particle is thrown with velocity u at an angle α from the horizontal. Another particle is thrown with the same velocity at an angle α from the vertical. What will be the ratio of times of flight of two particles?

3. The figure below shows four paths for a kicked football. Ignoring the effects of air on the flight, rank the paths according to the initial horizontal velocity component, highest first.



4. A projectile is thrown into space so as to have maximum horizontal range R. Taking the point of projection as origin, find out the co-ordinates of the point where the speed of the particle is minimum.

5. A large number of bullets are fired in all directions with same speed u. What is the maximum area on the ground on which these bullets will spread?

6. A projectile thrown with an initial speed u and the angle of projection 15° to the horizontal has a range R. If the same projectile is thrown at an angle of 45° to the horizontal with speed $2u$, what will be its range?

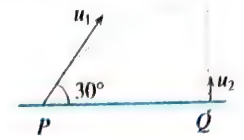
7. The range R of a projectile is same when its maximum heights are h_1 and h_2. What is the relation between R and h_1 and h_2?

8. A grasshopper can jump a maximum distance of 1.6 m. It spends negligible time on the ground. How far can it go in 10 s?

9. A projectile is thrown with an initial velocity of $v = a\hat{i} + b\hat{j}$. If the range of projectile is double the maximum height reached by it. Find the ratio $\frac{b}{a}$.

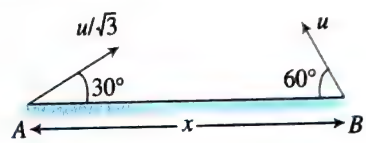
10. A ball is thrown at different angles with the same speed u and from the same points and it has same range in both the cases. If y_1 and y_2 be the heights attained in the two cases, then find the value of $y_1 + y_2$.

11. A particle P is projected with velocity u_1 at an angle of 30° with the horizontal. Another particle Q is thrown vertically upwards with velocity u_2 from a point vertically below the highest point of path of P. Determine the necessary condition for the two particles to collide at the highest point.



12. Two seconds after projection, a projectile is travelling in a direction inclined at 30° to the horizontal. After one more second, it is travelling horizontally. Find the magnitude and direction of its velocity.

13. Two particles are separated at a horizontal distance x as shown in the figure. They are projected at the same time as shown in the figure with different initial speed. Find the time after which the horizontal distance between the particles becomes zero.



14. The horizontal range (R) of a projectile becomes (R + $2H$) from R due to a wind in horizontal direction. Here H is the maximum height reached by the projectile. What constant horizontal acceleration is imparted by the wind?

15. A boy is running along positive x-axis with 8 m/s. While running he manages to throw a stone in a plane perpendicular to his direction of running with velocity 12 m/s at an angle 30° with vertical. Find the speed of the stone at the highest point of the trajectory.

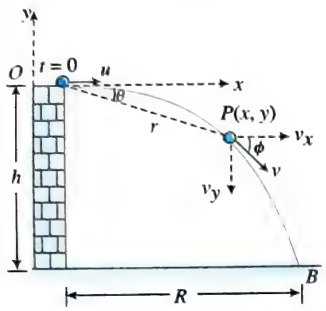
ANSWERS

1. 12.8 m	**2.** tan α
3. Graph 4 > 3 > 2 > 1	**4.** (R/2, R/4)
5. $\pi\frac{u^4}{g^2}$	**6.** 8R
7. $R = 4\sqrt{h_1 h_2}$	**8.** $20\sqrt{2}$ m
9. $b = 2a$	**10.** $\frac{u^2}{2g}$
11. $u_1 = 2u_2$	**12.** $u = 20\sqrt{3}$, $\theta = 60°$
13. x/u	**14.** $\frac{g}{2}$
15. 10 m/s	

HORIZONTAL PROJECTILE

A body is projected horizontally from a certain height y vertically above the ground with initial velocity u. If friction is considered to be absent, then there will be no other horizontal forces which can affect the horizontal motion. The horizontal velocity, therefore, remains constant and so the object covers equal distance in horizontal direction in equal intervals of time.

- **Trajectory of horizontal projectile:** The horizontal displacement x is governed by the equation



$$x = ut \Rightarrow t = \frac{x}{u} \qquad \text{...(i)}$$

The vertical displacement y is governed by

$$y = -\frac{1}{2}gt^2 \qquad \text{...(ii)}$$

(Since the initial vertical velocity is zero)

By substituting the value of t in (ii), $y = -\frac{1}{2}\frac{gx^2}{u^2}$

- **Displacement of projectile ($\vec{r}$):** After time t, horizontal displacement $x = ut$ and vertical displacement $y = \frac{1}{2}gt^2$.

So, position vector, $\vec{r} = ut\,\hat{i} - \frac{1}{2}gt^2\,\hat{j}$

$$\therefore \quad r = ut\sqrt{1 + \left(\frac{gt}{2u}\right)^2}$$

and $\quad \theta = \tan^{-1}\left(\frac{gt}{2u}\right) = \tan^{-1}\left(\sqrt{\frac{gy}{2}}\Big/u\right) \left(\text{as } t = \sqrt{\frac{2y}{g}}\right)$

- **Instantaneous velocity:** Throughout the motion, the horizontal component of the velocity is $v_x = u$.

The vertical component of velocity increases with time and is given by

$v_y = 0 - g\,t = -g\,t \qquad$ (from $v = u + gt$)

So, $\quad \vec{v} = v_x\hat{i} - v_y\hat{j} = \vec{v} = u\,\hat{i} - gt\,\hat{j}$

i.e., $\quad v = \sqrt{u^2 + (gt)^2} = u\sqrt{1 + \left(\frac{gt}{u}\right)^2}$

Again $\vec{v} = u\hat{i} - \sqrt{2gy}\,\hat{j}$

i.e., $\quad v = \sqrt{u^2 + 2gy}$

- **Direction of instantaneous velocity:** $\tan\phi = \frac{v_y}{v_x}$

$$\Rightarrow \quad \phi = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{\sqrt{2gy}}{u}\right) = \tan^{-1}\left(\frac{gt}{u}\right)$$

where ϕ is the angle of instantaneous velocity from the horizontal.

- **Time of flight:** If a body is projected horizontally from a height h with velocity u and the time taken by the body to reach the ground is T, then for vertical motion,

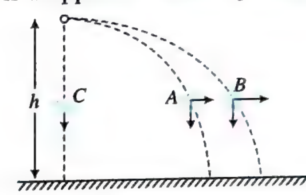
$$-h = 0 - \frac{1}{2}gT^2 \quad \Rightarrow \quad T = \sqrt{\frac{2h}{g}}$$

- **Horizontal range:** Let R be the horizontal distance travelled by the body. The acceleration in x-direction is zero. Hence

$$R = uT + \frac{1}{2}0T^2 \quad \Rightarrow \quad R = u\sqrt{\frac{2h}{g}}$$

Important Points:

1 If projectiles A and B are projected horizontally with different initial velocity from same height and a third particle C is dropped from same point, then



(a) All three particles will take equal time to reach the ground.
(b) Their net velocity would be different but all three particles possess same vertical component of velocity.
(c) The trajectory of projectiles A and B will be straight line w.r.t. particle C.

2. If a particle is released from an airplane moving horizontally at a certain height, the analysis of the motion of the particle will be same as the particle thrown from the certain height as discussed above.

ILLUSTRATION 5.22

A rubber ball escapes from the horizontal roof with a velocity $v = 5\text{ m s}^{-1}$. The roof is situated at a height, $h = 20$ m. If the length of each car is equal to $x_0 = 4$ m, with which car will the ball hit?

Sol. The initial velocity of the ball is in horizontal direction. The initial velocity in vertical direction is zero.

For vertical motion, $u_x = 5\text{ m s}^{-1}$, $u_y = 0$.

Displacement in vertical direction, $y = -h = -20$ m

$$-h = 0 - \frac{1}{2}gt^2 \Rightarrow \frac{2 \times 20}{10} = t^2 \Rightarrow t = 2\text{ s}$$

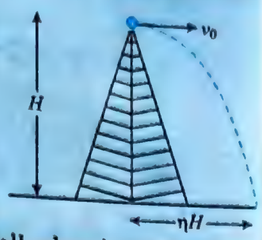
Horizontal displacement: Distance covered in 2 s,

$$x = d = u_x \times t = 5 \times 2 = 10\text{ m}$$

Since the length of one car is 4 m, it will hit the third car for which $8\text{ m} < d < 12\text{ m}$.

ILLUSTRATION 5.23

(a) With what velocity (v_0) should a ball be projected horizontally from the top of a tower so that the horizontal distance on the ground is ηH, where H is the height of the tower?



(b) Also determine the speed of the ball when it reaches the ground.

Sol.

(a) Given, horizontal distance $= \eta H$, where H is the height of the tower.

To find out the sufficient velocity as asked, we have to use the formula: distance = velocity × time. Here the time involved will be the time of flight as we are considering projectile motion, which is given by $T = \sqrt{\dfrac{2H}{g}}$.

So, $v_0 T = \eta H \Rightarrow v_0\sqrt{\dfrac{2H}{g}} = \eta H \Rightarrow v_0 = \eta\sqrt{\dfrac{gH}{2}}$

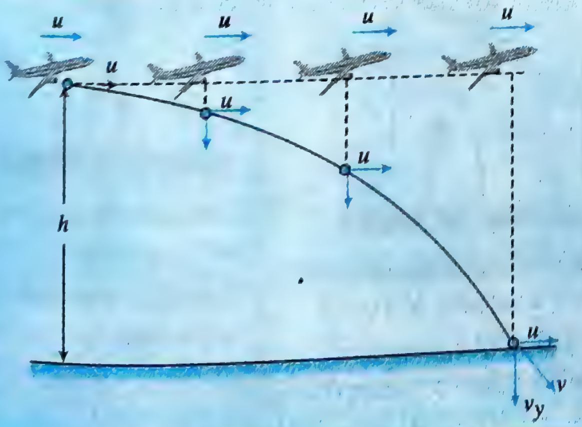
(b) During the projectile motion, the horizontal component of velocity remains same and its vertical component keeps on changing under the effect of gravity. So horizontal speed, $v_x = v_0$ and vertical speed, $v_y = \sqrt{2gH}$.

Total speed $= \sqrt{v_x^2 + v_y^2}$

$$= \sqrt{v_0^2 + 2gH} = \sqrt{\eta^2\frac{gH}{2} + 2gH} = \sqrt{\left(\frac{\eta^2+4}{2}\right)gH}$$

ILLUSTRATION 5.24

A relief food package is dropped from a airplane which is moving horizontal with a velocity of 30 m s^{-1} at a height h = 50 m. Find the (a) time of flight of the package, (b) location of the point of striking of the food package, (c) velocity of the package at the time of striking the ground, and (d) displacement of the food package.



Sol. $u_x = 30$ m s^{-1}, $u_y = 0$ m s^{-1}, and $h = 50$ m.

$$h = u_y t + \frac{1}{2}gt^2$$

$$\Rightarrow \frac{50\times 2}{10} = t^2 \Rightarrow t = \sqrt{10}\ \text{s}$$

Distance (horizontally) covered in $t = \sqrt{10}$ s

$$x = u_x t = 30\sqrt{10}\ \text{m}$$

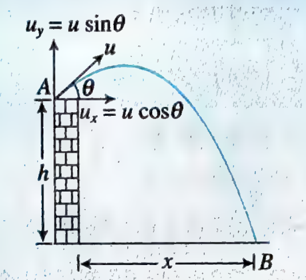
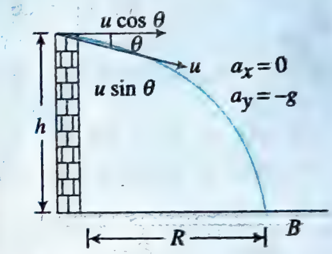
$$v_x = u_x = 30\ \text{m s}^{-1}$$

$$v_y = u_y + gt = 10\sqrt{10}\ \text{m s}^{-1}$$

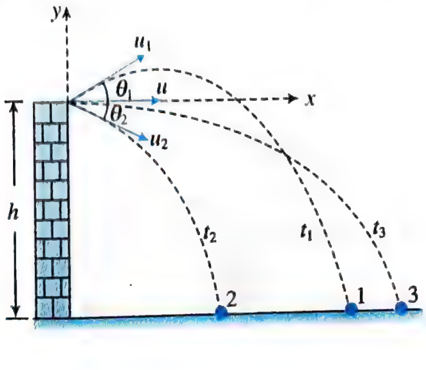
$$v = \sqrt{900+1000} = 10\sqrt{19}\ \text{m s}^{-1}$$

Displacement $= \sqrt{h^2 + x^2} = \sqrt{2500+9000} = 10\sqrt{115}$ m

PROJECTILE FROM HEIGHT AT CERTAIN ANGLE WITH HORIZONTAL

Case I	Case II
Projection at an angle θ above horizontal	Projection at an angle θ below horizontal
$u_x = u\cos\theta$; $a_y = -g$	$u_x = u\cos\theta$; $u_y = -u\sin\theta$; $a_y = -g$
	
Equation of motion between A and B (in Y direction) $S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$ $S_y = u_y t + \frac{1}{2}a_y t^2$ $\Rightarrow -h = (u\sin\theta)T - \frac{1}{2}gT^2$ Solving the above equation, we will get the time of flight, T. Range, $R = u_x T = u\cos\theta\, T$ Also, $v_y^2 = u_y^2 + 2a_y S_y$ $= u^2\sin^2\theta + 2gh$ $v_x = u\cos\theta$ $v_B = \sqrt{v_y^2 + v_x^2}$ $v_B = \sqrt{u^2 + 2gh}$	$S_y = u_y t + \frac{1}{2}a_y t^2$ $S_y = -h$, $u_y = -u\sin\theta$ $t = T$, $a_y = -g$ $\Rightarrow -h = -(u\sin\theta)T - \frac{1}{2}gT^2$ $\Rightarrow h = (u\sin\theta)T + \frac{1}{2}gT^2$. Solving the above equation, we will get the time of flight, T. Range, $R = u_x T = u\cos\theta\, T$ $v_x = u\cos\theta$ $v_y^2 = u_y^2 + 2a_y S_y$ $= u^2\sin^2\theta + 2(-g)(-h)$ $= u^2\sin^2\theta + 2gh$ $v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$

Important observation: If three objects are thrown from the same height in different directions as shown in figure, then



if $u_1 = u_2 = u$ and $\theta_1 = \theta_2$.

All the three objects will strike the ground with the same final speed. If $u_1 \neq u_2 \neq u$, then all the three objects will strike the ground with the different final speeds.

ILLUSTRATION 5.25

A ball is thrown from the top of a building 45 m high with a speed 20 m s^{-1} above the horizontal at an angle of 30°. Find

(a) The time taken by the ball to reach the ground.

(b) The speed of ball just before it touches the ground.

Sol. Given $v = 20$ m s^{-1}, $\theta = 30°$, $H = 45$ m.

(a) As the ball has been projected at an angle of 30° above horizontal, so first of all we need to analyse the velocity horizontally and vertically. This will be useful while using distance–time relation in horizontal and vertical directions.

$$v_{xi} = v\cos 30° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3} \text{ m s}^{-1}$$

$$v_{yi} = v\sin 30° = 20 \times \frac{1}{2} = 10 \text{ m s}^{-1}$$

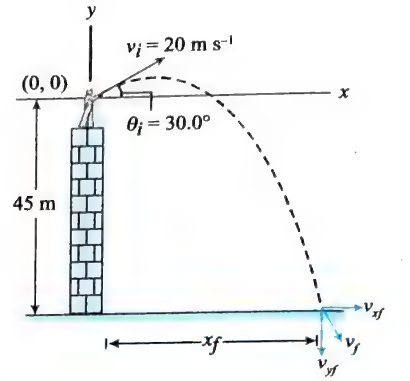
It will be easy for us to use distance–time relation in vertical as it will involve less calculation.

In y-direction: $-45 = 10t - \frac{1}{2} \times gt^2 \Rightarrow t^2 - 2t - 9 = 0$

which on solving gives $t = 1 + \sqrt{10}$ s (positive value), (other value is $1 - \sqrt{10}$ s, a negative value of time is not acceptable).

(b) $v_{yf} = 10 - 10 \times (1 + \sqrt{10}) = -10\sqrt{10}$ m s^{-1}

$$v_f = \sqrt{v_{yf}^2 + v_{xf}^2} = \sqrt{(10\sqrt{10})^2 + (10\sqrt{3})^2}$$
$$= 10\sqrt{13} \text{ m s}^{-1}$$



PROJECTILE FROM A MOVING FRAME

Consider a boy who throws a ball from a moving trolley. Let the velocity of ball relative to boy is u.

$$\vec{V}_{\text{ball, trolley}} = \vec{V}_{\text{ball}} - \vec{V}_{\text{trolley}} \Rightarrow \vec{V}_{\text{ball}} = \vec{V}_{\text{ball, trolley}} + \vec{V}_{\text{trolley}}$$

The above equation shows that absolute velocity of ball is vector sum of its velocity with respect to trolley and velocity of trolley.

Table: Horizontal and vertical component of ball's velocity w.r.t. observer B standing on the ground in four cases

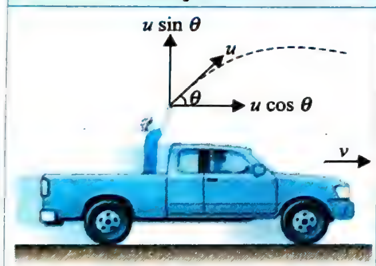
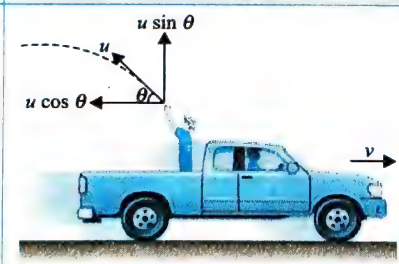
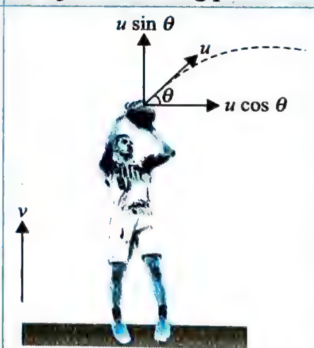
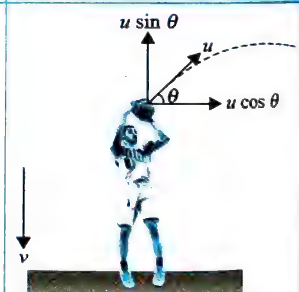
Case (1) Ball is projected in direction of motion of trolley	**Case (2)** Ball is projected opposite to the direction of motion of trolley	**Case (3)** Ball projected upwards from an upward moving platform	**Case (4)** Ball projected upwards from a downward moving platform
			
Horizontal component of ball's velocity $u\cos\theta + v$ Vertical component of ball's velocity $= u\sin\theta$	Horizontal component of ball's velocity $= u\cos\theta - v$ Vertical component of ball's velocity $= u\sin\theta$	Horizontal component of ball's velocity $= u\cos\theta$ Vertical component of ball's velocity $= u\sin\theta + v$	Horizontal component of ball's velocity $= u\cos\theta$ Vertical component of ball's velocity $= u\sin\theta - v$

ILLUSTRATION 5.26

A boy travelling in an open car moving on a levelled road with constant speed tosses a ball vertically up in the air and catches it back. Sketch the motion of the ball as observed by a boy standing on the footpath. Give explanation to support your diagram.

Sol. To the boy standing on the footpath, it will appear as if the ball has been thrown with some velocity at some inclination with the horizontal. Therefore, to him, the ball will appear to move along a parabolic path as shown in figure.

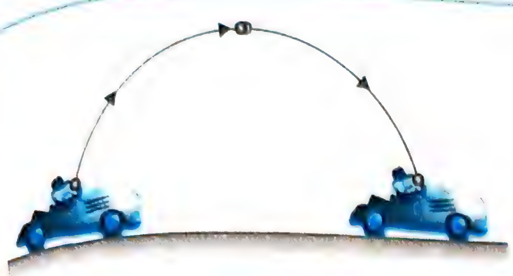


ILLUSTRATION 5.27

A boy throws a ball in air at 60° to the horizontal along a road with a speed of 10 ms^{-1}. Another boy sitting in a passing car observes the ball. Sketch the motion of the ball as observed by the boy in the car, if the car has a speed of 5 ms^{-1}. Give explanation to support your diagram.

Sol. The speed of the ball along horizontal,

$u_x = 10\cos 60° = 10 \times 0.5 = 5\ ms^{-1}$

The speed of the passing car, $v = 5\ ms^{-1}$

The ball thrown by the stationary boy moves along a parabolic path. Since $v = u_x = 5\ ms^{-1}$, to the boy sitting in a passing by car, the ball will appear to move vertically upwards and then vertically downwards as shown in figure.

ILLUSTRATION 5.28

A boy of height 1.5 m, making move on a skateboard due east with velocity 4 m s^{-1}, throws a coin vertically up with a velocity of 3 m s^{-1} relative to himself.

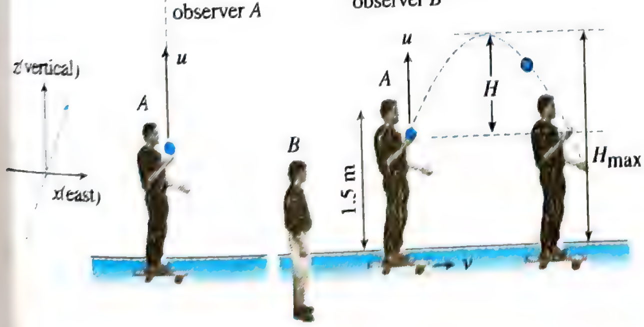
(a) Find the total displacement of the coin relative to ground till it comes to the hand of the boy.

(b) What is the maximum height attained by the coin w.r.t to ground?

Sol. The path of the coin as seen from boy A is a straight line (up and down), but that seen from ground observer B is parabola.

Velocity of coin, $\vec{v}_{\text{coin}} = \vec{v}_{\text{coin, boy}} + \vec{v}_{\text{boy,coin}} = 4\hat{i} + 3\hat{k}\ (\text{m s}^{-1})$

Acceleration of coin, $\vec{a}_{\text{coin}} = -10\hat{k}\ (\text{m s}^{-2})$



The displacement of coin in z-direction will be zero.

Using $\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$ in vertical direction (z-direction),

$$0 = 3(T) - \frac{1}{2} \times 10 \times (T)^2 \qquad (T = \text{Time of flight of coin})$$

or $$0 = T(3 - 5T) \Rightarrow T = \frac{3}{5}\text{s}$$

The displacement of the coin will be only x-direction.

Let the displacement in x-direction be x. Then

$$x = v_x \times T = 4 \times \frac{3}{5} = \frac{12}{5}\text{ m}$$

At maximum height, the final component of coin velocity in vertical direction will be zero.

Using $v^2 = u^2 + 2as$ in vertical direction,

$$0 = (3)^2 - 2.10.H \Rightarrow H = \frac{9}{20}\text{ m}$$

Hence, maximum height reached by coin

$$H_{\max} = \text{Height of boy} + H = 1.5 + \frac{9}{20} = \frac{39}{20}\text{ m}$$

(a) Hence the displacement of coin will be in horizontal direction and will be equal to 12/5 m.

(b) The maximum height attained by coin will be 39/20 m.

ILLUSTRATION 5.29

A shell is projected from a gun with a muzzle velocity, u. The gun is fitted with a trolley car at an angle θ as shown in figure. If the trolley car is made to move with constant velocity v towards right, find the

(a) horizontal range of the shell relative to ground.

(b) horizontal range of the shell relative to a person travelling with trolley.

Sol. The shell will follow parabolic path as seen from the observer travelling with trolley as well as seen from the observer on ground.

(a) The velocity of projection of the shell is $\vec{v}_s = \vec{v}_{sc} + \vec{v}_c$.

Substituting $\vec{u}_{sc} = u\cos\theta\hat{i} + u\sin\theta\hat{j}$ and $\vec{v}_c = v\hat{i}$, we have

$\vec{u}_s = (u\cos\theta + v)\hat{i} + u\sin\theta\hat{j}$

For horizontal range R of the shell, its displacement in horizontal direction can be given as $\vec{s} = R\hat{i}$, $\vec{a} = -g\hat{j}$ and $\vec{u}_s = (u\cos\theta + v)\hat{i} + u\sin\theta\hat{j}$

Using $\vec{s} = \vec{u}t + \frac{1}{2}\vec{a}t^2$, we have

$$R\hat{i} = (u\cos\theta + v)t\hat{i} + \left(ut\sin\theta - \frac{1}{2}gt^2\right)\hat{j}$$

Comparing the coefficient of $\hat{i}$ and $\hat{j}$, we obtain

$$R = (u\cos\theta + v)t \qquad \text{...(i)}$$

and $$ut\sin\theta - \frac{1}{2}gt^2 = 0 \qquad \text{...(ii)}$$

From (ii), we find $t = \dfrac{2u\sin\theta}{g}$

Or we would have directly used the formula for the time of flight directly to get same result.

Finally, substituting $t = \frac{2u\sin\theta}{g}$ in (i), we have

$$R = \frac{2u\sin\theta(u\cos\theta + v)}{g}$$

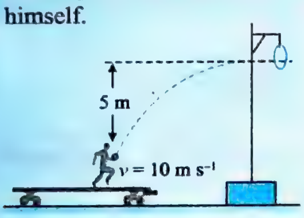
(b) The horizontal component of the initial velocity of the shell, as seen from trolley, $u_x = u\cos\theta$.
The vertical component, $u_y = u\sin\theta$.
The time of flight of shell will be same for both observers at trolley as well as ground.
Hence, range as seen from trolley is equal to the horizontal displacement in x direction w.r.t trolley.

$$R' = (u\cos\theta)t = u\cos\theta \times \left(\frac{2u\sin\theta}{g}\right)$$

$$= \frac{2u^2\sin\theta\cdot\cos\theta}{g} = \frac{u^2\sin 2\theta}{g}$$

ILLUSTRATION 5.30

A man is standing on a rail road car travelling with a constant speed of v = 10 ms^{-1} (figure). He wishes to throw a ball through a stationary hoop 5 m above the height of his hands in such a manner that the ball will move horizontally as it passes through the hoop. He throws the ball with a speed of 12.5 ms^{-1} w.r.t. himself.



(a) What must be the vertical component of the initial velocity of the ball?
(b) How many seconds after he releases the ball will it pass through the hoop?
(c) At what horizontal distance in front of the loop must he release the ball?

Sol. The important aspects to be noticed in this problem are:

- The velocity of the projection of ball is relative to man in motion.
- The ball clears the hoop when it is at the topmost point.

$$\vec{v}_{\text{ball,man}} = \vec{v}_{\text{ball}} - \vec{v}_{\text{man}}$$

$$\vec{v}_{\text{ball}} = \vec{v}_{\text{ball,man}} + \vec{v}_{\text{man}}$$

(a) Now we apply the above relation to x- as well as y-component of velocity. If the ball is projected with velocity v_0 and angle θ, then

x-component of $\vec{v}_{\text{ball}} = (v_0\cos\theta + 10)\,\text{ms}^{-1} = (12.5\cos\theta + 1)\,\text{ms}^{-1}$

y-component of $\vec{v}_{\text{ball}} = (v_0\sin\theta)\,\text{ms}^{-1} = 12.5\sin\theta\ \text{ms}^{-1}$

(b) Since the vertical component of the ball's velocity is unaffected by the horizontal motion of car, we can use the formula for the time of flight.

$$\frac{(12.5\sin\theta)^2}{2g} = 5\,\text{m}$$

$$\sin^2\theta = \frac{5\times(2\times10)}{12.5\times12.5}$$

$$\sin\theta = \frac{4}{5} \quad \text{and} \quad \cos\theta = \frac{3}{5}$$

$$v_0\sin\theta = (12.5)\times\left(\frac{4}{5}\right) = 10\ \text{m s}^{-1}$$

Time taken to reach the maximum height,

$$\frac{v_0\sin\theta}{g} = \frac{10}{10} = 1\,\text{s}$$

(c) The horizontal distance of loop from the point of projection $= (12.5\cos\theta + 10)\times 1 = 17.5$ m

ILLUSTRATION 5.31

A man is sitting in an open car which is travelling along a road at a speed of 30 m s^{-1}. The man stands up and throws a ball at a speed of 15 m s^{-1} relative to himself at an angle θ to the horizontal. The moment the man throws the ball, the driver of the car begins to accelerate at a constant rate of $10\sqrt{3}$ m s^{-2}, in the direction in which it was initially travelling.

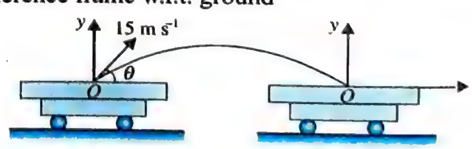
(a) Find the value of θ if the man in the car catches the ball.
(b) Find the time at which the man catches the ball if the instant at which the ball was thrown is $t = 0$.

Assume that the time at which the ball travels contains the line of the motion of the car, and that g = 10 m s^{-2}.

Sol. You can attempt the problem in a reference frame fixed to the car or in the reference frame fixed to the ground. Fundamentally, these two reference frames are equivalent in all respects, but in the reference frame fixed to the car, the calculations are slightly easier.

Analysis of the motion of the ball in the reference frame fixed to the car: Let x-axis of the coordinate system be along the motion of the car, and y-axis along the vertical direction.
Acceleration of the ball in the reference frame,

$\vec{a}_{BR}$ = Acceleration of the ball w.r.t. ground − Acceleration of the reference frame w.r.t. ground



In time interval $t = 0$ to $t = t$, $x = 0$, $y = 0$

$$\vec{a}_{BR} = (-g\hat{j}) - (a\hat{i}) = -a\hat{i} - g\hat{j}$$

$$\Rightarrow \quad (a_{BR})_x = -a,\ (a_{BR})_y = -g$$

Let u be the velocity of the ball at $t = 0$. Then $u_x = u\cos\theta$, $u_y = u\sin\theta$.

Using the equation, $s = ut + \frac{1}{2}at^2$, in both x- and y-directions,

$$0 = (u \cos \theta) t + \frac{1}{2}(-a)t^2 \quad \text{...(i)}$$

At $t = t$, $x = 0$, $y = 0$

and $$0 = (u \sin \theta) t + \frac{1}{2}(-g)t^2 \quad \text{...(ii)}$$

The man catches the ball.

From these equations, either $t = 0$, which refers to the initial condition, or $t = \dfrac{2u \sin \theta}{g}$ and $t = \dfrac{2u \cos \theta}{a}$.

$$\frac{2u \sin \theta}{g} = \frac{2u \cos \theta}{a}$$

$$\tan \theta = \frac{g}{a} = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \quad \theta = 30°$

$$t = \frac{2u \sin \theta}{g} = \frac{2 \times 15 \times \frac{1}{2}}{10} = 1.5 \text{ s}$$

Analysis of the motion of the ball in the ground reference frame: Consider a coordinate system fixed to the ground with x-axis along the direction of motion of the car and y-axis in vertically upward direction.

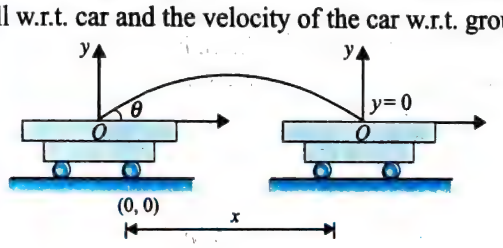
u = velocity of the ball relative to the man initially
 $= 15 \text{ ms}^{-1}$ (making an angle θ with the horizontal)

v = velocity of the car initially $= 30 \text{ ms}^{-1}$, in horizontal direction.

t = time at which the man catches the ball

a = constant acceleration of the car (between $t = 0$ and $t = t$)
 $= 10\sqrt{3} \text{ ms}^{-2}$.

Velocity of the ball w.r.t. ground is the vector sum of the velocity of the ball w.r.t. car and the velocity of the car w.r.t. ground.



$$\vec{V}_{BG} = \vec{V}_{BC} + \vec{V}_{CG}$$
$$= (u \cos \theta \hat{i} + u \sin \theta \hat{j}) + (v\hat{i})$$
$$= (u \cos \theta + v)\hat{i} + (u \sin \theta)\hat{j}$$

$$(V_x)_{t=0} = (u \cos \theta + v), \; (v_y)_{t=0} = u \sin \theta$$

Vertical displacement of the ball in time t must equal zero.

$$(u \sin \theta)t + 1/2(-g)t^2 = 0 \quad (a_y = -g \text{ and } t \neq 0)$$

$$t = \frac{2u \sin \theta}{g} \quad \text{...(i)}$$

The horizontal distance travelled by the car in time t must equal the horizontal distance travelled by the ball in the same time.

$$(u \cos \theta + v)t = vt + \frac{1}{2}at^2$$

$$t = \frac{2u \cos \theta}{a} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\dfrac{2u \sin \theta}{g} = \dfrac{2u \cos \theta}{a}$

$$\tan \theta = \frac{g}{a} = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$

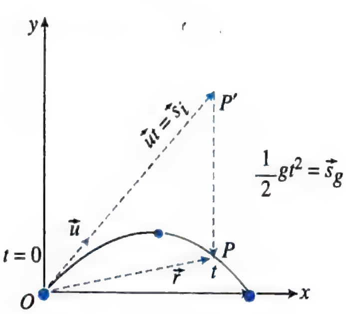
$\theta = 30°$

$$t = \frac{2u \sin \theta}{g} = \frac{2 \times 15 \times \frac{1}{2}}{10} = 1.5 \text{ s}$$

Physical significance of the formula $\vec{s} = \vec{u}t + \frac{1}{2}\vec{g}t^2$ in projectile motion: The expression for the position vector of the projectile as a function of time, with its acceleration being that due to gravity, is $\vec{a} = \vec{g} \Rightarrow \vec{r}_f = \vec{v}_i t + \frac{1}{2}\vec{g}t^2$

where the initial x and y components of the velocity of the projectile are $v_{xi} = v_i \cos \theta$ and $v_{yi} = v_i \sin \theta$, respectively.

Motion can be explained in the following way: If we put $g = 0$ in the above formula, only the first term $\vec{u}t$ will remain. It is then evident that if there were no gravity the body would have undergone a displacement $\vec{u}t$ in a straight line OP' due to the inertia of its motion during time t. For the sake of simplicity, let us call it *inertial displacement* $\vec{s}_i$.



Now, look at the second term $\frac{1}{2}\vec{g}t^2$. Only this term will remain when $v_0 = 0$. This means when the body is just dropped from rest, the gravity pull it down through a vertical displacement $\frac{1}{2}\vec{g}t^2$. You can call it as *gravitational displacement* $\vec{s}_g$ because it happens due to gravity.

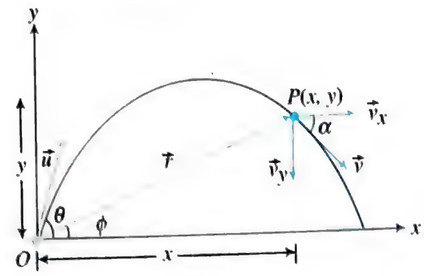
When the particle moves in gravity, it will experience both inertial and gravitational displacements simultaneously. Hence, the net displacement $\vec{s}$ of the particle is given as the vector sum of the inertial displacement $\vec{s}_i$ and gravitational displacement $\vec{s}_g$.

That means, $\vec{s} = \vec{s}_i + \vec{s}_g = \vec{u}t + \frac{1}{2}\vec{g}t^2$

Motion of a projectile as seen from another projectile: The position of projectile 1 relative to projectile 2 is, where $\vec{r}_1 = \vec{v}_1 t + \frac{1}{2}\vec{g}t^2$ and $\vec{r}_2 = \vec{v}_2 t + \frac{1}{2}\vec{g}t^2$. This gives $\vec{r}_{12} = (\vec{v}_1 - \vec{v}_2)t$, where $\vec{v}_1$ and $\vec{v}_2$ are the initial velocities of the projectiles. Hence, projectile 1 seems to move relative to projectile 2 in a straight line.

Displacement of projectile $\vec{r}$: Let the particle acquires a position P having the coordinates (x, y) just after time t from the instant

of projection. The corresponding position vector of the particle at time t is $\vec{r}$ as shown in figure



$$\vec{u} = u\cos\theta\hat{i} + u\sin\theta\hat{j}$$

$$\vec{a} = -g\,\hat{j}$$

Using $\vec{r} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

$$= (u\cos\theta)t\,\hat{i} + \left((u\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}$$

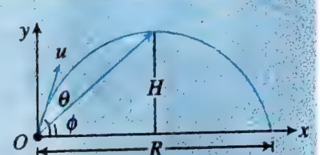
$$= \sqrt{(ut\cos\theta)^2 + \left((ut\sin\theta) - \frac{1}{2}gt^2\right)^2}$$

$$= ut\sqrt{1 + \left(\frac{gt}{2u}\right)^2 - \frac{gt\sin\theta}{u}}$$

and $\phi = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{ut\sin\theta - 1/2gt^2}{(ut\cos\theta)}\right)$

or $\phi = \tan^{-1}\left(\frac{2u\sin\theta - gt}{2u\cos\theta}\right)$

Note: The angle of elevation ϕ of the highest point of the projectile and the angle of projection θ are related to each other as $\tan\phi = \frac{1}{2}\tan\theta$



Application of $\vec{v} = \vec{u} + \vec{a}t$ in projectile motion: In projectile motion, the vertical component of velocity changes but the horizontal component of velocity remains always constant.

Let $\vec{v}$ be the instantaneous velocity of projectile at time t direction of this velocity is along the tangent to the trajectory at point P.

$$\vec{v} = \vec{u} + \vec{a}t = (u\cos\theta\hat{i} + u\sin\theta\hat{j}) + (-g\,\hat{j})t$$

$$= u\cos\theta\hat{i} + (u\cos\theta - gt)\,\hat{j}$$

$$|\vec{v}| = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2} = \sqrt{u^2 + g^2t^2 - 2ugt\sin\theta}$$

Direction of instantaneous velocity

$$\tan\alpha = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta} \text{ or } \alpha = \tan^{-1}\left[\tan\theta - \frac{gt}{u}\sec\theta\right]$$

Change in velocity: Initial velocity (at projection point), $\vec{u} = u\cos\theta\hat{i} + u\sin\theta\hat{j}$

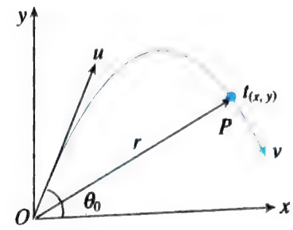
Final velocity (at highest point), $\vec{v}_h = u\cos\theta\hat{i} + 0\,\hat{j}$

Change in velocity (Between projection point and highest point), $\Delta\vec{v} = \vec{v}_h - \vec{u} = -u\sin\theta\hat{j}$

When the body reaches the ground after completing its motion then final velocity, $\vec{v}_f = u\cos\theta\hat{i} - u\sin\theta\hat{j}$

Change in velocity (Between complete projectile motion), $\Delta u = u_f - u_i = -2u\sin\theta\hat{j}$

Average velocity: The average velocity is $\vec{v}_{av} = \frac{\vec{r}}{t}$



Substituting $\vec{r} = \vec{u}t + \frac{1}{2}\vec{g}t^2$, we have $\vec{v}_{av} = \dfrac{\vec{u}t + \frac{1}{2}\vec{g}t^2}{t}$

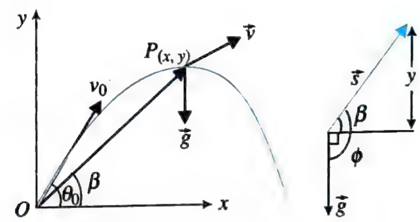
This gives $\vec{v}_{av} = \vec{u} + \frac{1}{2}\vec{g}t$

Application of $v^2 - u^2 = 2\vec{a} \times \vec{s}$ in projectile motion: Let us consider an upward position of a projectile possessed by a point $P(x, y)$. Then $\vec{s} = x\hat{i} + y\hat{j}$

Now substituting $\vec{s} = x\hat{i} + y\hat{j}$, $u = v_0$, and $\vec{a} = -g\,\hat{j}$ in the formula $v^2 - u^2 = 2\vec{a}\cdot\vec{s}$, we have

$$v^2 - v_0^2 = 2(-g\hat{j})\cdot(x\hat{i} + y\hat{j}) = -2gy$$

This yields $v^2 = v_0^2 - 2gy$



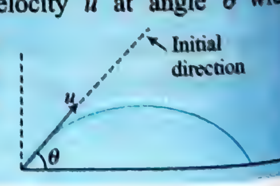
Similarly, for any downward position $P(x, y)$, we can prove that

$$v^2 = v_0^2 + 2gy$$

$\vec{a}.\vec{s} = \vec{g}.\vec{s} = gs\cos\phi = gs\cos(90° + \beta) = gs\sin\beta = gy$ (because $s\sin\beta = y$)

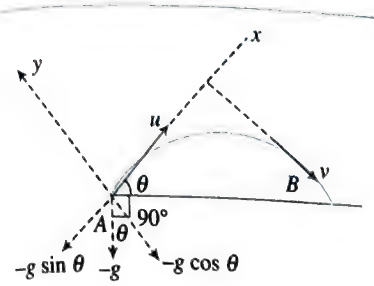
ILLUSTRATION 5.32

A particle is projected with relocity u at angle θ with horizontal. Calculate the time when it is moving perpendicular to initial direction. Also calculate the velocity at this position.



Method 1: Using properties of projectile motion

As we have to calculate the time between two positions A and B where the final direction of movement is perpendicular to the initial direction of movement. So for our own comfortability, we can choose the initial direction of motion as x-axis. Also let us assume the velocity at position B to be v.



Now analyzing motion in x- and y-direction, we have

$u_x = u;\quad u_y = 0$

$a_x = -g \sin \theta;\ a_y = -g \cos \theta$

Here we can use the following formula $v = u + at$ in x-direction. As we have the values of initial velocity, final velocity, and acceleration we can find t. Therefore,

$v_x = u_x + a_x t$

At position B, $v_x = 0$, as the final velocity is equal to the y-component of velocity. Therefore,

$0 = u - g \sin \theta \cdot t$

Thus, $t = \dfrac{u}{g\sin\theta}$ which is the required time to travel.

Mehtod 2: Using vectors

As u and v both are perpendicular to each other. We can use the orthogonality property of dot product, i.e., if two vectors are perpendicular to each other their dot product is zero, in order to find out the time of travel to the desired position. So,

So. $\vec{u}\cdot\vec{v} = 0 \Rightarrow \vec{u}\cdot(\vec{u} + \vec{a}t) = 0 \Rightarrow \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{a}t = 0$

$\Rightarrow\quad u^2 + ug\cos(90° + \theta)\, t = 0$

[Because angle between u and g is $90° + \theta$, as from the figure above]

$u^2 + u \cdot g\,(-\sin\theta)\, t = 0 \cdot t = 0$

So, $t = \dfrac{u}{g\sin\theta}$ is the desired time.

To find out the velocity we can use the same relation as used in this question. But as at final position (considered) only the y-component of velocity is present, so we need to use the same relation in y-direction.

$v_y = u_y + a_g t$

$\Rightarrow\quad v = 0 - g \cdot \cos \theta t$

$= -g\cos\theta\dfrac{u}{g\sin\theta} = -u\cot\theta$ is the velocity at position B.

Method 3: If the initial velocity u and velocity at time t are perpendicular, then the final velocity will be at an angle θ with the vertical.

The horizontal component of velocity is unchanged throughout the motion. Therefore,

$u \cos \theta = v \sin \theta$

or $\quad v = u \cot \theta$

The vertical component of velocity after time $t = -v \cos \theta$.

From the equation, $v_y = u \sin \theta - gt - v\cos\theta = u \sin \theta - gt$

$$t = \frac{u\sin\theta + v\cos\theta}{g} = \frac{u\sin\theta + u\cot\theta\cos\theta}{g}$$

$$= \frac{u}{g}\left[\frac{\sin^2\theta + \cos^2\theta}{\sin\theta}\right] = \frac{u}{g}\operatorname{cosec}\theta$$

Method 4: The slope of trajectory at the point of projection, $m_1 = \tan \theta$.

Slope of trajectory after time t,

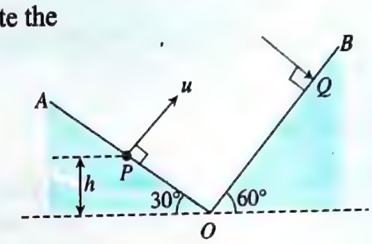
$$m_2 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_y}{v_x} = \frac{u\sin\theta - gt}{u\cos\theta}$$

Slopes are perpendicular, $\left(\dfrac{u\sin\theta - gt}{u\cos\theta}\right)(\tan\theta) = -1$

or $\quad t = \dfrac{u}{g\sin\theta}$

ILLUSTRATION 5.33

Two inclined planes OA and OB having inclination (with horizontal) 30° and 60°, respectively, intersect each other at O as shown in the figure. A particle is projected from point P with velocity $u = 10\sqrt{3}\ \text{m s}^{-1}$ along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicularly at Q, calculate the



(a) velocity with which particle strikes the plane OB.

(b) time of flight.

(c) vertical height h of P from O.

(d) maximum height from O, attained by the particle.

(e) distance PQ.

Sol. Two given planes are mutually perpendicular and the particle is projected perpendicularly from plane OA. It means $\vec{u}$ is parallel to plane OB.

At the instant of collision of the particle with OB, its velocity is perpendicular to OB or the velocity component parallel to OB is zero. For considering the motion of particle parallel to plane OB,

$\vec{u} = 10\sqrt{3}\ \text{m s}^{-1}$

Acceleration $= -g \sin 60° = -5\sqrt{3}\ \text{m s}^{-2}$

$u = 0,\ t = ?\ S = ?$

Using $v = u + at$, $t = 2$ s

$s = ut + \frac{1}{2}at^2$ or $OQ = 10\sqrt{3}$ m

Now considering the motion of the particle normal to plane OB, Initial velocity = 0, acceleration = $g \cos 60° = 5\ \text{ms}^{-2}$

$T = 2$ s, $v = ?$, $s = PO = ?$

Using $v = u + at$, $v = 10\ \text{ms}^{-1}$

$s = ut + \frac{1}{2}at^2$ or $OP = 10$ m

$h = PO \sin 30° = 10 \times \sin 30° = 5$ m

The inclination of $\vec{u}$ with the vertical is 30°. Therefore, its vertical component is $u \cos 30° = 15\ \text{ms}^{-1}$ (upward).

Considering vertically upward motion of the particle from P, initial velocity = $15\ \text{ms}^{-1}$, acceleration = $g - 10\ \text{ms}^{-2}$, $v = 0$, $S = H = ?$

Using, $v^2 = u^2 + 2as$, $H = 11.25$ m

Therefore, maximum height reached by the particle above

$$O = h + H = 16.25 \text{ m}$$

$$\text{Distance, } PQ = \sqrt{(PO)^2 + (OQ)^2} = \sqrt{(10)^2 + (10\sqrt{3})^2} = 20 \text{ m}$$

ILLUSTRATION 5.34

A truck starts from origin, accelerating with a m s^{-2} in positive x-axis direction. After 2 s, a man standing at the starting point of the truck projected a ball at an angle 30° with velocity v m s^{-1}. Find the relation between a and v such that the ball hits the truck. (Assume that truck is moving on horizontal plane and the man projected the ball from the same horizontal level of truck).

Sol. If the ball hits the truck, for the horizontal motion of truck and ball,

Displacement of ball in horizontal = Displacement of truck in horizontal direction

$$v\cos 30° \times (t-2) = \frac{1}{2}at^2$$

For the vertical motion of ball, the vertical displacement should be zero

$$v\sin 30° \times (t-2) - \frac{1}{2}g(t-2)^2 = 0$$

$$\Rightarrow \quad (t-2) = \frac{2v\sin 30}{g} = \frac{v}{g} \Rightarrow t = 2 + \frac{v}{g}$$

$$\frac{\sqrt{3}\,v}{2} \times \frac{v}{g} = \frac{1}{2}a\left(2 + \frac{v}{g}\right)^2$$

$$\frac{\sqrt{3}\,v^2}{g} = a\left(2 + \frac{v}{g}\right)^2 \Rightarrow a = \frac{\sqrt{3}v^2}{g\left(2 + \frac{v}{g}\right)^2}$$

ILLUSTRATION 5.35

A body is thrown at an angle θ_0 with the horizontal such that it attains a speed equal to $\sqrt{\frac{2}{3}}$ times the speed of projection when the body is at half of its maximum height. Find the angle θ_0.

Sol. At any height y, the speed of the projectile is

$$v = \sqrt{v_0^2 - 2gy}, \text{ where } y = \frac{y_{max}}{2}$$

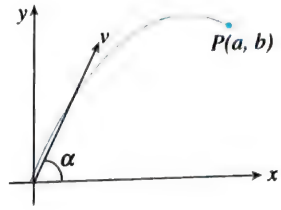
$$\text{Substituting } y_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}, \text{ we have } v = v_0\left(\sqrt{1 - \frac{\sin^2\theta_0}{2}}\right)$$

$$\text{Since } v = \sqrt{\frac{2}{3}}v_0 \text{ (given), we have } 1 - \frac{\sin^2\theta_0}{2} = \frac{2}{3}$$

$$\text{This gives } \sin\theta_0 = \sqrt{\frac{2}{3}}. \text{ Hence, } \theta_0 = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right).$$

MINIMUM VELOCITY AND ANGLE TO HIT A GIVEN POINT

Let us project the particle with a speed v at an angle α such that it passes through a point $P(a, b)$. We have to find the minimum speed. To minimize the speed v, we have to adjust the angle of projection α.



Equation of the trajectory of a particle is $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

If the projectile passes through (a, b), then

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha} = a\tan\alpha - \frac{ga^2}{2u^2}(1 + \tan^2\alpha)$$

$$ga^2\tan^2\alpha - 2au^2\tan\alpha + (ga^2 + 2bu^2) = 0 \quad \text{...(i)}$$

This quadratic equation in $\tan\alpha$ must give real roots for a particle to pass through (a, b). Thus,

Discriminant ≥ 0

$$4a^2u^4 - 4ga^2(ga^2 + 2bu^2) \geq 0$$

$$u^4 - 2gbu^2 - g^2a^2 \geq 0$$

$$u^4 - 2gbu^2 + b^2g^2 \geq b^2g^2 + a^2g^2$$

$$(u^2 - bg)^2 \geq (b^2 + a^2)g^2$$

$$u \geq \sqrt{bg + g\sqrt{a^2 + b^2}}$$

Hence, the minimum speed of projection is

$$u = \sqrt{g(b + \sqrt{a^2 + b^2})} \quad \text{...(ii)}$$

Equation (i) is the quadratic equation in $\tan\alpha$. We can find the value of $\tan\alpha$ corresponding to this velocity.

Substituting the value of u [from (ii)] in (i), we can find

$$\tan\alpha = \frac{u^2}{ga}$$

ILLUSTRATION 5.36

A mango in a tree is located at a horizontal and vertical distance of 30 m and 40 m, respectively, from the point of projection of a stone. Find the minimum speed and the critical angle of projection of the stone so as to hit the mango.

Sol. Here the coordinate of P(30 m, 40 m).

As we know the minimum velocity required to hit the mango is

$$v > \sqrt{g(y + \sqrt{x^2 + y^2})}$$

$$> \sqrt{10\left(40+\sqrt{30^2+40^2}\right)}$$

$$> \sqrt{900} \Rightarrow v > 30 \text{ m s}^{-1}$$

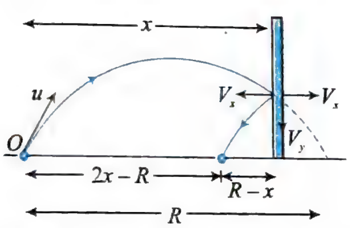
Also $\tan\theta = \dfrac{u^2}{gx} = \dfrac{(30)^2}{10 \times 30} = 3$

Hence, $\theta = \tan^{-1}(3)$

Elastic Collision of a Projectile with a Wall

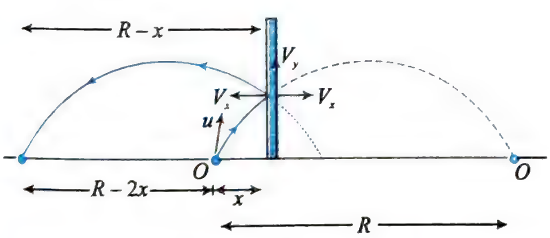
Let us consider a particle is projected with speed u at an angle θ from point O on the ground. The range of the projectile is R. If a vertical, smooth wall is present in the path of the projectile at a distance x from the point of projection, the collision of the projectile with the wall will be elastic i.e., the speed of the particle remains unchanged. Due to collision, the direction of x component of velocity is reversed but its magnitude remains the same and the y component of velocity remains unchanged. It means the remaining distance $(R - x)$ is covered in the backward direction and the projectile lands at a distance of $R - x$ in front of the wall. As the time of flight and maximum height depends only on y component of velocity, hence they do not change despite collision with the wall.

Case I: If $x \geq \dfrac{R}{2}$



Here distance of landing place of projectile from its point of projection is $2x - R$.

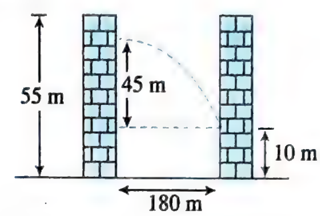
Case II: If $x < \dfrac{R}{2}$



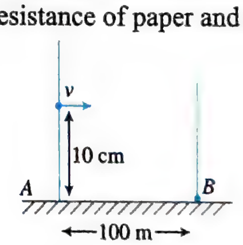
Here distance of landing place of projectile from its point of projection is $R - 2x$.

Concept Application Exercise 5.3

1. A ball rolls off top of a staircase with a horizontal velocity u m s^{-1}. If the steps are h metre high and b mere wide, the ball will just hit the edge of nth step. Find the value of n.
2. A body is projected horizontally from the top of a tower with initial velocity 18 m s^{-1}. It hits the ground at angle 45°. What is the vertical component of velocity when it strikes the ground?
3. A man standing on the roof of a house of height h throws one particle vertically downwards and another particle horizontally with the same velocity u. Find the ratio of their velocities when they reach the earth's surface.
4. A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m s^{-1}. Find the motorcycle's position, distance from the edge of the cliff and velocity after 0.5 s.
5. An object is thrown between two tall buildings 180 m from each other. The object thrown horizontally from a window 55 m above the ground from one building strikes a window 10 m above the ground in another building. Find out the speed of projection.



6. A fighter plane moving with a speed of $50\sqrt{2}$ m s^{-1} upward at an angle of 45° with the vertical releases a bomb when it was at a height 1000 m from ground. Find
   (a) The time of flight
   (b) The maximum height of the bomb above ground
7. Two paper screens A and B are separated by a distance of 100 m. A bullet pierces A and then B. The hole in B is 10 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting the screen A, calculate the velocity of the bullet when it hits the screen A. Neglect the resistance of paper and air.



8. Two stones A and B are projected simultaneously from the top of a 100-m high tower. Stone B is projected horizontally with speed 10 m s^{-1}, and stone A is dropped from the tower. Find out the following:
   (a) Time of flight of the two stone
   (b) Distance between two stones after 3 s
   (c) Angle of strike with ground
   (d) Horizontal range of particle B
9. Two guns are situated at the top of a hill firing the shots at the same speed. One gun fires the shot horizontally and other gun fires the shot at an angle of 60° with horizontal. The two shots collide in air. If the time taken by horizontal shot to reach the point of collision is 3 seconds, find the time interval between two shots.

10. A ball is thrown horizontally from the top of a tower and strikes the ground in 3 s at an angle of 30° with the vertical.
   (a) Find the height of the tower.
   (b) Find the speed with which the body was projected.

ANSWERS

1. $\dfrac{2hu^2}{gb^2}$ 2. 18 m s^{-1} 3. 1 : 1

4. Motorcycle is below its starting point; $\dfrac{\sqrt{349}}{4}$ m; $\sqrt{106}$ m s^{-1}

5. 60 m s^{-1}

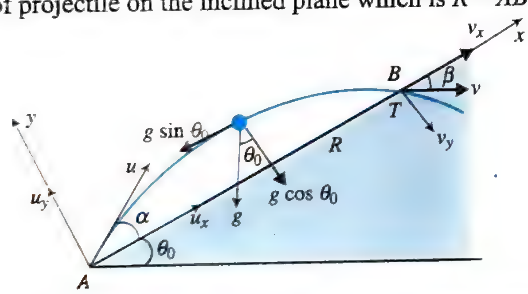
6. (a) 20 s (b) 1125 m 7. 700 m s^{-1}

8. (a) $2\sqrt{5}$ s (b) 45 m
   (c) $\tan^{-1}(2\sqrt{5})$ (d) $20\sqrt{5}$ m

9. 3 sec 10. 45 m, $10\sqrt{3}$ m s^{-1}

PROJECTILE MOTION ON AN INCLINED PLANE

Consider a plane is inclined at an angle θ_0 with horizontal. Let a projectile is projected from the foot of the inclined pane A, with velocity u and at an angle α with the inclined plane as shown in figure. The particle strikes the plane at point B. We want to find the range of projectile on the inclined plane which is $R = AB$.



Take x-axis parallel to the inclined plane and y-axis perpendicular to the inclined plane as shown in figure. We can write the following:

$$u_x = u \cos \alpha,\ u_y = u \sin \alpha,\ a_x = -g \sin \theta_0,\ a_y = -g \cos \theta_0$$

From A to B: $s_x = R$, $s_y = 0$ and let time taken from A to B is T.

From A to B, apply $s_y = u_y t + \frac{1}{2} a_y t^2$, we get

$$0 = (u \sin \alpha)T - \frac{1}{2} g \cos \theta_0 T^2 \Rightarrow T = \frac{2u \sin \alpha}{g \cos \theta_0} \quad \text{...(i)}$$

Now apply $s_x = u_x t + \frac{1}{2} a_x t^2$, we get

$$R = (u \cos \alpha)T - \frac{1}{2} g \sin \theta_0 T^2 \quad \text{...(ii)}$$

Put the value of T from (i) in (ii), we get

$$R = u \cos \theta \frac{2u \sin \alpha}{g \cos \theta_0} - \frac{1}{2} g \sin \theta_0 \left(\frac{2u \sin \alpha}{g \cos \theta_0}\right)^2$$

$$= \frac{2u^2 \sin \alpha \cos \alpha}{g \cos \theta_0} - \frac{2u^2 \sin^2 \alpha \sin \theta_0}{g \cos^2 \theta_0}$$

$$= \frac{u^2}{g \cos^2 \theta_0} [\sin 2\alpha \cos \theta_0 - \sin \theta_0 2 \sin^2 \alpha]$$

$$= \frac{u^2}{g \cos^2 \theta_0} [\sin 2\alpha \cos \theta_0 - \sin \theta_0 (1 - \cos 2\alpha)]$$

$$= \frac{u^2}{g \cos^2 \theta_0} [\sin(2\alpha + \theta_0) - \sin \theta_0] \quad \text{...(iii)}$$

The above equation (iii) gives the range of a projectile on an inclined plane.

Maximum range: We want to find angle α for which the range of the projectile is maximum on an inclined plane.

Range will be maximum if $\sin(2\alpha + \theta_0) = 1$

$$\Rightarrow \quad 2\alpha + \theta_0 = \pi/2 \Rightarrow \alpha = \frac{\pi}{4} - \frac{\theta_0}{2} \quad \text{...(iv)}$$

The above equation (iv) gives the angle α for which the range is maximum. Put the vale of α in (iii), we get

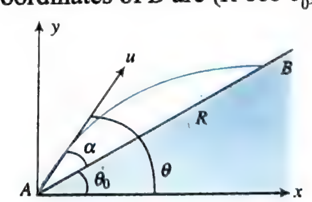
$$R_{\max} = \frac{u^2}{g \cos^2 \theta_0} \left[\sin\left[2\left(\frac{\pi}{4} - \frac{\theta_0}{2}\right) + \theta_0\right] - \sin \theta_0\right]$$

$$= \frac{u^2}{g \cos^2 \theta_0} [1 - \sin \theta_0]$$

$$= \frac{u^2 (1 - \sin \theta_0)}{g(1 - \sin^2 \theta_0)} = \frac{u^2}{g(1 + \sin \theta_0)} \quad \text{...(v)}$$

The above equation (v) gives the maximum range.

Alternate method to find range: Take x-and y-axis in horizontal and vertical directions, respectively. Angle of projection $\theta = \alpha + \theta_0$. The coordinates of B are $(R \cos \theta_0, R \sin \theta_0)$.



Apply the equation of trajectory: $y = x \tan \theta - \dfrac{gx^2}{2u^2 \cos^2 \theta}$

$$x = R \cos \theta_0 \text{ and } y = R \sin \theta_0$$

$$R \sin \theta_0 = R \cos \theta_0 \tan(\alpha + \theta_0) - \frac{g(R \cos \theta_0)^2}{2u^2 \cos^2(\alpha + \theta_0)}$$

$$\Rightarrow \quad \frac{Rg \cos^2 \theta_0}{2u^2 \cos^2(\alpha + \theta_0)} = \cos \theta_0 \tan(\alpha + \theta_0) - \sin \theta_0$$

$$= \frac{\cos \theta_0 \sin(\alpha + \theta_0) - \sin \theta_0 \cos(\alpha + \theta_0)}{\cos(\alpha + \theta_0)} = \frac{\sin \alpha}{\cos(\alpha + \theta_0)}$$

$$\Rightarrow \quad R = \frac{2u^2 \sin \alpha \cos(\alpha + \theta_0)}{g \cos^2 \theta_0}$$

$$= \frac{u^2}{g \cos^2 \theta_0} [2 \sin \alpha (\cos \alpha \cos \theta_0 - \sin \alpha \sin \theta_0)]$$

$$= \frac{u^2}{g\cos^2\theta_0}[\sin 2\alpha \cos\theta_0 - \sin\theta_0 2\sin^2\alpha]$$

$$= \frac{u^2}{g\cos^2\theta_0}[\sin 2\alpha \cos\theta_0 - \sin\theta_0 (1-\cos 2\alpha)]$$

$$= \frac{u^2}{g\cos^2\theta_0}[\sin(2\alpha+\theta_0) - \sin\theta_0]$$

which is same as equation (iii) obtained earlier.

Velocity at point *B*: In the given figure, from A to B:

Apply $v_x = u_x + a_x t$

$\Rightarrow$ $v\cos\beta = u\cos\alpha - g\sin\theta_0 \cdot T$

$$= u\cos\alpha - g\sin\theta_0 \cdot \frac{2u\sin\alpha}{g\cos\theta_0}$$

$$= u\cos\alpha(1 - 2\tan\alpha\tan\theta_0) \quad \text{...(vi)}$$

Apply $v_y = u_y + a_y t$

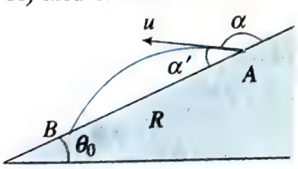
$$\Rightarrow \quad -v\sin\beta = u\sin\alpha - g\cos\theta_0 \cdot \frac{2u\sin\alpha}{g\cos\theta_0}$$

$$\Rightarrow \quad v\sin\beta = u\sin\alpha \quad \text{...(vii)}$$

Note:

1. From (vii), we find that the magnitude of velocity along y-axis is same at both the points A and B. This can also be concluded from the equation $v_y^2 = u_y^2 + 2a_y s_y$. Here $s_y = 0$ from A to B. So $v_y^2 = u_y^2 \Rightarrow |v_y| = |u_y| \Rightarrow v\sin\beta = u\sin\alpha$
2. From (vi), we find that the velocity component parallel to the inclined plane goes on decreasing.
3. From (vi) and (vii), $\tan\beta = \dfrac{\tan\alpha}{1-2\tan\alpha\tan\theta_0}$
   (a) If $\beta < 90°$, then $1 - 2\tan\alpha\tan\theta_0 > 0$ and in this condition $\tan\beta > \tan\alpha \Rightarrow \beta > \alpha$
   (b) If $\beta > 90°$, then $1 - 2\tan\alpha\tan\theta_0 < 0$ and in this case also, $\beta > \alpha$.

If the projectile were projected down the inclined plane: Replacing R with $(-R)$ and α with $180° - \alpha'$ in (iii), we get



$$\Rightarrow \quad -R = \frac{u^2}{g\cos^2\theta_0}[\sin\{2(180° - \alpha') + \theta_0\} - \sin\theta_0]$$

$$= \frac{u^2}{g\cos^2\theta_0}[\sin\{360° - (2\alpha' - \theta_0)\} - \sin\theta_0]$$

$$= \frac{u^2}{g\cos^2\theta_0}[-\sin(2\alpha' - \theta_0) - \sin\theta_0]$$

$$= \frac{u^2}{g\cos^2\theta_0}[\sin(2\alpha' - \theta_0) + \sin\theta_0] \quad \text{...(viii)}$$

The above equation (viii) gives the range down the inclined plane.

For maximum range: $2\alpha' - \theta_0 = \dfrac{\pi}{2} \Rightarrow \alpha' = \dfrac{\pi}{4} + \dfrac{\theta_0}{2}$

and $$R_{max} = \frac{u^2}{g(1-\sin\theta_0)} \quad \text{...(ix)}$$

Important Points:

As the projectile accelerates down perpendicular to the inclined plane with an acceleration $g\cos\beta$, its speed perpendicular to the inclined plane decreases from $v_0\sin\phi$ at O to zero at P and gradually increases to $v_0\sin\phi$ at Q just before striking.

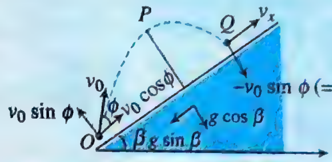
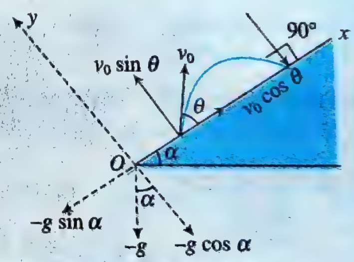


ILLUSTRATION 5.37

At what angle should a ball be projected up an inclined plane with a velocity v_0 so that it may hit the incline normally. The angle of the inclined plane with the horizontal is α.



Sol. As the ball has to hit the inclined plane normally, in that position the x-component of velocity will be zero and the velocity will have y-component only.

The ball will hit the incline normally if its parallel component of velocity reduces to zero during the time of flight.

By analyzing this motion along incline, i.e., x-direction,

$$v_x = u_x + a_x t$$

Here $v_x = 0$, $u_x = v_0\cos\theta$, $a_x = -g\sin\alpha$

$$0 = v_0\cos\theta - (g\sin\alpha)T \Rightarrow \quad = \frac{v_0\cos\theta}{g\sin\alpha} \quad \text{...(i)}$$

Also the displacement of the particle in y-direction will be zero.

Using $y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow 0 = v_0\sin\theta\, T - \frac{1}{2}g\cos\alpha\, T^2$

This gives $$T = \frac{2v_0\sin\theta}{g\cos\alpha} \quad \text{...(ii)}$$

From (i) and (ii), we have

$$\frac{v_0\cos\theta}{g\sin\alpha} = \frac{2v_0\sin\theta}{g\cos\alpha} \Rightarrow \frac{\cos\theta}{\sin\alpha} = \frac{2\sin\theta}{\cos\alpha}$$

$$2\tan\theta\tan\alpha = 1 \Rightarrow \tan\theta = \left[\frac{1}{2}\cot\alpha\right]$$

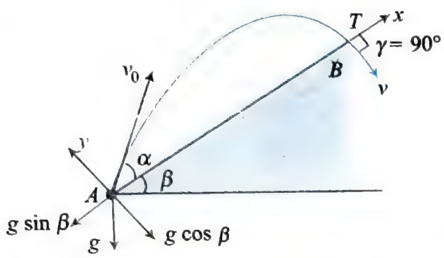
$$\theta = \tan^{-1}\left(\frac{1}{2}\cot\alpha\right)$$

which is the required angle of projection.

ILLUSTRATION 5.38

A body is projected up with a speed v_0 along the line of greatest slope of an inclined plane of angle of inclination β. If the body collides elastically perpendicular to the inclined plane, find the time after which the body passes through its point of projection.

Sol. On striking the plane at B, the velocity along the x-axis should be zero. So using $v_x = u_x + a_x t$, we get



$0 = v_0\cos\alpha - g\sin\beta\, T$, where T is the time of flight

$$T = \frac{v_0\cos\alpha}{g\sin\beta} \quad \text{...(i)}$$

Also $S_y = 0$ from A to B. So using $S_y = u_y t + \frac{1}{2}a_y t^2$, we get

$$0 = (v_0\sin\alpha)T - \frac{1}{2}(g\cos\beta)T^2 \Rightarrow T = \frac{2v_0\sin\alpha}{g\cos\beta} \quad \text{...(ii)}$$

From (i) and (ii), $\dfrac{v_0\cos\alpha}{g\cos\beta} = \dfrac{2v_0\sin\alpha}{g\cos\beta} \Rightarrow \tan\alpha = \dfrac{1}{2}\cot\beta$

$$\Rightarrow \quad \cos\alpha = \frac{2}{\sqrt{4+\cot^2\beta}} = \frac{2\sin\beta}{\sqrt{1+3\sin^2\beta}}$$

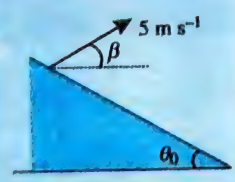
Put the value of $\cos\alpha$ in (i) to get $T = \dfrac{2v_0}{g\sqrt{1+3\sin^2\beta}}$

Since the body collides elastically, it will rebound perpendicular to the inclined plane with the same speed v. In consequence, it retraces its path elapsing equal time T for backward journey to the point of projection. Hence, the required time of motion is

$$T' = 2T = \frac{4v_0}{g\sqrt{1+3\sin^2\beta}}$$

ILLUSTRATION 5.39

An inclined plane makes an angle $\theta_0 = 30°$ with the horizontal. A particle is projected from this plane with a speed of 5 m s^{-1} at an angle of elevation $\beta = 30°$ with the horizontal as shown in figure.



(a) Find the range of the particle on the plane when it strikes the plane.

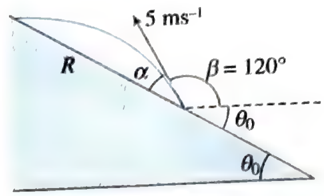
(b) Find the range of the particle for $\beta = 120°$.

Sol.
(a) $\alpha' = \theta_0 + \beta = 60°$

$$R = \frac{u^2}{g\cos^2\theta_0}[\sin(2\alpha' - \theta_0) + \sin\theta_0]$$

$$= \frac{5^2}{10\cos^2 30}[\sin(2\times 60° - 30°) + \sin 30°] = 5\text{ m}$$

(b) $\alpha + \theta_0 + 120° = 180° \quad \Rightarrow \quad \alpha = 30°$



$$R = \frac{u^2}{g\cos^2\theta_0}[\sin(2\alpha + \theta_0) - \sin\theta_0]$$

$$= \frac{5^2}{10\cos^2 30°}[\sin(2\times 30° + 30°) - \sin 30°] = 5/3\text{ m}$$

ILLUSTRATION 5.40

A body has maximum range R_1 when projected up the plane. The same body when projected down the inclined plane, it has maximum range R_2. Find the maximum horizontal range. Assume equal speed of projection in each case and the body is projected onto the inclined plane in the line of the greatest slope.

Sol. Let θ_0 be the angle of inclined plane with horizontal. Then for upward projection,

$$R_{max} = \frac{u^2}{g(1+\sin\theta_0)} = R_1 \quad \text{...(i)}$$

For downward projection,

$$R_{max} = \frac{u^2}{g(1-\sin\theta_0)} = R_2 \quad \text{...(ii)}$$

For a projection on horizontal surface, we have

$$R_{max} = \frac{u^2}{g} = R\text{ (say)} \quad \text{...(iii)}$$

To establish a relation between R, R_1, and R_2, we need to eliminate θ_0. From (i) and (ii), we get

$$\frac{2}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \Rightarrow \quad R = \frac{2R_1R_2}{R_1 + R_2}$$

CONCEPT APPLICATION EXERCISE 5.4

1. For a given velocity of projection from a point on the inclined plane, the maximum range down the plane is three times the maximum range up the incline. Then find the angle of inclination of the inclined plane.

2. A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is $\alpha = 30°$ and the angle of the barrel to the horizontal $\beta = 60°$. The

initial velocity v of the shell is 21 m s^{-1}. Then find the distance of point from the gun at which the shell will fall.

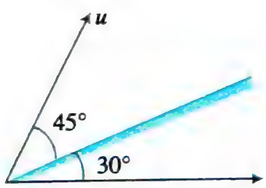
3. The maximum range of a rifle bullet on the horizontal ground is 6 km. Find its maximum range on an inclined of 30°.

4. A bullet is fired from the bottom of the inclined plane at angle $\theta = 37°$ with the inclined plane. The angle of incline is 30° with the horizontal. Find the (a) position of the maximum height of the bullet from the inclined plane; (b) time of flight; (c) range along the incline; (d) the value of θ at which the range will be maximum; (e) maximum range.

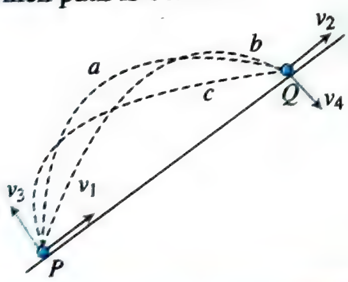
5. On an inclined plane two particles A and B are projected with same speed at the same angle with the horizontal, particle A down and particle B up the plane. If the ratio of time of flight of A and B is $\cot\theta$, where θ is the angle at which B is projected measured from inclined plane, find the angle at which particles are projected.

6. A particle is projected with velocity u on an inclined plane at an angle θ with the plane. If the particle lands on the plane at right angle its time of flight is $u \operatorname{cosec}\theta/2g$. Find the angle of inclination of the plane with horizontal.

7. A particle is projected on an inclined plane with a speed u as shown in figure. Find the range of the particle on the inclined plane.



8. There are three paths a, b and c of a projectile projected from point P as shown in figure. Prove that $v_1 > v_2$ and $v_3 = v_4$. Which path is correct?



ANSWERS

1. 30° 2. 30 m 3. 4 km

4. (a) $30\sqrt{3}$ m (b) $2 \times 2\sqrt{3}$ s

(c) $40(4\sqrt{3}-3)$m (d) 30° (e) $\dfrac{500}{3}$ m

5. 45° 6. $\alpha = 2\theta$ 7. $\dfrac{2u^2(\sqrt{3}-1)}{3g}$

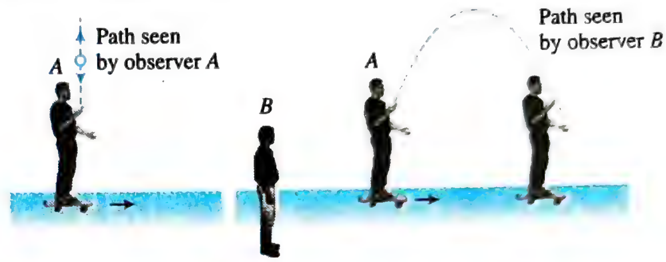
8. Path b

RELATIVE MOTION

RELATIVE VELOCITY IN TWO DIMENSIONS

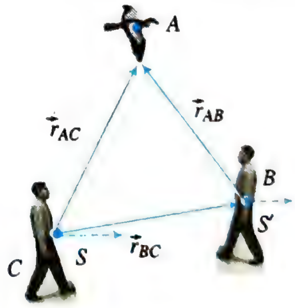
In this section, we describe how observations made by different observers in different frames of reference are related to each other. We find that observers in different frames of reference may measure different positions, velocities, and accelerations for a given particle. That is, two observers moving relative to each other generally do not agree on the outcome of a measurement.

Suppose a person riding on a skateboard (observer A) throws a ball in such a way that it appears in this person's frame of reference to move first straight upward and then straight downward along the same vertical line, as shown in figure. An observer B on the ground sees the path of the ball as a parabola, as illustrated in figure. Relative to observer B, the ball has a vertical component of velocity (resulting from the initial upward velocity and the downward acceleration due to gravity) and a horizontal component.



ANALYSIS OF RELATIVE MOTION IN GENERAL

Let us assume that observers B and C are fixed with the reference frames S' and S, respectively, and observe the motion of object A.



Let the observers B and C measure the position vectors of A as $\vec{r}_{AB}$ and $\vec{r}_{AC}$, respectively. If the position vector of B relative to C is $\vec{r}_{BC}$, following the triangle law of vectors, we have $\vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$

Differentiating the above expression with time, we have

$$\frac{d\vec{r}_{AC}}{dt} = \frac{d\vec{r}_{AB}}{dt} + \frac{d\vec{r}_{BC}}{dt}$$

Substituting $\dfrac{d\vec{r}_{AB}}{dt} = \vec{v}_{AB}$, $\dfrac{d\vec{r}_{AC}}{dt} = \vec{v}_{AC}$, and $\dfrac{d\vec{r}_{BC}}{dt} = \vec{v}_{BC}$, we have

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

The above expression tells us that simultaneously different observers (B and C) will record different velocities ($\vec{v}_{AB}$ and $\vec{v}_{BC}$) of the object (A).

$$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$$

If observer B is located on ground $\vec{v}_{AC} = \vec{v}_{A,\text{Earth}} + \vec{v}_{\text{Earth},C}$

or $\quad \vec{v}_{A,C} = \vec{v}_{A,\text{Earth}} - \vec{v}_{C,\text{Earth}} = \vec{v}_A - \vec{v}_C$

or $\quad \vec{v}_{\text{Object,Observer}} = \vec{v}_{\text{Object}} - \vec{v}_{\text{Observer}}$

or $\quad \vec{v}_{\text{rel}} = \vec{v}_{\text{Object}} - \vec{v}_{\text{Observer}}$

Graphical Method to Find Relative Velocity

Let two bodies A and B are moving with velocities $\vec{v}_A$ and $\vec{v}_B$ respectively at angle θ with each other. We can write the velocity of B with respect to A as $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$

When two bodies move at an angle θ with each other, then their relative velocity is given by:

Magnitude: $|\vec{v}_{B/A}| = |\vec{v}_B - \vec{v}_A|$

$$= \sqrt{v_A^2 + v_B^2 + 2v_A v_B \cos(180^\circ - \theta)} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

Direction: $\tan\alpha = \dfrac{v_B \sin(180^\circ - \theta)}{v_A + v_B \cos(180^\circ - \theta)}$

$$\Rightarrow \quad \tan\alpha = \frac{v_B \sin\theta}{v_A - v_B\cos\theta}$$

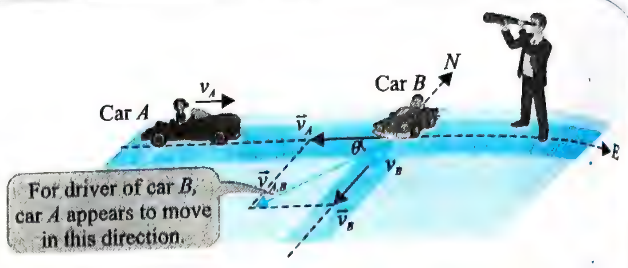
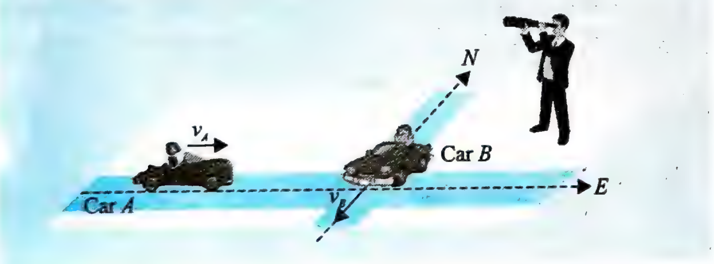


Illustration 5.41

Two cars are moving, car A along East and car B is moving along South. Draw the direction of the motion of

(a) car A as seen from car B

(b) car B as seen from car A

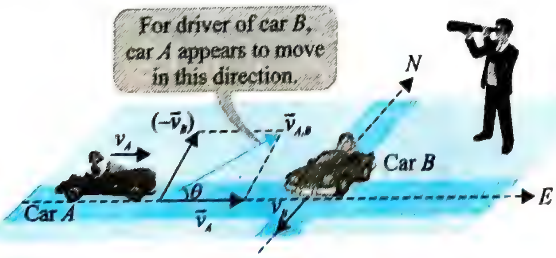


Sol. **Direction of motion of car A as seen from car B**

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$$

Magnitude: $|\vec{v}_{A,B}| = \sqrt{v_A^2 + v_B^2}$

Direction: $\tan\theta = \dfrac{v_B}{v_A} \Rightarrow \theta = \tan^{-1}\left(\dfrac{v_B}{v_A}\right)$



Direction of motion of car B as seen from car A

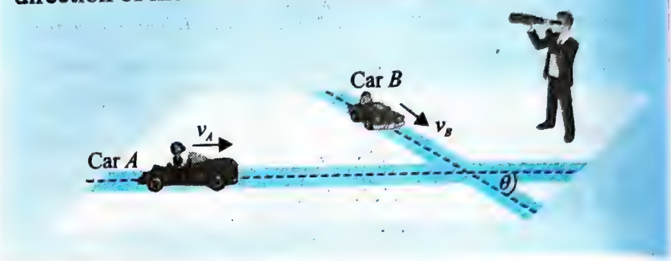
$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

Magnitude: $|\vec{v}_{B,A}| = \sqrt{v_A^2 + v_B^2}$

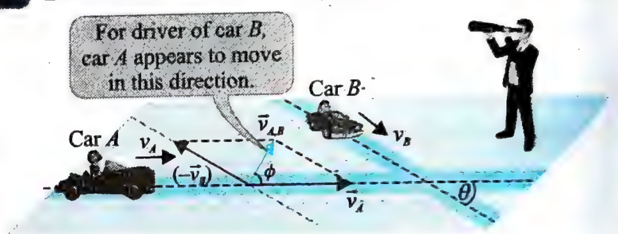
Direction: $\tan\theta = \dfrac{v_A}{v_B} \Rightarrow \theta = \tan^{-1}\left(\dfrac{v_A}{v_B}\right)$

Illustration 5.42

Two cars A and B are moving as shown in figure. Calculate the relative velocity of A with respect to B. Also draw the direction of motion of car A as seen from car B.



Sol. **Direction of motion of car A as seen from car B**



Velocity of car A as seen from car B

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$$

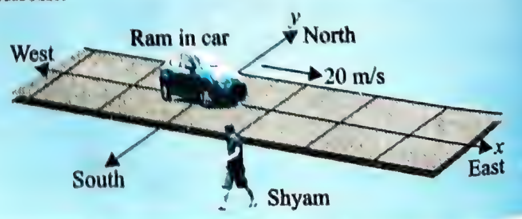
Magnitude: $|\vec{v}_{A,B}| = \sqrt{v_A^2 + v_B^2 + 2v_A \cdot v_B \cdot \cos(180^\circ - \theta)}$

Direction:

$$\tan\phi = \frac{v_B \sin(180^\circ - \theta)}{v_A + v_B \cos(180^\circ - \theta)} \Rightarrow \phi = \tan^{-1}\left(\frac{v_B \sin\theta}{v_A - v_B\cos\theta}\right)$$

Illustration 5.43

Ram was riding in a car traveling at a speed of 20 m/s towards east. Shyam was standing on the sideway when the car passed him. Just before the car passed close to Shyam (when it was north-west of him), Ram threw a banana peel relative to him with velocity $\vec{v}_{\text{Peel,Ram}} = [(-10)\hat{i} + (-10)\hat{j}]$ m/s. Shyam accused that Ram had thrown the banana peel at him. Justify his statement.



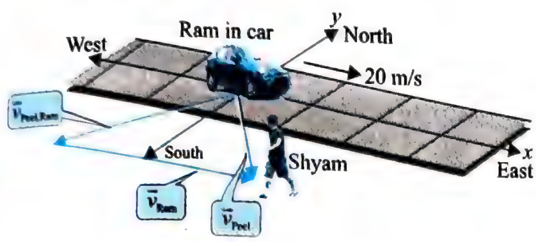
Sol. The velocity of banana peel relative to Ram,

$$\vec{v}_{\text{Peel,Ram}} = [(-10)\hat{i} + (-10)\hat{j}] \text{ m/s}$$

As East and North directions are taken along x-axis and y-axis, respectively, it means the banana peel was thrown from a point in south-west direction relative to car (reference frame of Ram).

As $\vec{v}_{\text{Peel, Ram}} = \vec{v}_{\text{Peel}} - \vec{v}_{\text{Ram}}$, hence velocity of banana peel w.r.t ground, $\vec{v}_{\text{Peel}} = \vec{v}_{\text{Peel, Ram}} + \vec{v}_{\text{Ram}}$

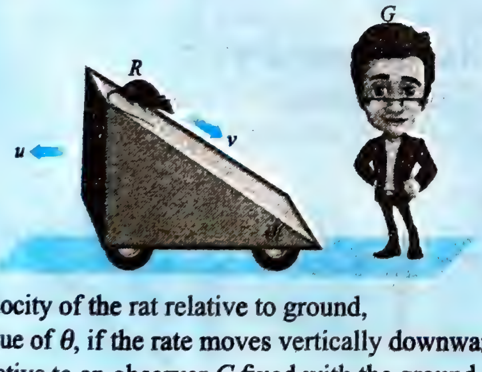
$$\vec{v}_{\text{Peel}} = \left[(-10)\hat{i} + (-10)\hat{j}\right] + 20\hat{i} = \left(10\hat{i} - 10\hat{j}\right)\text{m/s}$$



Shyam is on the ground; velocity of banana peel relative to him can be considered to be the absolute velocity of banana peel, i.e., the absolute velocity of banana peel is directed towards south-east, towards Shyam. From this illustration, it is clear that the two observers moving at constant velocity relative to one another find that their velocity measurements yield different results.

ILLUSTRATION 5.44

A rat is moving down the slant of a wedge of angle of inclination θ, with a velocity $\vec{v}$, as shown in the figure. If the wedge moves towards left with a velocity $\vec{u}$, find

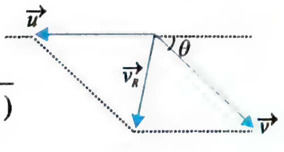


(a) velocity of the rat relative to ground,
(b) value of θ, if the rate moves vertically downward relative to an observer G fixed with the ground.

Sol.

(a) The velocity of rat R relative to the person G standing on ground

$$\vec{v}_R = \vec{v}_{R,W} + \vec{v}_W = \vec{u} + \vec{v}.$$



$$v_R = \sqrt{u^2 + v^2 + 2uv\cos(180° - \theta)}$$

$$= \sqrt{u^2 + v^2 - 2uv\cos\theta}$$

(b) If the rat appears to move vertically downward relative to G.

$$u = v\cos\theta$$

$$\cos\theta = \frac{u}{v} \Rightarrow \theta = \cos^{-1}\frac{u}{v}$$

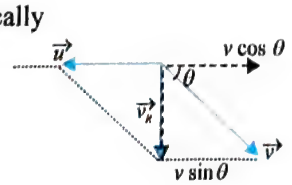


ILLUSTRATION 5.45

A truck is moving a constant velocity of $u = 54$ km/hr. In what direction should a stone be projected up with a velocity of $v = 30$ m/s, from the floor of the truck, so as to appear at right angles to the truck, for a person standing on earth?

Sol.

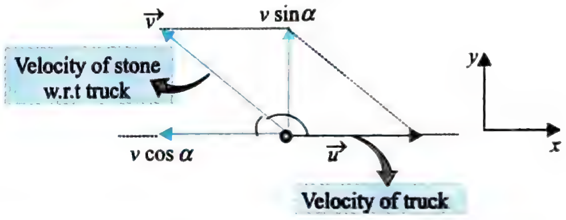
The velocity of stone relative to truck, $\vec{v} = \left(-v\cos\alpha\,\hat{i} + v\sin\alpha\,\hat{j}\right)$

The velocity of truck, $\vec{u} = u\,\hat{i}$

The velocity of stone relative to the person G standing on ground

$$\vec{v}_{\text{stone}} = \vec{v}_{\text{stone, truck}} + \vec{v}_{\text{truck}} = \vec{v} + \vec{u} = \left(-v\cos\alpha\,\hat{i} + v\sin\alpha\,\hat{j}\right) + u\,\hat{i}x$$

$$\Rightarrow \quad \vec{v}_{\text{stone}} = \left(u - v\cos\alpha\right)\hat{i} + v\sin\alpha\,\hat{j}$$



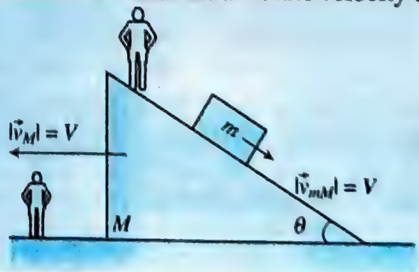
For ground observer, the stone appears at right angles to the truck. It means $u - v\cos\alpha = 0$

$$u = v\cos\alpha$$

$$\cos\alpha = \frac{u}{v} = \frac{15}{30} = \frac{1}{2} \Rightarrow \alpha = 60° \text{ or } \theta = 120°$$

ILLUSTRATION 5.46

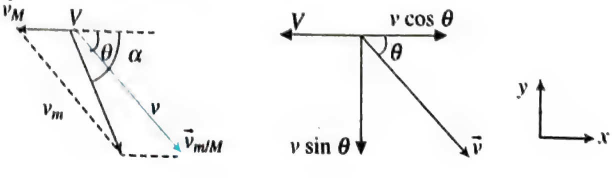
A block slips along an incline of a wedge. Due to the reaction of the block on the wedge, it slips backwards. An observer on the wedge will see the block moving straight down the incline. Discuss how to find the absolute velocity of the block.



Sol. We know that $\vec{v}_{m/M} = \vec{v}_m - \vec{v}_M \Rightarrow \vec{v}_m = \vec{v}_{m/M} + \vec{v}_M$

Note that a single subscript implies absolute velocity. The absolute velocity of block is the vector sum of its velocity relative to the wedge and velocity of wedge relative to ground.

The absolute velocity of block (ground reference frame) is shown in the vector diagram given in figure.



$$|\vec{v}_m| = \sqrt{v^2 + V^2 + 2vV\cos(\pi - \theta)} = \sqrt{v^2 + V^2 - 2vV\cos\theta}$$

We can derive this result by resolving v into its components.

Sum of x-components $V_x = v\cos\theta - V$

Sum of y-components $V_y = v\sin\theta$

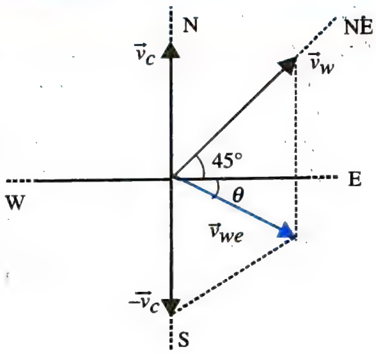
$$\text{Resultant velocity} = \sqrt{V_x^2 + V_y^2} = \sqrt{(v\cos\theta - V)^2 + (v\sin\theta)^2}$$

$$= \sqrt{v^2 + V^2 - 2vV\cos\theta}$$

$$\tan\alpha = \frac{V_y}{V_x} = \frac{v\sin\theta}{v\cos\theta - V}$$

A political party has to start its procession in an area where wind is blowing at a speed of $30\sqrt{2}$ km h^{-1} and party flags on the cars are fluttering along north-east direction. If the procession starts with a speed of 40 km h^{-1} towards north, find the direction of flags on the cars.

When the procession is stationary, the flags flutter along the north-east direction. It means wind is flowing along the north-east direction. The flags will start fluttering along the direction of the relative velocity of wind w.r.t. procession.



$$\vec{v}_{wc} = \vec{v}_w - \vec{v}_c$$

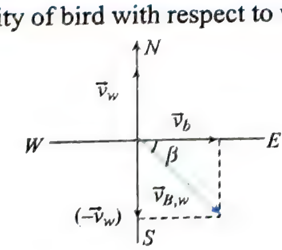
$$= (30\sqrt{2}\cos 45°\hat{i} + 30\sqrt{2}\sin 45°\hat{j}) - 40\hat{j} = 30\hat{i} - 10\hat{j}\ (\text{m s}^{-1})$$

$$\tan\theta = \frac{10}{30} = \frac{1}{3}$$

So the flag will flutter in a direction at $\theta = \tan^{-1}(1/3)$ S of E.

A bird is flying due east with a velocity of 4 m s^{-1}. The wind starts to blow with a velocity of 3 m s^{-1} due north. What is the magnitude of relative velocity of bird w.r.t. wind? Find out its direction also.

The velocity of bird with respect to wind can be given as



$$\vec{v}_{b,w} = \vec{v}_b - \vec{v}_w = \vec{v}_b + (-\vec{v}_w)$$

$$= 4\hat{i} + (-3\hat{j})\ (\text{m s}^{-1}) = 4\hat{i} - 3\hat{j}\ (\text{m s}^{-1})$$

$$|\vec{v}_{b,w}| = \sqrt{(4)^2 + (3)^2} = 5\ \text{m s}^{-1}$$

Here the direction of the relative velocity of the bird is

$$|\tan\beta| = \frac{3}{4} \Rightarrow \beta = \tan^{-1}\left(\frac{3}{4}\right)$$

Hence, the relative velocity of the bird with respect to wind is 5 m s^{-1} and in the direction $\tan^{-1}\left(\frac{3}{4}\right)$ from east toward south.

Relative Motion in River Flow

Case I: Consider a swimmer in still water. The swimmer can generate a velocity due to its own effort. We call this velocity, *velocity of swimmer in still water.*

Velocity of swimmer relative to water $= \vec{v}_{s,w}$.

Next, consider a person with a life jacket in a river flowing with a velocity. The person makes no effort to swim; he just drifts due to river flow. The velocity imparted due to river flow is called the velocity of water relative to ground, i.e., it denotes the rate at which water flows.

The velocity of water flow relative to ground $= \vec{v}_w$.

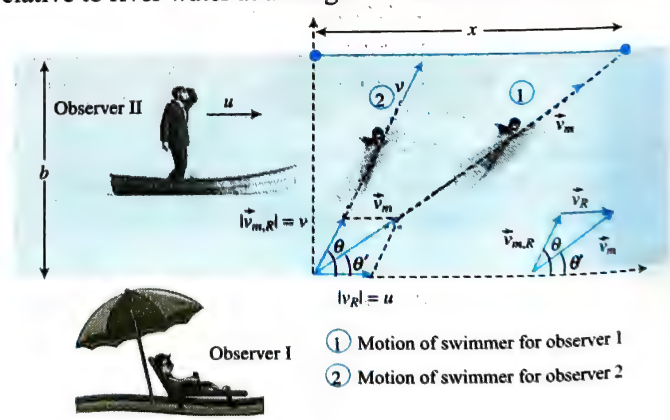
Case II: Next, consider a swimmer applying his effort in flowing water. In this case, swimmer's net velocity resultant velocity will be decided by two factors: (1) his own effort and (2) water flow. Thus, resultant motion is obtained by the vector sum of two velocities imparted to swimmer.

The resultant velocity of swimmer relative to ground = Velocity of swimmer relative to water + Velocity of water flow relative to ground

or $\quad \vec{v}_s = \vec{v}_{s,w} + \vec{v}_w$

Motion of Man Swimming in a River

Consider a man swimming in a river with a velocity of $\vec{v}_{m,R}$ relative to river water at an angle of θ with the river flow.



The velocity of river water is $\vec{V}_R$.

Let there be two observers I and II, observer I is on ground and observer II is on a raft floating along with the river and hence moving with the same velocity as the river. Hence motion w.r.t. observer II is same as motion w.r.t. river. i.e., the man will appear to swim at an angle θ with the river flow for observer II.

For observer I, the velocity of man will be

$$\vec{v}_m = \vec{v}_{m,R} + \vec{v}_R$$

Hence, the swimmer will appear to move at an angle θ' with the river flow.

River Problem in Two Dimensions (Crossing River)

Consider a man swimming in a river with a velocity of $\vec{v}_{m,R}$ relative to river at an angle of θ with the river flow.

The velocity of river is $\vec{v}_R$ and the width of the river is d.

$$\vec{v}_m = \vec{v}_{m,R} + \vec{v}_R$$

$$= (v\cos\theta\,\hat{i} + v\sin\theta\,\hat{j}) + u\hat{i}$$

$$= (v\cos\theta + u)\hat{i} + v\sin\theta\,\hat{j}$$

Here $v\sin\theta$ is the component of the velocity of man in the direction perpendicular to the river flow.

This component of velocity is responsible for the man crossing the river. Hence, if the time to cross the river is t, then

$$t = \frac{d}{v_y} = \frac{d}{v\sin\theta}$$

It is defined as the displacement of man in the direction of river flow (see figure).

It is simply the displacement along x-axis during the period the man crosses the river. ($v\cos\theta + u$) is the component of the velocity of man in the direction of river flow, and this component of velocity is responsible for drift along the river flow. If the drift is x, then

Drift $= v_x \times t$

$$x = (v\cos\theta + u) \times \frac{d}{v\sin\theta}$$

Crossing the River in Shortest Time

As we know that $t = d/v\sin\theta$. Clearly t will be minimum when $\theta = 90°$, i.e., time to cross the river will be minimum if the man swims perpendicular to the river flow. Which is equal to d/v.

Crossing the River in Shortest Path, Minimum Drift

The minimum possible drift is zero. In this case, the man swims in the direction perpendicular to the river flow as seen from the ground. This path is known as *shortest path.*

Here $x_{min} = 0 \Rightarrow (v\cos\theta + u) = 0$

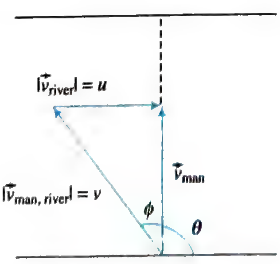
or $\quad \cos\theta = -\dfrac{u}{v}$

Since $\cos\theta$ is –ve, therefore, $\theta > 90°$, i.e., for minimum drift the man must swim at some angle ϕ with the perpendicular in upstream direction, where

$$\sin\phi = \frac{|\vec{v}_R|}{|\vec{v}_{m,R}|} = \frac{u}{v}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{-v_R}{v_{m,R}}\right) \Rightarrow \frac{u}{v} \le 1$$

i.e., $u \le v$



i.e., minimum drift is zero if and only if the velocity of man in still water is greater than or equal to the velocity of river.

Time to cross the river along the shortest path,

$$t = \frac{d}{v\sin\theta} = \frac{d}{\sqrt{v^2 - u^2}}$$

Note: If $|\vec{v}_R| > |\vec{v}_{m,R}|$, then it is not possible to have zero drift. In this case, the minimum drift (corresponding to shortest possible path is non-zero and the condition for minimum drift can be proved to be $\cos\theta = -\dfrac{v}{u}$ or $\sin\phi = \dfrac{u}{v}$ or $\cos\theta = \dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{u}{v}\right)$ for minimum but non-zero drift.

ILLUSTRATION 5.49

A man can row a boat 4 km/hr in still water. He is crossing a river where the current is 2 km/hr.

(a) In what direction will his boat be headed if he wants to reach a point on the other bank, directly opposite to starting point?

(b) If width of the river is 4 km, how long will it take him to cross the river with the condition in part *a*?

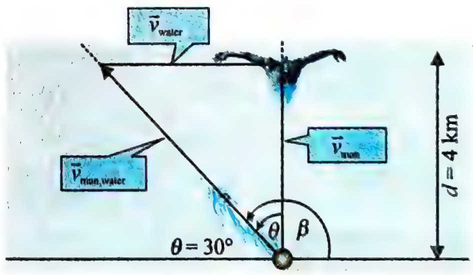
(c) In what direction should he head the boat if he wants to cross river in shortest time. Find the time taken by him to cross the river? Also find the location of the point where he lands on the other side of the river.

Sol.

(a) We know $\vec{v}_{\text{man,water}}$, the velocity of the boat relative to the river, and $\vec{v}_{\text{water}}$, the velocity of the river water relative to the Earth. What we must find is $\vec{v}_{\text{man}}$, the velocity of the boat/man relative to the Earth. The relationship between these three quantities is $\vec{v}_{\text{man}} = \vec{v}_{\text{man,water}} + \vec{v}_{\text{water}}$. The terms in the equation must be manipulated as vector quantities; the vectors are shown in figure.

The angle at which man should swim is order to reach just opposite end

$$\sin\theta = \frac{|v_{\text{water}}|}{|v_{\text{man,water}}|} = \frac{2}{4} = \frac{1}{2} \Rightarrow \theta = 30°$$



Hence to reach the point directly opposite to starting point, he should head the boat an angle $\beta = (90° + 30°) = 120°$ with river flow.

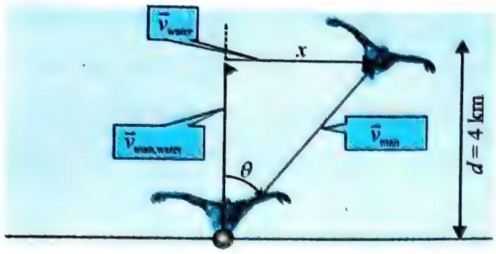
(b) Use the Pythagorean theorem to find the net velocity of man.

$$v_m = \sqrt{v^2 - u^2} = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}\text{ km/h.}$$

Time taken to cross the river

$$t = \frac{d}{v_{\text{man}}} = \frac{4}{2\sqrt{3}}\text{hr} = \frac{2}{\sqrt{3}}\text{ hr.}$$

(c) If man swims across the river, the water current pushes the man down the river. The man will not be able to move directly across the river, but will end up downstream with drift. The analysis now involves the new triangle shown in figure.



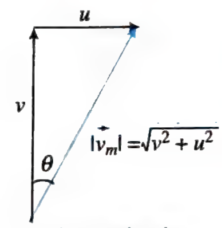
Time taken to cross the river $t_{\min} = \dfrac{d}{v_{\text{man,water}}} = \dfrac{4}{4} = 1\,\text{hr}$

Drifting in this case $= v_{\text{water}} \times t_{\min} = 2 \times 1 = 2\,\text{km}$

ILLUSTRATION 5.50

A man wishes to cross a river in a boat. If he crosses the river in minimum time he takes 10 min with a drift of 120 m. If he crosses the river taking shortest route, he takes 12.5 min. Find the velocity of the boat with respect to water.

Sol. **Case I:** If the man crosses the river in minimum time he should move perpendicular to bank or normal to the direction of water flow.



(a) Man crossing river taking minimum time

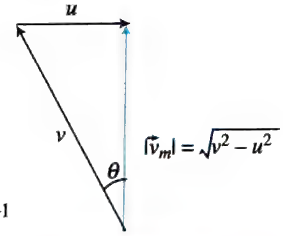
Let $|\vec{v}_{m,w}| = v$ and $|\vec{v}_w| = u$

Time to cross river, $t_1 = 10 = \dfrac{d}{v} \Rightarrow d = 10\,v$,

$x = ut_1 \Rightarrow 120u = u \times 10 \Rightarrow u = 12\ \text{m min}^{-1}$

Case 2: If the man crosses river taking the shortest route, the drift should be zero. Time to cross river,

$$t_2 = 12.5 = \frac{d}{\sqrt{v^2 - u^2}}$$



(b) Man crossing river taking shortest

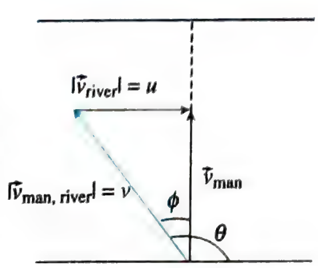
From case 1, $d = 10v$ and $v = 12\ \text{m s}^{-1}$

$$12.5 = \frac{10v}{\sqrt{v^2 - 12^2}} \Rightarrow v = 20\,\text{m s}^{-1}$$

ILLUSTRATION 5.51

A man can swim at the rate of 5 km h^{-1} in still water. A 1-km wide river flows at the rate of 3 km h^{-1}. The man wishes to swim across the river directly opposite to the starting point.
(a) Along what direction must the man swim?
(b) What should be his resultant velocity?
(c) How much time will he take to cross the river?

Sol.



(a) Velocity of man with respect to river water, $v = 5\ \text{km h}^{-1}$. This is greater than the river flow velocity. Therefore, he can cross the river directly (along the shortest path or no drift condition from flow velocity). The angle of swim,

$$\theta = \frac{\pi}{2} + \sin^{-1}\left(\frac{u}{v}\right) = 90^\circ + \sin^{-1}\left(\frac{u}{v}\right)$$

$$= 90^\circ + \sin^{-1}\left(\frac{3}{5}\right) = 90^\circ + 37^\circ = 127^\circ \text{ w.r.t. the river flow,}$$

or 37° w.r.t. perpendicular in upstream direction

(b) Resultant velocity or velocity of mass will be

$$v_m = \sqrt{v^2 - u^2} = \sqrt{5^2 - 3^2} = 4\ \text{km h}^{-1}$$

In the direction perpendicular to the river flow.

(c) time taken to cross the river

$$t = \frac{d}{\sqrt{v^2 - u^2}} = \frac{1\,\text{km}}{4\,\text{km h}^{-1}} = \frac{1}{4}\,\text{h} = 15\ \text{min}$$

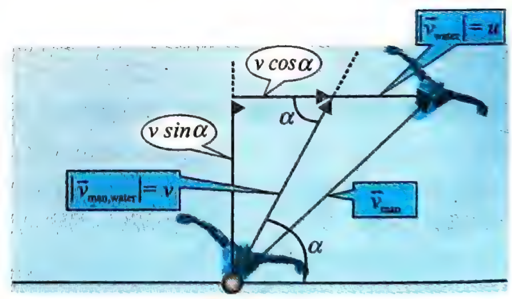
ILLUSTRATION 5.52

The ratio of the distance carried away by the water current downstream in crossing a river by a person making same angle with downstream and upstream are respectively as 2 : 1. Show that the ratio of the speed of person to the water current cannot be less than 1/3.

Sol.

Case I: Motion of the person, making an angle (say α) with the downstream.

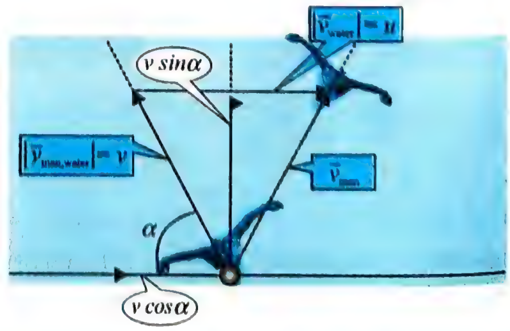
The time taken to cross the river $t = \dfrac{d}{v \sin \alpha}$



The distance carried away downstream in the same time.

Case II: Motion of the person, making an angle (say α) with the upstream.

The time taken to cross the river $= \dfrac{d}{v \sin\alpha}$



The distance carried away downstream in the same time.

$$x_2 = (u - v\cos\alpha)\frac{d}{v\sin\alpha}$$

Given $\dfrac{(u+v\cos\alpha)\dfrac{d}{v\sin\alpha}}{(u-v\cos\alpha)\dfrac{d}{v\sin\alpha}}=\dfrac{2}{1}\Rightarrow\dfrac{(u+v\cos\alpha)}{(u-v\cos\alpha)}=\dfrac{2}{1}$

$\Rightarrow\quad u+v\cos\alpha=2u-2v\cos\alpha$

$\Rightarrow\quad 3v\cos\alpha=u\Rightarrow\cos\alpha=\dfrac{u}{3v}$

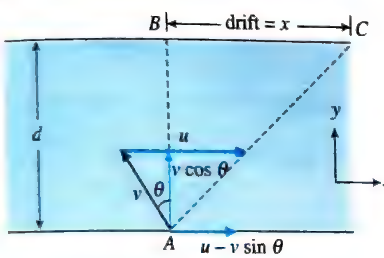
But $\quad \cos\alpha\le 1\Rightarrow\dfrac{u}{3v}\le 1\Rightarrow\dfrac{u}{3v}\le 1\Rightarrow\dfrac{v}{u}\ge\dfrac{1}{3}$

Hence, $\dfrac{v}{u}$ cannot be less than $\dfrac{1}{3}$.

ILLUSTRATION 5.53

A boat moves relative to water with a velocity v which is n times less than the river flow velocity u. At what angle to the stream direction must the boat move to minimize drifting?

Sol. In this case, the velocity of boat is less than the river flow velocity. Hence, boat cannot reach the point directly opposite to its starting point, i.e., drift can never be zero.



Suppose the boat starts at an angle θ from the normal direction up stream as shown in figure.

Component of the velocity of boat along the river,

$v_x=u-v\sin\theta$

and velocity perpendicular to the river, $v_y=v\cos\theta$.

Time taken to cross the river is $t=\dfrac{d}{v_y}=\dfrac{d}{v\cos\theta}$

Drift $\quad x=(v_x)t=(u-v\sin\theta)\dfrac{d}{v\cos\theta}=\dfrac{ud}{v}\sec\theta-d\tan\theta$

The drift x is minimum when $\dfrac{dx}{d\theta}=0$

or $\quad \left(\dfrac{ud}{v}\right)(\sec\theta\cdot\tan\theta)-d\sec^2\theta=0$

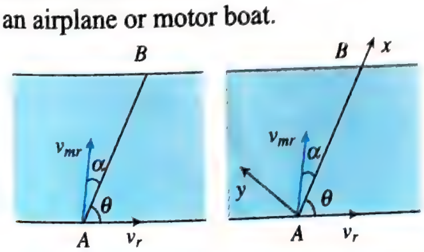
or $\quad \dfrac{u}{v}\sin\theta=1\Rightarrow\sin\theta=\dfrac{v}{u}$

i.e., for minimum drift, the boat must move at an angle $\theta=\sin^{-1}\theta$

$=\sin^{-1}\left(\dfrac{v}{u}\right)=\sin^{-1}\dfrac{1}{n}$ from the normal direction.

Swimming in a Directed Direction

Many times the person is not interested in minimizing the time or drift. But he has to reach a particular place. This is common in the cases of an airplane or motor boat.



The man desires to have this final velocity along AB. In other words, he has to move from A to B. We wish to find the direction in which he should make an effort so that his actual velocity is along line AB. In this method, we assume AB to be the reference line. The resultant of v_{mr} and v_r is along line AB. Thus, the components of v_{mr} and v_r in a direction perpendicular to line AB should cancel each other.

$$\vec{v}_m=\vec{v}_{m,r}+\vec{v}_r$$

$$=[v_{mr}\cos\alpha\,\hat{i}+v_{mr}\sin\alpha\,\hat{j}]+[v_r\cos\theta\,\hat{i}-v_r\sin\theta\,\hat{j}]$$

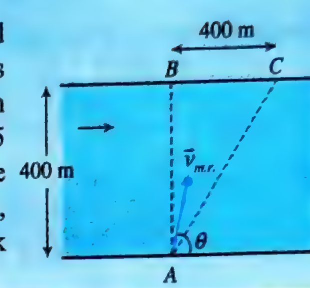
$$=(v_{mr}\cos\alpha+v_r\cos\theta)\hat{i}+(v_{mr}\sin\alpha-v_r\sin\theta)\hat{j}$$

The displacement of the person in y-direction is zero.

Hence $v_{mr}\sin\alpha=v_r\sin\theta$

ILLUSTRATION 5.54

A river is flowing with a speed of 1 km h^{-1}. A swimmer wants to go to point C starting from A. He swims with a speed of 5 km h^{-1} at an angle θ w.r.t. the river flow. If $AB=BC=400$ m, at what angle with the river bank should the swimmer swim?



Sol. Resultant path of the swimmer is at 45° with bank, therefore, x- and y-components of swimmer's resultant velocity must be equal.

Let velocity of swimmer $|\vec{v}_m|=v$

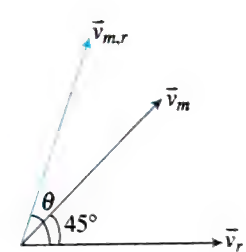
$\vec{v}_m=\vec{v}_{m,r}+\vec{v}_r$

$v_x=v_r+v_{m,r}\cos\theta$

$v_y=v_{m,r}\sin\theta$

Condition for reaching the point C,

$\tan 45°=\dfrac{v_y}{v_x},\ v_y=v_x$



$(v_r+v_{m,r}\cos\theta)=v_{m,r}\sin\theta$

$1+5\cos\theta=5\sin\theta$

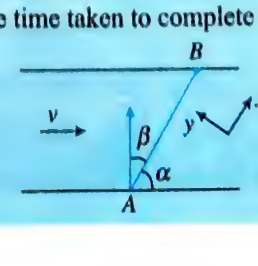
On squaring, $1+25\cos^2\theta+10\cos\theta=25-25\cos^2\theta$

$50\cos^2\theta+10\cos\theta-24=0$

On solving, we get $\theta=53°$.

ILLUSTRATION 5.55

A man wants to swim in a river from A to B and back from B to A always following line AB. The distance between points A and B is S. The velocity of the river current v is constant over the entire width of the river. The line AB makes an angle α with the direction of current. The man moves with velocity u at angle β to the line AB. The man swim to cover distance AB and back, find the time taken to complete the journey.

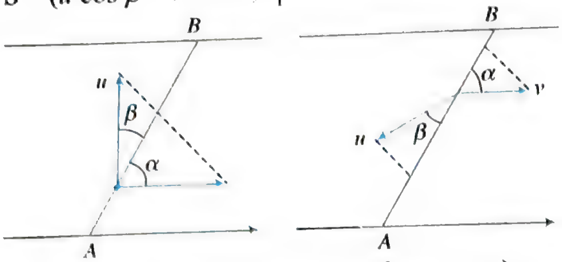


Sol. In this problem, we choose axis along AB and normal to it. Since the man moves along AB, the velocity components of current and man normal to AB must cancel out, i.e.,

$$u \sin \beta = v \sin \alpha$$

When the man moves from A to B, his resultant velocity along AB $= (u \cos \beta + v \cos \alpha)$

Hence, $S = (u \cos \beta + v \cos \alpha)\, t_1$



while for motion from B to A, $S = (u \cos \beta - v \cos \alpha)t_2$

From the condition of the problem, $t_1 + t_2 = t$

$$\frac{S}{u\cos\beta + v\cos\alpha} + \frac{S}{u\cos\beta - v\cos\alpha} = t$$

$$S\left[\frac{u\cos\beta - v\cos\alpha + u\cos\beta + v\cos\alpha}{u^2\cos^2\beta - v^2\cos^2\alpha}\right] = t$$

$$\frac{S(2u\cos\beta)}{u^2\cos^2\beta - v^2\cos^2\alpha} = t$$

RAIN–MAN PROBLEMS

If rain is falling vertically with a velocity $\vec{v}_r$ and an observer moving horizontally with velocity $\vec{v}_m$, the velocity of rain relative to the observer will be

$$\vec{v}_{r,m} = \vec{v}_r - \vec{v}_m \text{ or } v_{r,m} = \sqrt{v_r^2 + v_m^2}$$

and direction $\theta = \tan^{-1}\left(\dfrac{v_m}{v_r}\right)$ with the vertical as shown in the figure below.

Different Situations in Rain–Man Problems

The man is stationary and the rain is falling at his back to an angle ϕ with the vertical. The aim is to determine the angle at which the man should hold the umbrella to prevent himself from wetting.

Let $\vec{v}_{\text{man}}$ = velocity of man w.r.t. ground

$\vec{v}_{\text{rain}}$ = velocity of rain w.r.t. ground

$\vec{v}_{\text{rain,man}}$ = velocity of rain w.r.t. man

velocity of rain w.r.t. man $\vec{v}_{\text{rain,man}} = \vec{v}_{\text{rain}} - \vec{v}_{\text{man}} = \vec{v}_{\text{rain}} + (-\vec{v}_{\text{man}})$

The man is stationary and the rain is falling at his back at an angle ϕ with the vertical	The man starts moving forward. The relative velocity of rain w.r.t man shifts towards vertical direction.	As the man further increases his speed, then at a particular value the rain appears to be falling vertically.	If the man increases his speed further more, then the rain appears to be falling from the forward direction.

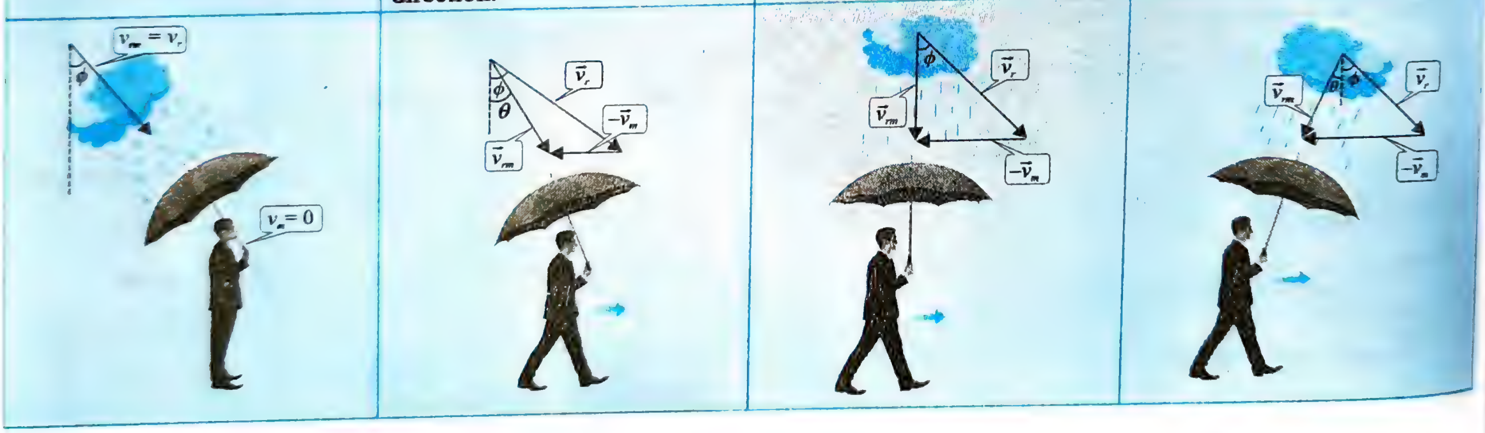
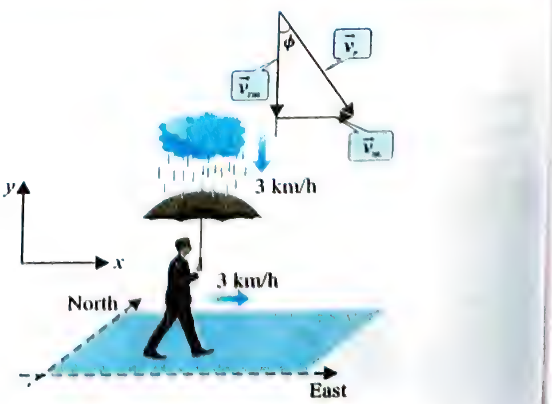


ILLUSTRATION 5.56

A man walking due east with 3 km/hr observes rain falling vertically at the rate of 3 km/hr. Find the actual speed and direction of the rain with respect to ground.



Sol.

We are given $\vec{v}_{r,m}$, $\vec{v}_r$ and $\vec{v}_m$. The relationship between these three quantities is $\vec{v}_{\text{rain}} = \vec{v}_{\text{rain,man}} + \vec{v}_{\text{man}}$. The terms in the equation can be represented as the vectors are shown in figure.

We can use the Pythagoras theorem to find the net velocity of rain $v_r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ km/h

Using trigonometry: $\tan\theta = \frac{3}{3} = 1 \Rightarrow \theta = 45°$

The rain is falling at 45° east of vertical with a velocity $3\sqrt{2}$ km/h

Alternate Method

Let us take east direction as positive x-direction and vertical up direction as positive y-direction.

We are given $\vec{v}_{r,m} = -3\hat{j}$ km/h, $\vec{v}_m = 3\hat{i}$ km/h

$$\vec{v}_r = \vec{v}_{r,m} + \vec{v}_m = (-3\hat{j}) + (3\hat{i})\,\text{km/h}$$

The magnitude of net velocity of rain $|\vec{v}_r| = 3\sqrt{2}$ km/h

The direction of rain made with y-axis: $\tan\theta = \frac{3}{3} = 1 \Rightarrow \theta = 45°$

ILLUSTRATION 5.57

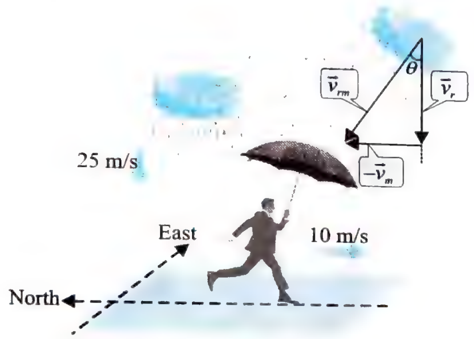
Rain is falling vertically with a velocity of 25 ms^{-1}. A man is walking with a speed of 10 ms^{-1} in the north to south direction. What is the direction in which he should hold his umbrella to safe him self from the rain?

Sol. We are given:

$\vec{v}_{man}$ = velocity of man w.r.t. ground=10 ms^{-1} (north to south)

$\vec{v}_{rain}$ = velocity of rain w.r.t. ground=25 ms^{-1} (vertically downward)

$\vec{v}_{rain,man}$ = velocity of rain w.r.t. man = ?



The relationship between these three quantities is

$$\vec{v}_{rain,man} = \vec{v}_{rain} - \vec{v}_{man} = \vec{v}_{rain} + (-\vec{v}_{man})$$

The terms in the equation can be represented as the vectors are shown in figure. We can use the Pythagoras theorem to find the velocity of rain with respect to man $v_r = \sqrt{10^2 + 25^2} = 5\sqrt{29}$ m/s.

Now let us find the angle made by rain with vertical.

Using trigonometry:

$$\tan\theta = \frac{|v_m|}{|v_r|} = \frac{10}{25} \Rightarrow \theta = \tan^{-1}(0.4) \Rightarrow \theta = 45°$$

The man should hold his umbrella at an angle $\tan^{-1}(0.4)$ with vertical towards south.

ILLUSTRATION 5.58

A standing man observes rain falling with the velocity of 20 ms^{-1} at an angle of 30° with the vertical.

(a) Find the velocity with which the man should move so that rain appears to fall vertically to him.

(b) Now if he further increases his speed, rain again appears to fall at 30° with the vertical. Find his new velocity.

Sol.

(a) Velocity of man, $\vec{v}_m = -v\hat{i}$ (let)

Velocity of rain, $\vec{v}_r = (-10\hat{i} - 10\sqrt{3}\,\hat{j})\,\text{ms}^{-1}$

Velocity of rain w.r.t to man, $\vec{v}_{r,m} = \vec{v}_r - \vec{v}_m$

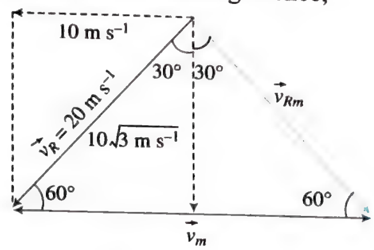
$$= (-10\hat{i} - 10\sqrt{3}\,\hat{j}) - (-v\hat{i})\,\text{ms}^{-1}$$

$$\vec{v}_{r,m} = -(10 - v)\hat{i} - 10\sqrt{3}\hat{j}\ \text{ms}^{-1}$$

For vertical fall, the horizontal component must be zero.

$\Rightarrow \quad -(10 - v) = 0$ or $v = 10$ m s^{-1}

(b) Velocity of rain will not change hence,



$$\vec{v}_r = (-10\hat{i} - 10\sqrt{3}\hat{j})\,\text{ms}^{-1}$$

Let the velocity of man in this case be $\vec{v}_m = -v_1\hat{i}$

Velocity of rain w.r.t. to man $\vec{v}_{r,m} = \vec{v}_r - \vec{v}_m$

$$= (-10\hat{i} - 10\sqrt{3}\,\hat{j}) - (-v_1\hat{i})\,\text{ms}^{-1}$$

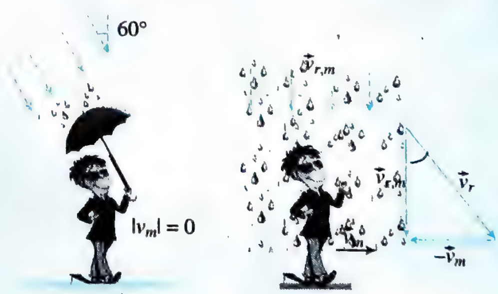
$$= -(10 - v_1)\hat{i} - 10\sqrt{3}\hat{j}\ \text{ms}^{-1}$$

Angle with the vertical = 30°

$$\Rightarrow \quad \tan 30° = \frac{10 - v_x}{-10\sqrt{3}} \Rightarrow v_x = 20\ \text{m s}^{-1}$$

ILLUSTRATION 5.59

A person standing on a road has to hold his umbrella at 60° with the vertical to keep the rain away. He throws the umbrella and starts running at 20 ms^{-1}. He find that rain drops are hitting his head vertically. Find the speed of the rain drops with respect to (a) the road and (b) the moving person.



Sol. $\vec{v}_{rg} = \vec{v}_{rm} + \vec{v}_{mg} \Rightarrow \vec{v}_{rm} = \vec{v}_{rg} - \vec{v}_{mg} = \vec{v}_{rg} + (-\vec{v}_{mg})$

When man is not moving he is observing actual velocity of the rain.

$$\vec{v}_{rain} = v\sin 60°\,\hat{i} - v\cos 60°\,\hat{j}$$

$$= \frac{\sqrt{3}}{2}v\hat{i} - \frac{v}{2}\hat{j}\,(\text{ms}^{-1}) \qquad \text{...(i)}$$

When the man starts running with speed 20 m s^{-1}, rain appears to

fall vertically as seen by man.
Hence, the velocity of rain with respect to man

$$\vec{v}_{r,m} = \vec{v}_{\text{rain}} - \vec{v}_{\text{man}} \quad \text{...(ii)}$$

Let the velocity of rain with respect to man has magnitude

$$|\vec{v}_{r,m}| = v' \Rightarrow -v'\hat{j} = \left(\frac{\sqrt{3}}{2}v\hat{i} - \frac{v}{2}\hat{j}\right) - 20\hat{i}$$

$$-v'\hat{j} = \left(-20 + \frac{\sqrt{3}}{2}v\right)\hat{i} - \frac{v}{2}\hat{j} \quad \text{...(iii)}$$

Comparing left side and right side terms,

$$-20 + \frac{\sqrt{3}}{2}v = 0 \Rightarrow v = \frac{40}{\sqrt{3}}\ (\text{m s}^{-1})$$

$$v' = \frac{v}{2} = \frac{1}{2}\left(\frac{40}{\sqrt{3}}\right) = \frac{20}{\sqrt{3}}\ (\text{m s}^{-1})$$

Hence, the actual velocity of rain (from (i)),

$$\vec{v}_{\text{rain}} = \frac{\sqrt{3}}{2}\left(\frac{40}{3}\right)\hat{i} - \frac{1}{2}\left(\frac{40}{\sqrt{3}}\right)\hat{j}\ (\text{m s}^{-1})$$

$$= 20\hat{i} - \frac{20}{\sqrt{3}}\hat{j}\ (\text{m s}^{-1})$$

Hence, magnitude of actual velocity of rain,

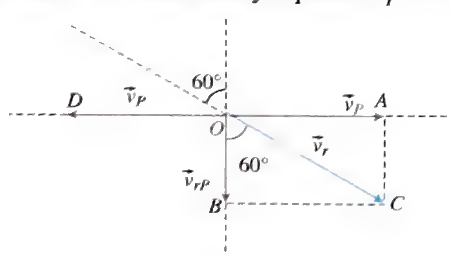
$$|\vec{v}_r| = \frac{40}{\sqrt{2}}\ (\text{m s}^{-1})$$

And the magnitude of velocity of rain with respect to man,

$$|\vec{v}_{r,m}| = \frac{20}{\sqrt{3}}\ (\text{m s}^{-1})$$

In relative velocity, the observer observes the velocity of an object considering himself at rest.
Consider the example of a man sitting in a moving train and observes the objects outside situated on the ground.

Method 2: Given $\theta = 60°$ and velocity of person $\vec{v}_P = \overrightarrow{OA} = 20\ \text{m s}^{-1}$.



This velocity is same as the velocity of person w.r.t. ground. First of all let us see how the diagram works out.

$\vec{v}_{rP} = \overrightarrow{OB}$ = Velocity of rain w.r.t. person.

$\vec{v}_r = \overrightarrow{OC}$ = Velocity of rain w.r.t. earth

Values of $\vec{v}_r$ and $\vec{v}_{rP}$ can be obtained by using simple trigonometric relations.

(a) Speed of rain drops w.r.t. earth = $\vec{v}_r = \overrightarrow{OC}$

From ΔOAB, $\dfrac{CB}{OC} = \sin 60°$

$$\Rightarrow \quad OC = \frac{CB}{\sin 60°} = \frac{20}{\sqrt{3}/2} = \frac{40}{\sqrt{3}} = \frac{40\sqrt{3}}{3}\ \text{m s}^{-1}$$

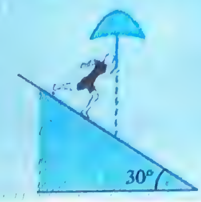
(b) Speed of rain w.r.t. the person, $\vec{v}_{rP} = \overrightarrow{OB}$

From, $\dfrac{OB}{CB} = \cot 60°$

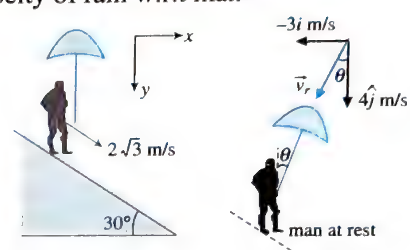
$$\Rightarrow \quad OB = CB \cot 60° = \frac{20}{\sqrt{3}} = \frac{20\sqrt{3}}{3}\ \text{m s}^{-1}$$

ILLUSTRATION 5.60

A man is coming down an incline of angle 30°. When he walks with speed $2\sqrt{3}\ \text{m s}^{-1}$ he has to keep his umbrella vertical to protect himself from rain. The actual speed of rain is $5\ \text{m s}^{-1}$. At what angle with vertical should he keep his umbrella when he is at rest so that he does not get drenched?



Sol. Velocity of rain w.r.t man



$$\vec{v}_{r,m} = x\hat{j} = \vec{v}_r - \vec{v}_m$$

$$\vec{v}_m = 2\sqrt{3}[\cos 30\hat{i} + \sin 30\hat{j}] = 3\hat{i} + \sqrt{3}\hat{j}$$

$$\vec{v}_r = -3\hat{i} + (x - \sqrt{3})\hat{j}$$

$$5 = \sqrt{3^2 + (x-\sqrt{3})^2}$$

$$16 = (x - \sqrt{3})^2 \Rightarrow 4 + \sqrt{3} = x$$

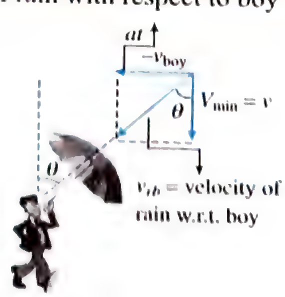
$$\vec{v}_r = -3\hat{i} + 4\hat{j}$$

$$\tan\theta = \frac{3}{4} \Rightarrow \theta = 37°$$

ILLUSTRATION 5.61

During a rainy day, rain is falling vertically with a velocity $2\ \text{m s}^{-1}$. A boy at rest starts his motion with a constant acceleration of $2\ \text{m s}^{-2}$ along a straight road. Find the rate at which the angle of the axis of umbrella with vertical should be changed so that the rain always falls parallel to the axis of the umbrella.

Sol. Velocity of rain with respect to boy



$$\vec{v}_{\text{rain,boy}} = \vec{v}_{\text{rain}} - \vec{v}_{\text{boy}} = \vec{v}_{\text{rain}} + (-\vec{v}_{\text{boy}})$$

At any time t, rain will appear to the boy as shown in the figure.

$$\tan\theta = \frac{at}{v}$$

Boy should hold his umbrella at an angle θ from the vertical

$$\therefore \quad \tan\theta = \frac{at}{v}$$

To find the rate at which the angle of the axis of umbrella with vertical, we need to find $d\theta/dt$.

$$\frac{d\tan\theta}{dt} = \frac{d\left(\frac{at}{v}\right)}{dt} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{a}{v}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{a}{v\sec^2\theta} = \frac{a}{v[1+\tan^2\theta]}$$

$$= \frac{a}{v\left[1+\frac{a^2t^2}{v^2}\right]} = \frac{av}{v^2+a^2t^2} = \frac{2\times 2}{4+4t^2} = \frac{1}{1+t^2}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{1+t^2}$$

WIND–AIRPLANE PROBLEMS

This is very similar to boat–river flow problems. The only difference is that boat is replaced by aeroplane and river is replaced by wind. Thus, the velocity of aeroplane with respect to wind

$$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w$$

or $\quad \vec{v}_a = \vec{v}_{aw} + \vec{v}_w$

where $\vec{v}_a$ is the velocity of aeroplane w.r.t. ground and $\vec{v}_w$ is the velocity of wind.

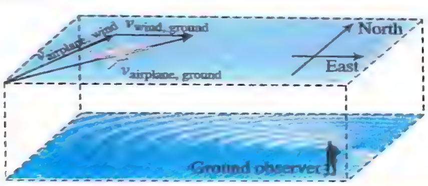


ILLUSTRATION 5.62

An aeroplane pilot wishes to fly due west. A wind of 100 km h^{-1} is blowing towards south.

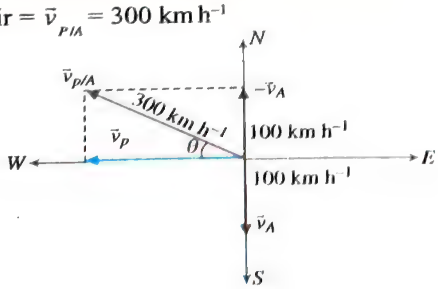
(a) If the speed of the plane (its speed in still air) is 300 kmh^{-1}, in which direction should the pilot head?

(b) What is the speed of the plane with respect to ground? Illustrate with a vector diagram.

Sol. Velocity of air (wind) $= \vec{v}_A = 100$ km h^{-1}

Velocity of plane w.r.t. air $= \vec{v}_{P/A} = 300$ km h^{-1}

$$\vec{v}_P = \vec{v}_{P/A} + \vec{v}_A$$



The net velocity of the plane will be the vector sum of two velocities.

Velocity of air and velocity of plane w.r.t air: If the plane is to move towards west finally, then the N-S component of velocity should be zero. For this,

$$\vec{v}_{P/A}\sin\theta = \vec{v}_A$$

$$\Rightarrow \quad 300\sin\theta = 100$$

$$\Rightarrow \quad \sin\theta = \frac{1}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right)$$

So the pilot should head in direction $\theta = \sin^{-1}\left(\frac{1}{3}\right)$ N of W.

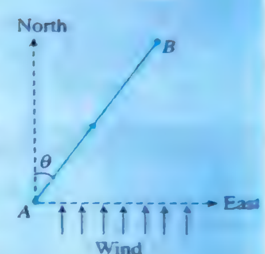
Speed of plane w.r.t ground,

$$\vec{v}_p = \vec{v}_{P/A}\cos\theta$$

$$= 300\sqrt{1-\sin^2\theta} = 300\sqrt{1-\left(\frac{1}{3}\right)^2} = 200\sqrt{2}\text{ km h}^{-1}$$

ILLUSTRATION 5.63

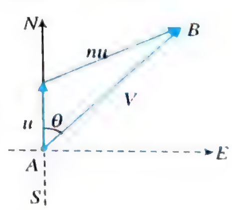
A helicopter flies horizontally with constant velocity in a direction θ east of north between two points A and B, at distance d apart. Wind is blowing from south with constant speed u; the speed of helicopter relative to air is nu, where $n > 1$. Find the speed of the helicopter along AB. The helicopter returns from B to A with same speed nu relative to air in same wind. Find the total time for the journeys.



Sol. Velocity of helicopter w.r.t. ground is given by

$$\vec{V}_{\text{Helicopter}} = \vec{V}_{\text{Helicopter,air}} + \vec{V}_{\text{Air}} = \vec{V}_{\text{Air}} + \vec{V}_{\text{Helicopter,air}}$$

$$|\vec{V}_{HA}| = nu, \quad |\vec{V}_A| = u, \ |\vec{V}_H| = V$$



Hence, the actual velocity of helicopter is the vector sum of its velocity of helicopter relative to air and velocity of air.

Using cosine rule, for motion from A to B,

$$n^2u^2 = u^2 + v^2 - 2uv\cos\theta$$

or $\quad v^2 - 2uv\cos\theta = u^2(n^2 - 1)$

or $\quad (v - u\cos\theta)^2 = u^2(n^2 - 1) + u^2\cos^2\theta$

$$= u^2(n^2 - \sin^2\theta)$$

$$v - u\cos\theta = \pm u\sqrt{n^2 - \sin^2\theta}$$

$$v = u\cos\theta \pm u\sqrt{n^2 - \sin^2\theta}$$

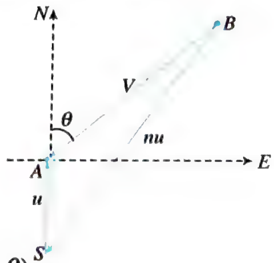
As $n \geq 1$,

$$n^2 - \sin^2\theta \geq 1 \quad \sin^2\theta \geq \cos^2\theta$$

Hence, speed of helicopter is $v = u\cos\theta + u\sqrt{n^2 - \sin^2\theta}$

Now we consider motion from B to A.

Similar to first case, the helicopter will return to A along BA when its velocity after being deflected by wind is along BA.



Once again using cosine rule in the vector triangle, we get

$$n^2u^2 = u^2 + V^2 - 2uV\cos(180° - \theta)$$

$$= u^2 + V^2 + 2uV\cos\theta$$

$$V = -u\cos\theta + u\sqrt{n^2 - \sin^2\theta}$$

The total time for motion A to B and B to A,

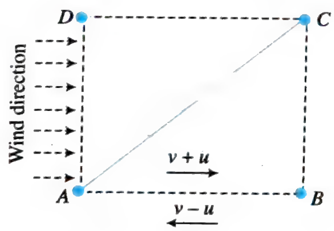
$$t = \frac{d}{v} + \frac{d}{V} = \frac{d(u+V)}{uV} = \frac{2du\sqrt{n^2 - \sin^2\theta}}{u^2(n^2 - \sin^2\theta) - u^2\cos^2\theta}$$

$$= \frac{2du\sqrt{n^2 - \sin^2\theta}}{u^2(n^2 - 1)} = \frac{2d\sqrt{n^2 - \sin^2\theta}}{u(n^2 - 1)}$$

ILLUSTRATION 5.64

A, B, C, and D are four trees, located at the vertices of a square. Wind blows from A to B with uniform speed. The ratio of times of flight of a bird from A to B and from B to A is n. At what angle should the bird fly from the direction of wind flow, in order that it starts from A and (a) reaches C, (b) reaches D?

Sol. Let the velocity of bird w.r.t. air is v and the velocity of wind is u.



The velocity of bird from A to B,

$$|\vec{v}_{AB}| = v + u$$

Velocity of bird from B to A,

$$|\vec{v}_{BA}| = v - u$$

$$t_{AB} = \frac{l}{v+u}, t_{BA} = \frac{l}{v-u}$$

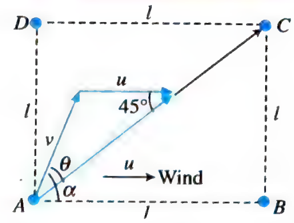
$$\frac{t_{AB}}{t_{BA}} = n \Rightarrow n = \frac{v-u}{v+u}$$

$$\frac{u}{v} = \frac{1-n}{1+n} \quad \text{...(i)}$$

(a) Motion of bird from A to C

$$\vec{v}_{\text{bird}} = \vec{v}_{\text{bird,wind}} + \vec{v}_{\text{wind}} = \vec{v} + \vec{u}$$

From velocity diagram, we can use sine rule:



$$\frac{u}{\sin\theta} = \frac{v}{\sin 45°} \Rightarrow \sin\theta = \frac{u}{\sqrt{2}\,v} = \frac{1-n}{\sqrt{2}(1+n)}$$

$$\sin(\alpha - 45°) = \frac{1-n}{\sqrt{2}(1+n)}$$

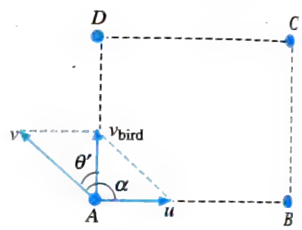
$$\sin\alpha\cos 45° - \cos\alpha\sin 45° = \frac{1-n}{\sqrt{2}(1+n)}$$

$$\sin\alpha - \cos\alpha = \frac{1-n}{1+n}$$

Squaring both sides, $1 - 2\sin\alpha\cos\alpha = \left(\frac{1-n}{1+n}\right)^2$

$$\sin 2\alpha = \frac{4n}{(1+n)^2} \Rightarrow \alpha = \frac{1}{2}\sin^{-1}\left[\frac{4n}{(1+n)^2}\right]$$

(b) For motion of bird from A to D: From vector diagram, it is clear



$$\sin\theta' = \frac{u}{v} = \frac{1-n}{1+n}$$

$$\sin(\alpha - 90°) = \frac{1-n}{1+n}$$

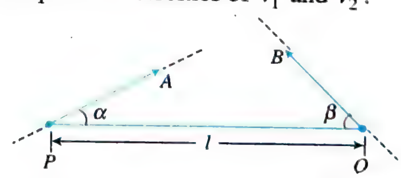
$$\cos\alpha = -\frac{(1-n)}{1+n}$$

$$\alpha = \cos^{-1}\left[\frac{-(1-n)}{1+n}\right]$$

SHORTEST DISTANCE BETWEEN TWO MOVING PARTICLES

If two particles move with constant velocities in any directions, the usual method to determine the shortest distance between them is to apply the technique of calculus. However, the concept of relative velocity yields an effective method of obtaining it, without applying calculus.

Consider, for example, two particles P and Q moving along PA and QB with respective velocities of $\vec{v}_1$ and $\vec{v}_2$.

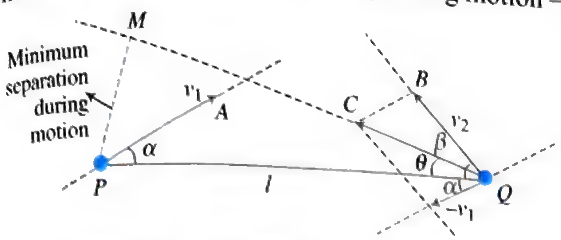


Let us bring P virtually to rest by applying $-\vec{v}_1$ velocity to the bodies in the system.

Compounding a velocity $-\vec{v}_1$ to the particle P, it appears to have a relative velocity along QC, moving in that direction. Thus, the situation is similar to particle P being at rest and particle Q moving along QC. Since the minimum distance between a fixed point and any point on a line is the perpendicular distance of the fixed point from line, therefore, dropping a perpendicular from P

to line Q produced (if necessary) as PM, it can be seen that PM is the minimum (shortest) distance between the two particles.

Minimum separation between P and Q during motion $= l \sin\theta$



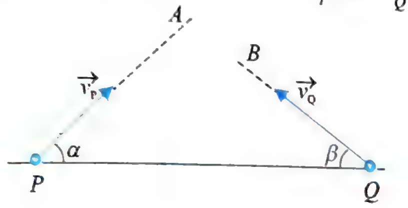
The time taken for the distance to be minimum is the time elapsed before the particle Q reaches point M, travelling with a speed equal to the magnitude of the relative velocity of Q w.r.t. P.

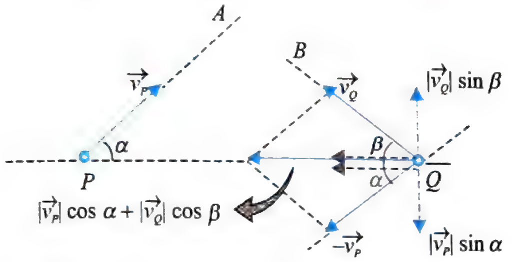
Time taken to reach minimum separation, $t_{\min} = \dfrac{PM}{|v_{PQ}|}$

$$|\vec{v}_{PQ}| = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos(\alpha+\beta)}$$

Condition for Two Particles to Collide

Consider, for example, two particles P and Q moving along PA and QB with respective velocities of $\vec{v}_P$ and $\vec{v}_Q$ as shown in figure.





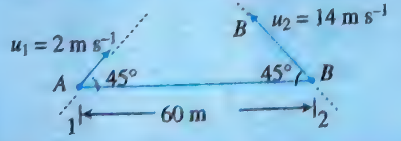
For two particles to collide, their relative velocity must be directed along the line joining them. Therefore, their relative velocity along the perpendicular to this line must be zero.

$$|\vec{v}_P|\sin\alpha = |\vec{v}_Q|\sin\beta$$

Finding time of collision $t = \dfrac{PQ}{|\vec{v}_{Q,P}|} = \dfrac{PQ}{|\vec{v}_P|\cos\alpha + |\vec{v}_Q|\cos\beta}$

ILLUSTRATION 5.65

Two particles are located on a horizontal plane at a distance 60 m. At $t = 0$ both the particles are simultaneously projected at angle 45° with velocities 2 m s^{-1} and 14 m s^{-1}, respectively.

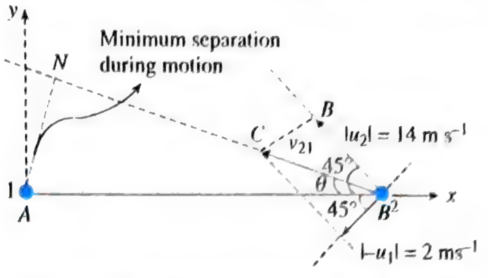


(a) Find the minimum separation between them during motion.

(b) At what time is the separation between them minimum?

Sol. In relative motion, observer considers himself at rest and observes the motion of object. Graphically, we can draw the direction of motion of particle 2 w.r.t. particle 1.

Both the particles are moving in gravitation field with same acceleration g. Hence, relative acceleration of particle 2 as seen from particle 1 will be zero. It means the relative velocity of the particle 2 w.r.t. particle 1 will be constant and will be equal to initial relative velocity. Graphically, we can draw the situation as shown in figure.



AN is the minimum separation between the particles and BN is the relative separation between the particles when the distance between 1 and 2 is shortest. From figure, we can write

$$v_{12}\cos\theta = 14\cos 45° + 2\cos 45° \quad \text{...(i)}$$

$$v_{12}\sin\theta = 14\sin 45° - 2\sin 45° \quad \text{...(ii)}$$

From (i) and (ii), $v_{12} = 10\sqrt{2}$ m s^{-1}

$$\cos\theta = \frac{4}{5} \text{ and } \sin\theta = \frac{3}{5}, \text{ as } \theta = 37°$$

Hence, minimum separation between the particles

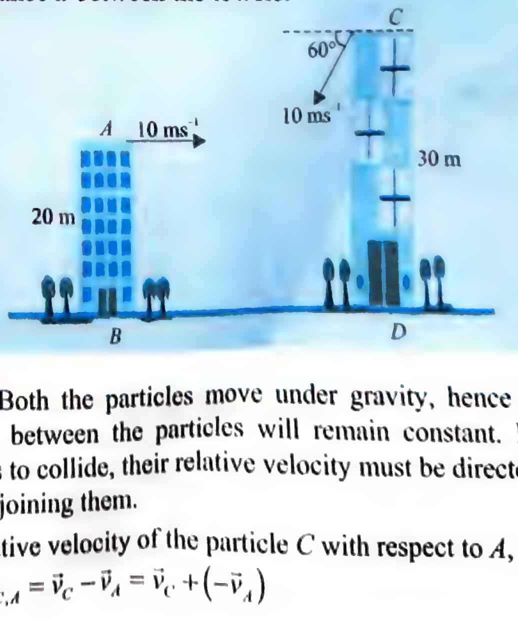
$$= AN = AB\sin\theta = 60 \times \frac{3}{5} = 36 \text{ m}$$

The time when separation between the particles is minimum,

$$t = \frac{BN}{|\vec{v}_{12}|} \Rightarrow t = \frac{60\cos 37°}{10\sqrt{2}} = \frac{12\sqrt{2}}{5}\text{ s}$$

ILLUSTRATION 5.66

Two towers AB and CD are situated at distance d apart as shown in figure. AB is 20 m high and CD is 30 m high from the ground. A particle is thrown from the top of AB horizontally with a velocity of 10 ms^{-1} towards CD. Simultaneously, another particle is thrown from the top of CD at an angle 60° to the horizontal towards AB with the same magnitude of initial velocity as that of the first object. The two particles moving in the same vertical plane collide in mid-air. Calculate the distance d between the towers.



Sol. Both the particles move under gravity, hence relative velocity between the particles will remain constant. For two particles to collide, their relative velocity must be directed along the line joining them.

The relative velocity of the particle C with respect to A,

$$\vec{v}_{C,A} = \vec{v}_C - \vec{v}_A = \vec{v}_C + (-\vec{v}_A)$$

In given figure $\vec{v}_{C,A}$ represents the relative velocity of the particle C with respect to A. As $|\vec{v}_C| = |\vec{v}_A|$, hence the angle between $\vec{v}_C$ and $\vec{v}_{C,A}$ should be 30°. Hence $\angle CAE = 30°$. Now in ΔCAE,

$$\tan 30° = \frac{10}{d} \Rightarrow d = \frac{10}{\tan 30°} = 10\sqrt{3}\text{ m}$$

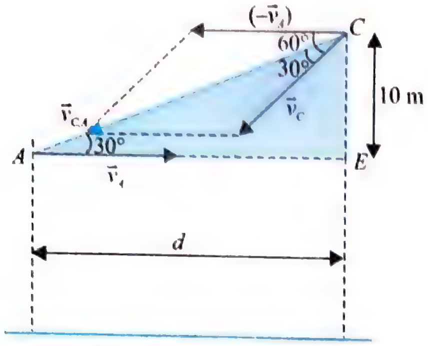


ILLUSTRATION 5.67

Two cars A and B are moving west to east and south to north, respectively, along crossroads. A moves with a speed of 20 m s^{-1} and is 500 m away from the point of intersection of cross roads and B moves with a speed of 15 m s^{-1} and is 400 m away from the point of intersection of cross roads. Find the shortest distance between them.

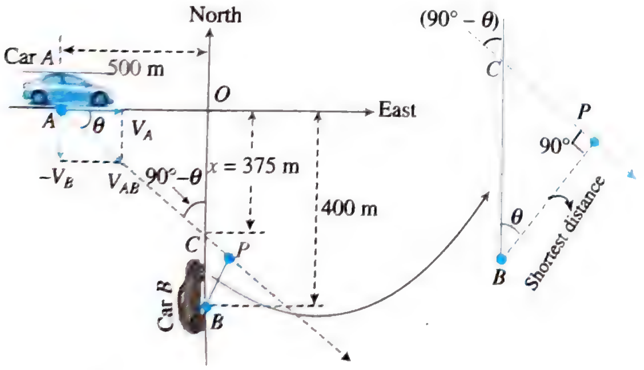
Sol. Method I (Using the concept of relative velocity)

Let us assume the velocity of A w.r.t. B. To do this, we plot the resultant velocity, $\vec{V}_{AB}$.

$$\vec{V}_{AB} = \vec{V}_A - \vec{V}_B = \vec{V}_A + (-\vec{V}_B)$$

As the accelerations of both the cars is zero, so the relative acceleration between them is also zero. Hence, the relative velocity will remain constant. So the path of A with respect to B will be straight line and along the direction of relative velocity of A with respect to B. The shortest distance between A and B is a perpendicular from B on the line of motion of A with respect to B. From figure,

$$\tan\theta = \frac{V_B}{V_A} = \frac{15}{20} = \frac{3}{4} \qquad \text{...(i)}$$



This θ is the angle made by the resultant velocity vector $|\vec{V}_{AB}|$ with the x-axis. Let $OC = x$

Also we know that from figure, In ΔAOC,

$$\tan\theta = \frac{x}{500} = \frac{3}{4} \qquad \text{...(ii)}$$

From equation (i) and (ii), we get $x = 375$ m.

$\therefore \quad BC = OB - OC = 400 - 375 = 25$ m

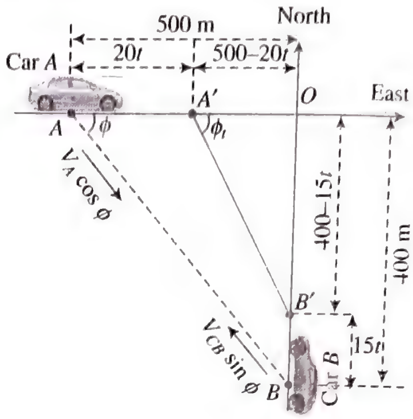
But the shortest distance is BP.

From diagram, it is clear that $BP = BC\cos\theta = 25 \times \frac{4}{5}$

$\therefore \quad BP = 20$ m

Method II (Using the concept of relative velocity of approach)

At any time t, let us plot the components of velocity of A and B in the direction along AB. From these components, we can find the velocity of approach between them at any time which is the relative velocity between them along line joining between instantaneous positions. When the distance between the two is minimum, the relative velocity of approach is zero.



$\therefore \quad V_A\cos\phi_t + V_B\sin\phi_t = 0$

(where α_t is the angle made by the line $A'B'$ with the x-axis)

$$20\cos\phi_t = -15\sin\phi_t$$

$$\therefore \quad \tan\phi_t = -\frac{20}{15} = -\frac{4}{3}$$

Here we should not confuse this angle (ϕ) with the angle θ in method (I) because that θ is the angle made by the net relative velocity with x-axis, but ϕ is the angle made by the line joining the two cars with x-axis when the velocity of approach is zero.

$$\frac{400 - 15t}{500 - 20t} = -\frac{4}{3}$$

$$t = \frac{128}{5}\text{ s}$$

So, $OB' = 16$ m and $OA' = -12$ m

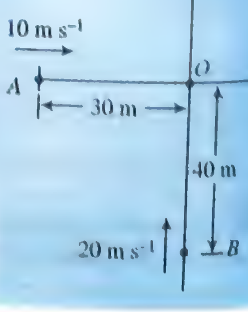
$$A'B' = \sqrt{16^2 + (-12)^2} = 20\text{ m}$$

ILLUSTRATION 5.68

Two particles A and B are moving with uniform velocity as shown in figure given below at $t = 0$.

(a) Will the two particles collide?

(b) Find out the shortest distance between two particles.



Sol. Observing A from the frame of B

Let us draw $\vec{v}_{AB}$,

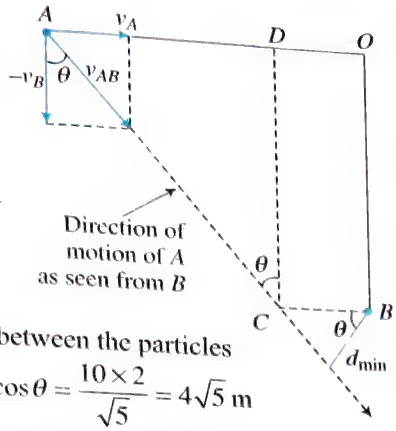
$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = \vec{v}_A + (-\vec{v}_B)$

$\tan\theta = \dfrac{10}{20} = \dfrac{1}{2}$

Again $\tan\theta = \dfrac{AD}{CD} = \dfrac{AD}{40} = \dfrac{1}{2}$

$\Rightarrow \quad AD = 20 \Rightarrow DO = 10$

$\Rightarrow \quad BC = 10$



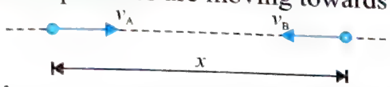
Hence, the shortest distance between the particles

$$d_{\text{short}} = BC\cos\theta = 10\cos\theta = \frac{10\times 2}{\sqrt{5}} = 4\sqrt{5}\text{ m}$$

Since the closest distance is non-zero, therefore, the particles will not collide.

VELOCITY OF APPROACH

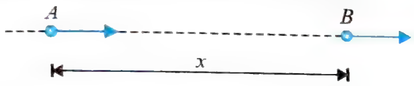
Let us consider two particles are moving towards each other.



We can say they are approaching towards each other. Then the relative velocity is called **velocity of approach.**

In this case $|\vec{v}_{\text{app}}| = v_A + v_B$ ...(i)

If both the particles move in same direction



If $v_A > v_B$, then we can say particle A is approaching towards particle B.

$|\vec{v}_{\text{app}}| = v_A - v_B$ ...(ii)

The separation between the particles will decrease with time. Here we can express the velocity of approach as

$$v_{\text{app}} = -\frac{dx}{dt} \qquad \text{...(iii)}$$

Here x is the separation between the particles at any instant.

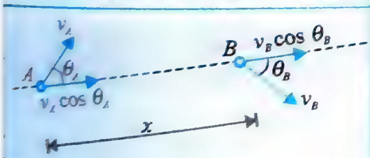
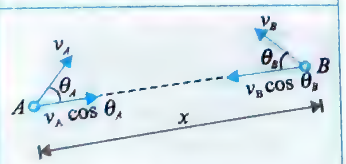
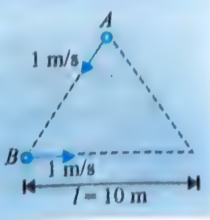
Situation 1	Situation 2
	
Velocity of approach of A towards particle B	Velocity of approach of A towards particle B
$v_A\cos\theta_A - v_B\cos\theta_B = -\dfrac{dx}{dt}$	$v_A\cos\theta_A + v_B\cos\theta_B = -\dfrac{dx}{dt}$

ILLUSTRATION 5.69

At $t = 0$, two particles A and B start moving with same speed along the sides of an equilateral triangle as shown in figure. Find the velocity of approach of A w.r.t. B at time (i) $t = 0$ and (ii) $t = 5$ sec



Sol.

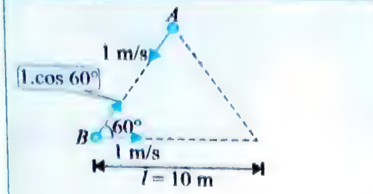
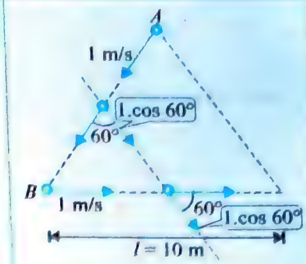
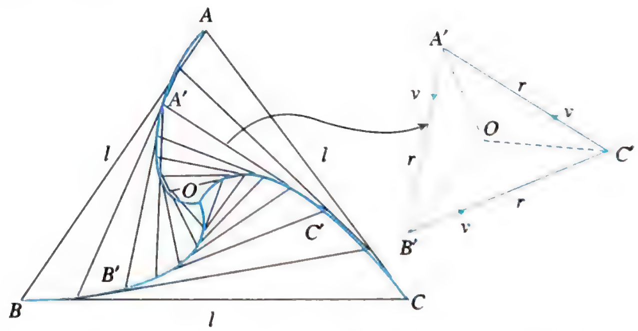
(i) $t = 0$	(ii) $t = 5$ sec
	
Velocity of approach of A w.r.t. B $= 1 + 1.\cos 60° = 1 + \dfrac{1}{2} = 1.5\text{ m/s}$	Velocity of approach of A w.r.t. B $= 1.\cos 60° - 1.\cos 60° = 0$

ILLUSTRATION 5.70

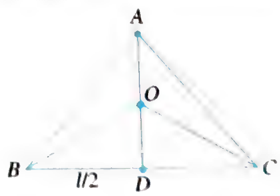
Three particles A, B and C are situated at the vertices of an equilateral triangle of side l at $t = 0$. The particle A heads towards B, B towards C, C towards A with constant speeds v. Find the time of their meeting.

Sol. The motion of the particles is roughly sketched in figure. By symmetry they will meet at the centroid O of the triangle. As the speed of all the particles is equal they will cover equal distance in any given interval of time.

If we join the instantaneous position of the particle at any time, the particles will form an equilateral triangle of same centroid as initial triangle.



Let us consider the motion of any one particle say A. At any instant, its velocity makes an angle 30° with line AO.



The component of the velocity along AO is $v\cos 30°$. This component will be equal to the rate of change of distance between A and O.

At $t = 0$, distance between A and O,

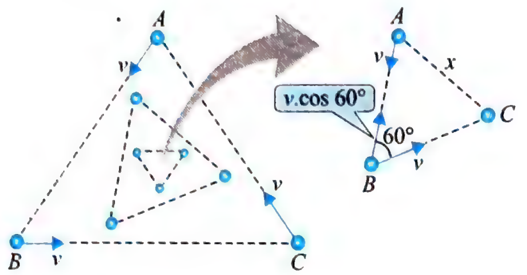
$$AO = \frac{2}{3}AD = \frac{2}{3}\sqrt{l^2 - \left(\frac{l}{2}\right)^2} = \frac{l}{\sqrt{3}}$$

At time $t = T$, the separation between A and O is zero.
Hence, time taken for AO to become zero,

$$T = \frac{(l/\sqrt{3})}{v\cos 30^\circ} = \frac{(l/\sqrt{3})}{v(\sqrt{3}/2)} = \frac{2l}{3v}$$

Second approach

As both the particles are approaching towards each other, here we can apply velocity of approach. Let the side of the equilateral triangle formed by joining the instantaneous positions of the particles be x. Then from the figure,



Velocity of approach of A w.r.t. B at given instant

$$v + v\cos 60^\circ = -\frac{dx}{dt} \Rightarrow \frac{3}{2}v = -\frac{dx}{dt}$$

or $$\frac{3}{2}v.dt = -dx$$

Integrating both sides, $\frac{3}{2}v\int_0^T dt = -\int_l^0 dx$

$$\frac{3}{2}vT = -[x]_l^0$$

$$\frac{3}{2}vT = -[0-l]$$

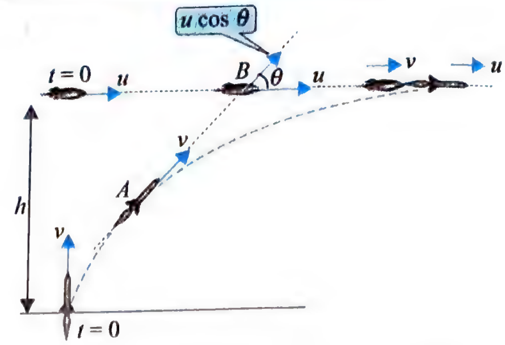
$$\frac{3}{2}vT = l \Rightarrow T = \frac{2l}{3v}$$

ILLUSTRATION 5.71

A rocket moves horizontally with a constant velocity u at a height h. A guided missile is fired vertically with a speed v when the rocket passes above it.

Assuming that the missile always aims at the rocket with the constant speed v, find the time after the missile strikes the rocket.

Sol. Let at any instant the velocity vector of missile makes an angle θ with the velocity vector of the rocket and the time after missile strikes the rocket is t.



Let r = distance of separation between the rocket and missile after a time t.

Velocity of approach of A w.r.t. B at given instant

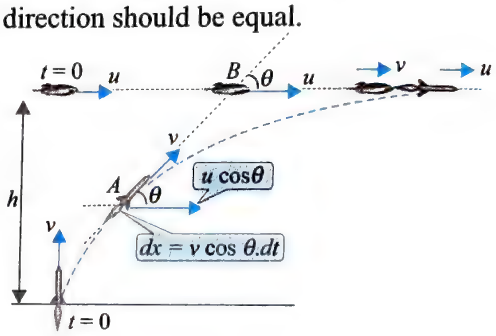
$$(v - u\cos\theta) = -\frac{dr}{dt}$$

or $$(v - u\cos\theta)dt = -dr$$

Integrating above equation both sides, $\int_0^T v dt - \int_0^T u\cos\theta dt = -\int_h^0 dr$

$$v.T - u\int_0^T \cos\theta dt = h \qquad \text{...(i)}$$

When missile strikes the rocket, the displacement of both in horizontal direction should be equal.



$$\int_0^T v\cos\theta dt = u.T \Rightarrow v\int_0^T \cos\theta dt = u.T$$

or $$\int_0^T \cos\theta dt = \frac{uT}{v} \qquad \text{....(ii)}$$

From (i) and (ii), $v.T - u\left(\frac{uT}{v}\right) = h \Rightarrow T = \frac{hv}{v^2 - u^2}$

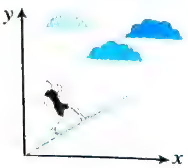
CONCEPT APPLICATION EXERCISE 5.5

1. A 400-m wide river is flowing at a rate of 2.0 m s^{-1}. A boat is sailing with a velocity of 10 m s^{-1} with respect to the water, in a direction perpendicular to the river.
   (a) Find the time taken by the boat to reach the opposite bank.
   (b) How far from the point directly opposite to the starting point does the boat reach the opposite bank?
   (c) In what direction does the boat actually move?
2. A man wishing to cross a river flowing with velocity u jumps at an angle θ with the river flow.
   (a) Find the net velocity of the man with respect to ground if he can swim with speed v in still water.
   (b) In what direction does the boat actually move?

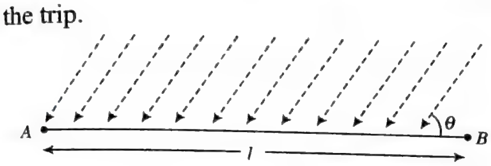
(c) Find how far from the point directly opposite to the starting point does the boat reach the opposite bank, if the width of the river is d.

3. Find the time an aeroplane having velocity v takes to fly around a square with side a if the wind is blowing at a velocity u along one side of the square.

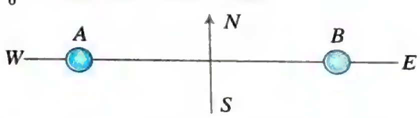
4. To a man running upwards on the hill, the rain appears to fall vertically downwards with 4 m s^{-1}. The velocity vector of the man w.r.t. earth is $(2\hat{i}+3\hat{j})\,\text{m s}^{-1}$. If the man starts running down the hill with the same speed, then determine the relative speed of the rain w.r.t. man.



5. An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is l and the aeroplane maintains the constant speed v w.r.t. wind. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.



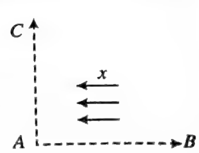
6. A pilot is supposed to fly a certain distance AB, due east from A to B and then due west from B to A. The velocity of plane is v and that of air is u. The time for the round trip is t_0 in still air. Show that



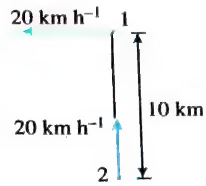
(a) if the air velocity be due east or west, the time for round trip will be $t_1 = \dfrac{t_0}{\left(1-\dfrac{u^2}{v^2}\right)}$.

(b) if the air velocity is due north or south, the time for a round trip will be $t_2 = \dfrac{t_0}{\sqrt{\left(1-\dfrac{u^2}{v^2}\right)}}$.

7. Wind blows with a velocity x in the direction shown in figure. Two aeroplanes start out from point A and fly with a constant speed y. The first flies from A to B and the other along A to C. Both of them return back to A. If $AB = AC$, then which plane will return to point A first, and what will be the ratio of the times of flight of the two planes?



8. Two ships are 10 km apart on a line joining south to north. The one farther north is steaming west at 20 km h^{-1}. The other is steaming north at 20 km h^{-1}. What is their distance of closest approach? How long do they take to reach it?



9. A boatman finds that he can save 6 s in crossing a river by the quickest path than by the shortest path. If the velocity of the boat and the river be, respectively, 17 m s^{-1} and 8 m s^{-1}, find the river width.

10. A man directly crosses a river in time t_1 and swims down the current a distance equal to the width of the river in time t_2. If u and v be the speed of the current and the man respectively, show that $t_1 : t_2 :: \sqrt{v+u} : \sqrt{v-u}$.

11. A person rows a boat across a river making an angle of 60° with the downstream. Find the percentage time he would have saved, had he crossed the river in the shortest possible time.

12. A person, intending to cross a river by the shortest path, starts at an angle α with the downstream. If the speed of the person be less than that of water current, show that α must be obtuse.

ANSWERS

1. (a) 40 s **(b)** 80 m

(c) $\tan^{-1} 5$, (downstream) with the river flow.

2. (a) $\sqrt{u^2+v^2+2vu\cos\theta}$ **(b)** $\dfrac{v\sin\theta}{u+v\cos\theta}$

(c) $(u+v\cos\theta)\dfrac{d}{v\sin\theta}$

3. $\dfrac{2a}{v^2-u^2}\left(v+\sqrt{v^2-u^2}\right)$ **4.** $\sqrt{20}$ m s^{-1}

5. $\dfrac{2vl\sqrt{1-\dfrac{u^2\sin^2\theta}{v^2}}}{v^2-u^2}$

7. $\dfrac{1}{\sqrt{1-x^2/y^2}} > 1$; first plane takes more time.

8. $5\sqrt{2}$ km, 15 min **9.** 765 m **11.** 15.47%

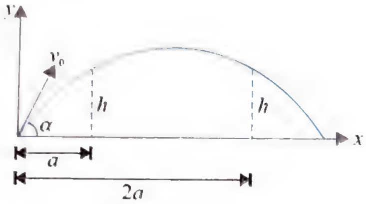
Solved Examples

EXAMPLE 5.1

A particle is projected from a point on the level ground and its height is h when at horizontal distances a and $2a$ from its point of projection. Find the velocity of projection.

Sol.

If v_0 is the velocity of projection and α the angle of projection, the equation of trajectory is



$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha} \qquad \text{...(i)}$$

With origin at the point of projection,

$$gx^2 - 2v_0^2\sin\alpha\cos\alpha\cdot x + 2v_0^2\cos^2\alpha . y = 0 \qquad \text{...(ii)}$$

Since the projectile passes through two points (a, h) and $(2a, h)$, then a and $2a$ must be roots of equation (ii).

Sum of roots,

$$a + 2a = \frac{2v_0^2\sin\alpha\cos\alpha}{g} \qquad \text{...(iii)}$$

Multiplication of roots,

$$\text{and} \quad a\times 2a = \frac{2v_0^2\cos^2\alpha h}{g} \qquad \text{... (iv)}$$

Dividing equations (iii) by (iv), we get

$$\frac{3a}{2a^2} = \frac{\tan\alpha}{h} \quad \text{or} \quad \tan\alpha = \frac{3h}{2a}$$

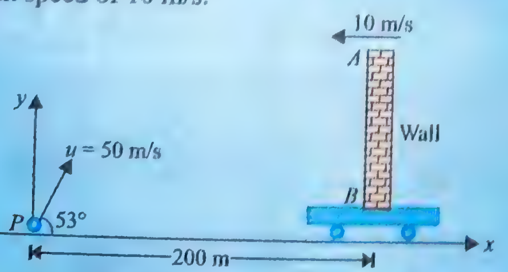
From equation (iv),

$$v_0^2 = \frac{ga^2}{h}\sec^2\alpha = \frac{ga^2}{h}(1+\tan^2\alpha) = \frac{ga^2}{h}\left(1+\frac{9h^2}{4a^2}\right)$$

$$= \frac{g}{4}\left(\frac{4a^2}{h}+9h\right) \quad \text{or} \quad v_0 = \frac{1}{2}\sqrt{\left(\frac{4a^2}{h}+9h\right)g}$$

EXAMPLE 5.2

A ball is fired from point P, with an initial speed of 50 m/s at an angle of 53° with the horizontal. At the same time, a long wall AB at 200 m from point P, starts moving towards P with a constant speed of 10 m/s.



(a) Find the time when the ball collides with wall AB.

(b) Find the coordinates of point C where the ball collides, taking point P as origin.

Sol.

(a) Horizontal component of relative velocity of the ball w.r.t wall

$$\vec{v}_{b,w} = \vec{v}_b - \vec{v}_w = 50\cos 53° - (-10) = 50\times\frac{3}{5}+10 = 40\,\text{m/s}$$

Hence the time when the ball hits the wall

$$t = \frac{200}{|\vec{v}_{b,w}|} = \frac{200}{4} = 5\,\text{sec}$$

(b) During 5 seconds, the displacement of the ball in x-direction

$$x = (u\cos 53°)\times 5 = 50\times\frac{3}{5}\times 5 = 150\,\text{m}$$

Hence, x-coordinate of point C = 150 m

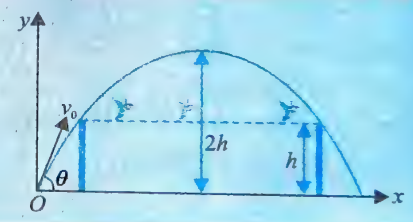
For finding y-coordinate we can use,

$$x = (u\sin 53°)\times 5 - \frac{1}{2}10\times 5^2 = 75\,\text{m}$$

Hence the coordinates of point C (150 m, 75 m).

EXAMPLE 5.3

A stone is projected from a point on the ground in such a direction so as to hit a bird on the top of a telegraph post of height h, and then attain a height $2h$ above the ground. If, at the instant of projection, the bird were to fly away horizontal with a uniform speed, find the ratio of the horizontal velocities of the bird and the stone if the stone still hits the bird.



Sol. Maximum height of the projectile is given by the expression

$$h_{\max} = \frac{v_0^2\sin^2\theta}{2g} \quad \Rightarrow \quad 2h = \frac{v_0^2\sin^2\theta}{2g}$$

$$\Rightarrow \quad v_0 = \frac{2\sqrt{gh}}{\sin\theta} \qquad \text{...(i)}$$

$$\because \quad y = v_0\sin\theta t - \frac{1}{2}gt^2 \Rightarrow h = \frac{2\sqrt{gh}}{\sin\theta}\sin\theta t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad gt^2 - 4\sqrt{gh}t + 2h = 0$$

$$\Rightarrow \quad t = \frac{4\sqrt{gh}\pm\sqrt{16gh-8gh}}{2g} \Rightarrow t = \frac{4\sqrt{gh}\pm 2\sqrt{2gh}}{2g}$$

$$\Rightarrow \quad t_1 = \sqrt{\frac{h}{g}}\left(2-\sqrt{2}\right) \Rightarrow t_2 = \sqrt{\frac{h}{g}}\left(2+\sqrt{2}\right)$$

At these two times the projectile is at the same height as the bird. Let v be the speed of the bird, for bird to be hit.

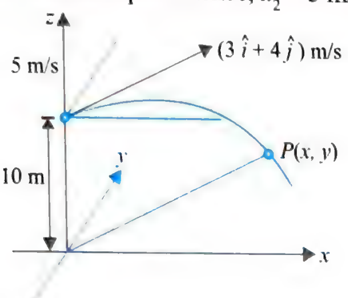
$$vt_2 = v_0\cos\theta\,(t_2 - t_1)$$

$$\Rightarrow \quad \frac{v}{v_o\cos\theta} = \frac{t_2 - t_1}{t_2} = \frac{2}{\sqrt{2}+1}$$

EXAMPLE 5.4

A helicopter is moving vertically upwards with a velocity 5 m/s. When the helicopter is at a height 10m from ground, a stone is thrown with a velocity $(3\hat{i}+4\hat{j})$ m/s from the helicopter w.r.t. the man in it. Considering the point on ground vertically below the helicopter as the origin of coordinates, and the ground below as xy plane, find the coordinates of the point where the stone will fall. Its distance from origin and the distance between the helicopter and the stone, at the instant the stone strikes the ground, assuming helicopter moves upwards with constant velocity.

Sol. The initial velocity of the stone in vertical direction will be equal to velocity of the helicopter. Hence, $u_z = 5$ m/s.



Now applying initially, $s_z = u_z t + \frac{1}{2}a_z t^2$

Here $s_z = -10$ m and $a_z = -10 \text{ m/s}^2$

$\Rightarrow \quad -10 = 5t - \frac{1}{2}10t^2 \Rightarrow t^2 - t - 2 = 0$

$\Rightarrow \quad t = 2$ s = time of flight of stone

In horizontal direction, the velocity of stone will remain constant as helicopter is moving vertically.

The velocity of stone in horizontal direction = velocity of the stone w.r.t helicopter.

The stone falls at P. The x-coordinate of point P, $x = u_x t = 3 \times 2 = 6$ m

and the y-coordinate of point P, $y = u_y t = 4 \times 2 = 8$ m

EXAMPLE 5.5

Show that the notion of one projectile as seen from another projectile will always be a straight line.

Sol. If (x_A, y_A) is the position of projectile A after time t projected with a velocity v_0^A, then

$$x_A = v_{x0}^A t \text{ and } y_A = v_{y0}^A t - \frac{1}{2}gt^2$$

Similarly, for the projectile B with projection velocity v_0^B,

$$x_B = v_{x0}^B t \text{ and } y_B = v_{y0}^B t - \frac{1}{2}gt^2$$

Thus, $(x_B - x_A) = (v_{x0}^B - v_{x0}^A)$

and $(y_B - y_A) = (v_{y0}^B - v_{y0}^A)$

$$\frac{y_B - y_A}{x_B - x_A} = \frac{v_{y0}^B - v_{y0}^A}{v_{x0}^B - v_{x0}^A} = m(\text{say})$$

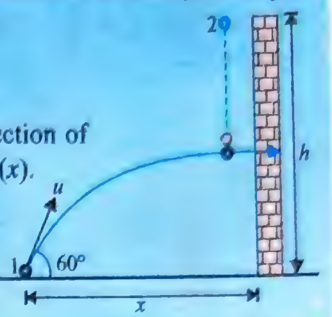
Let (x, y) represents the position of projectile B relative to projectile A. Then $x_B - x_A = x$ and $y_B - y_A = y$

Thus, $\frac{y}{x} = m$ or $y = mx$, which is the equation of a straight line.

EXAMPLE 5.6

Ball I is thrown towards a tower at an angle of 60° with the horizontal with unknown speed (u). At the same moment, ball II is released from the top of tower as shown. Both balls collide after two seconds and at the moment of collision, velocity of ball I is horizontal. Find

(a) speed u.
(b) distance of point of projection of ball I from base of tower (x).
(c) height of tower



Sol.

(a) As at the time of collision ball I is moving horizontally. It means at the time of collision the ball I is at the highest position of its path. Time taken by ball I to reach its highest position should be half the time of flight ($T/2$).

$$t = \frac{1}{2}\left(\frac{2u\sin 60°}{g}\right) = 2 \quad \text{or} \quad u = \frac{40}{\sqrt{3}} \text{ m/s}$$

(b) Distance of point of projection from base of the tower

$$x = (u\cos 60°)t = \left(\frac{40}{\sqrt{3}}\right)\times\left(\frac{1}{2}\right)\times 2 = \frac{40}{\sqrt{3}} \text{ m}$$

(c) Height ascended by ball I in 2 s:

$$h_1 = u\sin 60° \times 2 - \frac{1}{2}10(2)^2 = 20 \text{ m}$$

Height descended by ball II in 2 s:

$$h_2 = \frac{1}{2}gt^2 = \frac{1}{2}10(2)^2 = 20 \text{ m}$$

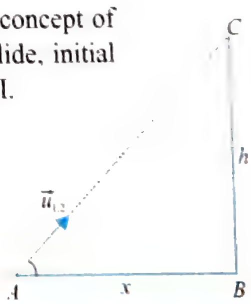
Now, $h = h_1 + h_2 = 20 + 20 = 40$ m

Second Approach:

The height of tower can be found using concept of relative velocity. If the balls have to collide, initial velocity of ball I should be towards ball II.

For this $\frac{h}{x} = \tan 60°$

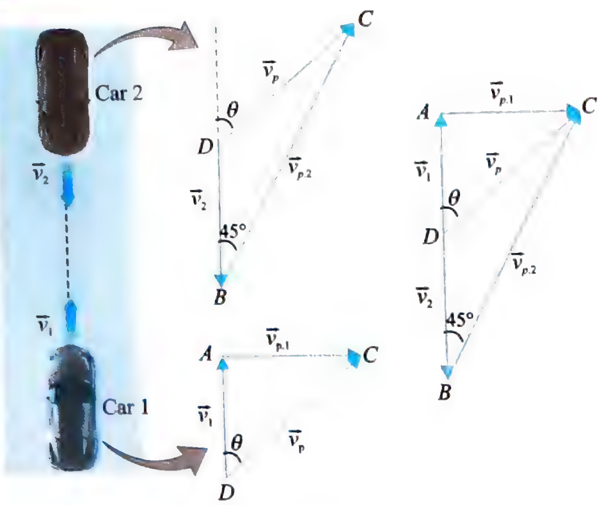
$$\Rightarrow \quad h = x\tan 60° = \frac{40}{\sqrt{3}}\sqrt{3} = 40 \text{ m}$$



EXAMPLE 5.7

An airplane is observed by two observers traveling at 60 km/hour in two vehicles moving in opposite directions on a straight road. To an observer, in one vehicle, the plane appears to cross the road track at right angles while to the other, it appears to be 45°. At what angle does the plane actually cross the road track and what is its speed relative to ground?

Sol.



Let be $\vec{v}_p$ the velocity of plane relative to the ground, at angle θ to velocity $\vec{v}_1$ of observer in car 1.

In case (i), $\vec{v}_{p_1} = \vec{v}_p - \vec{v}_1$

$$\vec{v}_p = \vec{v}_{p_1} + \vec{v}_1$$

Vector diagram is shown in the figure. Note that according to observer in car 1 the plane crosses the road at right angles.

Similarly, in case 2, $\vec{v}_p = \vec{v}_{p2} + \vec{v}_2$

We can combine figures (a) and (b) $\tan 45° = \dfrac{AC}{AB}$

$$v_{p_1} = (v_1 + v_2)\tan 45° = 120 \times 1 = 120 \text{ km/hr}$$

In ΔDAC, $CD^2 = AD^2 + AC^2$

$$v_p^2 = v_1^2 + v_{p,1}^2 = 60^2 + 120^2 \Rightarrow v_p = 60\sqrt{5} \text{ km/h}$$

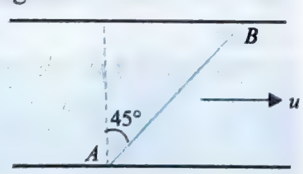
$$\tan\theta = \frac{v_{p_1}}{v_1} = \frac{120}{60} = 2$$

Hence $\theta = \tan^{-1} 2$

EXAMPLE 5.8

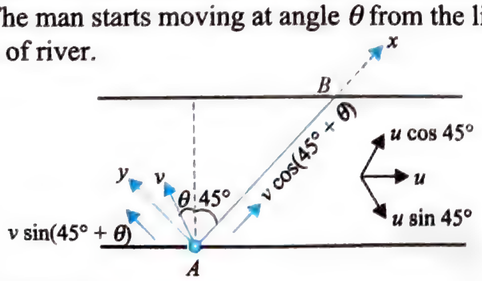
A man wants to reach point B on the opposite bank of a river flowing at a speed as shown in the figure.

What minimum speed relative to water should the man have so that he can reach directly to point B? In which direction should he swim?



Sol.

Let us consider the direction AB as x-axis and normal to AB line as y-axis. The man starts moving at angle θ from the line normal to the bank of river.



Velocity of the man w.r.t river

$$\vec{v}_{m,w} = v\cos(45° + \theta)\hat{i} + v\sin(45° + \theta)\hat{j}$$

Velocity of water $\vec{v}_w = u\cos 45°\hat{i} - u\sin 45°\hat{j}$

Hence velocity of the man, $\vec{v}_m = \vec{v}_{m,w} + \vec{v}_w$

$$\vec{v}_m = \left(v\cos(\theta + 45°)\hat{i} + v\sin(\theta + 45°)\hat{j}\right) + \left(u\cos 45°\hat{i} - u\sin 45°\hat{j}\right)$$

$$\vec{v}_m = (v\cos(\theta + 45°) + u\cos 45°)\hat{i} + (v\sin(\theta + 45°) - u\sin 45°)\hat{j}$$

To reach point B, y-component of the man's velocity should be zero.

$$v\sin(\theta + 45°) = u\sin 45° \Rightarrow v = \frac{v\sin 45°}{\sin(\theta + 45°)} \quad \text{...(i)}$$

For minimum speed of the man relative to water,

$$\theta + 45° = 90° \Rightarrow \theta = 45°$$

Substituting $\theta = 45°$ in (i), we get $v_{\min} = \dfrac{v\sin 45°}{\sin(45° + 45°)} = \dfrac{u}{\sqrt{2}}$

Hence, the minimum speed relative to water should the man have so that he can reach directly to point B is $u/\sqrt{2}$, at angle 45° with upstream.

EXAMPLE 5.9

A man wants to cross a river 500 m wide. The rowing speed of the man relative to water is 3 km/hr and river flows at the speed of 2 km/hr. If man's walking speed on the shore is 5 km/hr, then in which direction he should start rowing in order to reach the directly opposite point on the other bank in shortest time.

Sol. Velocity of the man relative to ground,

$$\vec{v}_m = (u - v\sin\theta)\hat{i} + v\cos\theta\,\hat{j}$$

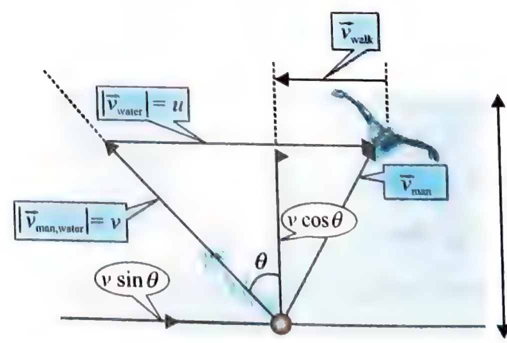
Hence, time taken by the man to cross the river is

$$t_1 = \frac{d}{v\cos\theta} = \frac{1}{2(v\cos\theta)}$$

$\therefore$ Drift of the man along the river is

$$x = (u - v\sin\theta)t_1 = (u - v\sin\theta)\frac{1}{2(v\cos\theta)}$$

$$\Rightarrow \quad x = (2 - 3\sin\theta)\frac{1}{2(3\cos\theta)} = \frac{1}{3}\sec\theta - \frac{1}{2}\tan\theta$$



Time taken by the man to cover the distance,

$$t_2 = \frac{x}{v_{walk}} = \frac{\frac{1}{3}\sec\theta - \frac{1}{2}\tan\theta}{5} = \frac{1}{15}\sec\theta - \frac{1}{10}\tan\theta$$

$$t_1 = \frac{1}{2(v\cos\theta)} = \frac{\sec\theta}{6}, \quad t_2 = \frac{1}{15}\sec\theta - \frac{1}{10}\tan\theta$$

Therefore, total time $T = t_1 + t_2 = \dfrac{1}{6}\sec\theta + \left(\dfrac{1}{15}\sec\theta - \dfrac{1}{10}\tan\theta\right)$

$$\Rightarrow \quad T = \frac{7}{30}\sec\theta - \frac{1}{10}\tan\theta$$

For T to be minimum, $\frac{dT}{d\theta}=0$

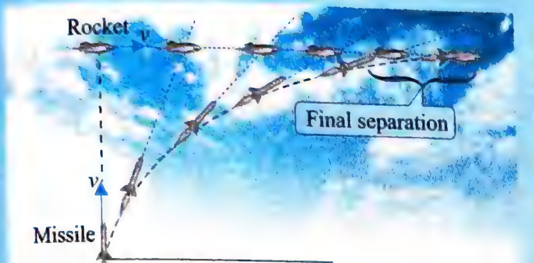
$$\frac{dT}{d\theta}=\frac{7}{30}\sec\theta\tan\theta-\frac{1}{10}\sec^2\theta=0$$

$$\frac{7}{30}\sec\theta\tan\theta=\frac{1}{10}\sec^2\theta \Rightarrow \frac{7}{3}\tan\theta=\sec\theta$$

$$\Rightarrow \sin\theta=\frac{3}{7}\Rightarrow \theta=\sin^{-1}\left(\frac{3}{7}\right)$$

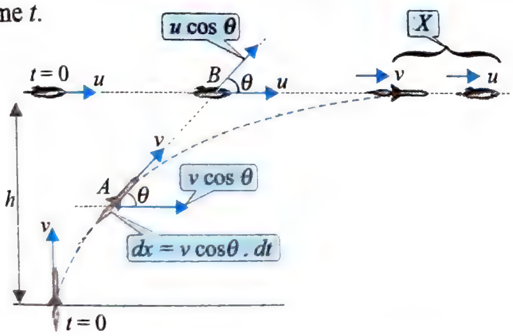
EXAMPLE 5.10

A rocket moves horizontally with a constant velocity v at a height h. A guided missile is fired vertically with same speed v when the rocket passes above it. Assuming that the missile always aims at the rocket with the constant seed u, find the final separation between both.



Sol. Let at any instant the velocity vector of missile makes an angle θ with the velocity vector of the rocket and the final separation between both is X.

Let r = distance of separation between the rocket and missile after a time t.



Velocity of approach of A w.r.t. B at given instant

$$(v-v\cos\theta)=-\frac{dr}{dt}$$

$$\Rightarrow \quad (v-v\cos\theta).dt=-dr$$

$$\int_0^T v.dt-\int_0^T v\cos\theta.dt=-\int_h^X dr$$

$$v.T-v.\int_0^T \cos\theta.dt=h-X \qquad \text{...(i)}$$

The difference in the displacements of rocket and horizontal component of displacement of missile is equal to final separation.

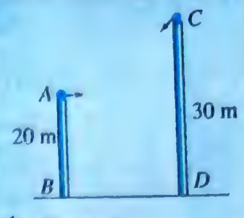
$$v.T-\int_0^T v\cos\theta.dt=X$$

or
$$v.T-v\int_0^T \cos\theta.dt=X \qquad \text{...(ii)}$$

From (i) and (ii), $h-x=x\Rightarrow x=\frac{h}{2}$

EXAMPLE 5.11

Two towers AB and CD are situated at distance d apart as shown in figure. AB is 20 m high and CD is 30 m high from the ground. An object of mass m is thrown from the top of AB horizontally with a velocity of 10 ms^{-1} towards CD. Simultaneously, another object of mass $2m$ is thrown from the top of CD at an angle 60° to the horizontal towards AB with the same magnitude of initial velocity as that of the first object. The two objects move in the same vertical plane, collide in mid-air, and stick to each other.



(a) Calculate the distance d between the towers.

(b) Find the position where the objects hit the ground.

Sol. **Method 1:**

(a) Let t be the time taken for collision. For mass m thrown horizontally from A for horizontal motion,

$$PM=10t \qquad \text{...(i)}$$

For vertical motion, $u_y=0$, $s_y=y$, $a_y=g$

$$S_y=u_yt+\frac{1}{2}a_yt^2$$

$$\Rightarrow \quad y=\frac{1}{2}gt^2 \qquad \text{...(ii)}$$

$$v_y=u_y+a_yt=gt \qquad \text{...(iii)}$$

For mass $2m$ thrown from C, for horizontal motion

$$QM=[10\cos 60°]t$$

$$\Rightarrow \quad QM=5t \qquad \text{...(iv)}$$

For vertical motion: $u_y=10\sin 60°=5\sqrt{3}$

$$\Rightarrow \quad v_y=5\sqrt{3}+gt \qquad \text{...(v)}$$

$$a_y=g,\ S_y=y+10,\ S=ut+\frac{1}{2}at^2$$

$$\Rightarrow \quad y+10=5\sqrt{3}\,t+\frac{1}{2}gt^2 \qquad \text{...(vi)}$$

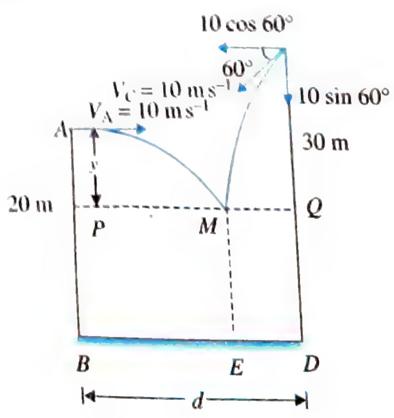
From (ii) and (vi),

$$\frac{1}{2}gt^2+10=5\sqrt{3}\,t+\frac{1}{2}gt^2\Rightarrow t=\frac{2}{\sqrt{3}}\text{s}$$

$$BD=PM+MQ=10t+5t$$

$$=15t=15\times\frac{2}{\sqrt{3}}=10\sqrt{3}=17.32\text{ m}$$

(b) Applying the conservation of linear momentum (during collision of the masses at M) in the horizontal direction, we have m × 10 − 2 m 10 cos 60°

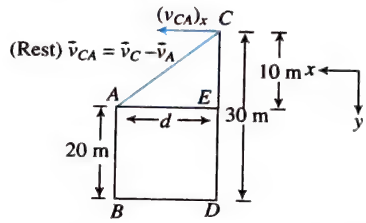


$= 3\,m \times v_x \quad \Rightarrow \quad 10\text{ m} - 10\text{ m} = 3\text{ m} \times v_x \quad \Rightarrow v_x = 0$

Since the horizontal momentum comes out to be zero, the combination of masses will drop vertically downwards and fall at E, $BE = PM = 10t = 10 \times \dfrac{3}{\sqrt{3}} = 11.547$ m

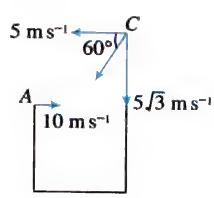
Method 2:

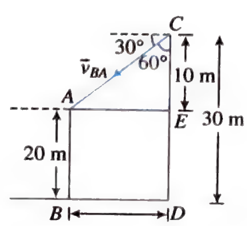
Acceleration of A and C both is 9.8 m s^{-2} downward.



Therefore, the relative acceleration between them is zero, i.e., the relative motion between them will be a straight line.

Now assuming A to be at rest, the condition of collision will be that $\vec{v}_{CA} = \vec{v}_C - \vec{v}_A$ = relative velocity of C w.r.t. A should be along CA.





$$\vec{v}_A = 10\hat{i}$$

$$\vec{v}_B = -5\hat{i} - 5\sqrt{3}\,\hat{j}$$

$$\vec{v}_{BA} = -5\hat{i} - 5\sqrt{3}\,\hat{j} - 10\hat{i}$$

$$\therefore \quad \vec{v}_{BA} = -15\hat{i} - 5\sqrt{3}\,\hat{j}$$

$$\therefore \quad \tan 60° = \frac{d}{10} \quad \text{or} \quad d = 10\sqrt{3}\text{ m}$$

EXAMPLE 5.12

Two guns, situated on the top of a hill of height 10 m, fire one shot each with the same speed $5\sqrt{3}$ m s^{-1} at some interval of time. One gun fires horizontally and the other fires upward at an angle of 60° with the horizontal. The shots collide in air at point P. Find (a) the time interval between the fringes and (b) the coordinates of point P. Take the origin of the coordinate system at the foot of the hill right below the muzzle and trajectories in the x-y plane.

Sol. **Method 1:**

For bullet *A*. Let t be the time taken by bullet A to reach P.

Vertical motion: $u_y = 0$, $S = ut + \dfrac{1}{2}at^2$, $S_y = -(10 - y)$

$$a_y = -10\text{ m s}^{-2}, \quad \Rightarrow \quad 10 - y = 5t^2 \qquad \text{...(i)}$$

Horizontal motion: $x = 5\sqrt{3}\,t$ ...(ii)

For bullet *B*. Let $(t + t')$ be the time taken by bullet B to reach P.

Vertical motion: $\uparrow + u_y = +5\sqrt{3}\sin 60° = +7.5$ m s^{-1}

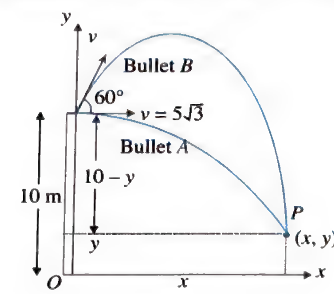
$$S = ut + \frac{1}{2}at^2;\ a_y = -10\text{ m s}^{-2}$$

$$y - 10 = 7.5(t + t') - 5(t + t')^2 \qquad \text{...(iii)}$$

Horizontal motion: $x = (5\sqrt{3}\cos 60°)(t + t')$

$$\Rightarrow \quad 5\sqrt{3}\,t + 5\sqrt{3}\,t' = 2x \qquad \text{...(iv)}$$

Substituting the value of x from (ii) in (iv), we get



$$5\sqrt{3}\,t + 5\sqrt{3}\,t' = 10\sqrt{3}\,t \quad \Rightarrow \quad t = t'$$

(a) Putting $t = t'$ in (iii),

$y - 10 = 15\,t - 20t^2$ ...(v)

Adding (i) and (v), $\quad 0 = 15t - 15\,t^2 \Rightarrow t = 1$ s

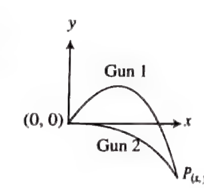
So time interval between the firings is $t' = 1$ s

(b) Putting $t = 1$ in (ii), we get $x = 5\sqrt{3}$ m.

Putting $t = 1$ in (i), we get $y = 5$.

Therefore, the coordinates of point P are $(5\sqrt{3}, 5)$ in meters.

Method 2: We take the point of firing as origin and x- and y-axis as shown in figure. Equation of trajectory of the projectile is



$$y = x\tan\theta - \frac{gx^2}{2v_i^2\cos^2\theta}$$

For gun 1, $\theta = 60°$. Therefore,

$$y = x\tan 60° - \frac{gx^2}{2v_i^2\cos^2 60°} = x\sqrt{3} - \frac{2gx^2}{v_i^2}$$

For gun 2, $\theta = 0°$. Therefore,

$$y = \frac{-gx^2}{2v_i^2}$$

Two shots collide at point P. Therefore, their coordinates must be same.

$$\frac{-gx^2}{2v_i^2} = x\sqrt{3} - \frac{2gx^2}{2v_i^2}$$

$$x\sqrt{3} = \frac{2gx^2}{v_i^2} - \frac{gx^2}{2v_i^2} = \frac{3gx^2}{2v_i^2}$$

$$x = 0 \quad \text{and} \quad x = \frac{2v_i^2}{\sqrt{3}g} = \frac{2(5\sqrt{3})^2}{\sqrt{3}(10)} = 5\sqrt{3}\text{ m}$$

$$y = \frac{-gx^2}{2v_i^2} = \frac{-10(5\sqrt{3})^2}{2(5\sqrt{3})^2} = -5\text{ m}$$

If the origin is assigned at ground, the coordinates of point P will be $(5\sqrt{3}\text{ m}, 5\text{ m})$.

Now we consider the x-component of displacement for both the shots.

Gun 1: $x = 5\sqrt{3}\text{ m} = v_i t = (5\sqrt{3}\text{ m s}^{-1})t$

or $t_1 = 1\text{ s}$

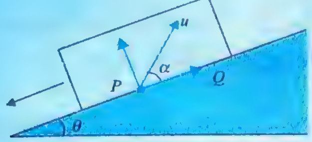
Gun 2: $x = 5\sqrt{3}\text{ m} = v_i \cos 60° t_2 = \dfrac{5\sqrt{3}}{2} t_2$

or $t_2 = 2\text{ s}$

Time interval between two shots, $\Delta t = t_2 - t_1 = 1\text{ s}$

EXAMPLE 5.13

A large, heavy box is sliding without friction down a smooth plane of inclination θ. From a point P on the bottom of the box, a particle is projected inside the box. The initial speed of the particle with respect to the box is u, and the direction of projection makes an angle α with the bottom as shown in figure.



(a) Find the distance along the bottom of the box between the point of projection P and the point Q where the particle lands. (Assume that the particle does not hit any other surface of the box. Neglect air resistance.)

(b) If the horizontal displacement of the particle, as seen by an observer on the ground, is 0, find the speed of the box with respect to the ground at the instant when the particle was projected.

Sol. **Method 1:**

(a) u is the relative velocity of the particle with respect to the box. Resolve u.

Let u_x is the relative velocity of the particle with respect to the box in x-direction. Let u_y is the relative velocity with respect to the box in y-direction.

y-direction motion (taking relative terms w.r.t. box)

$$u_y = +u\sin\alpha,\ a_y = -g\cos\theta,\ S_y = 0$$

$$S_y = u_y t + \frac{1}{2}a_y t^2 \Rightarrow 0 = (u\sin\alpha)t - \frac{1}{2}g\cos\theta \times t^2$$

or $$t = \frac{2u\sin\alpha}{g\cos\theta}$$

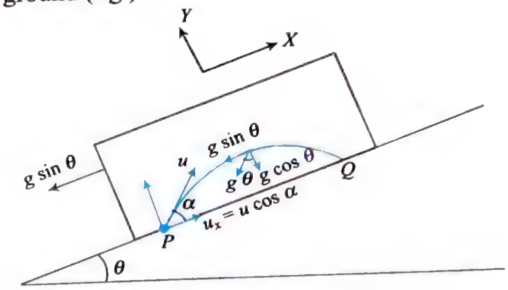
x-direction motion (taking relative terms w.r.t. box):

$$u_x = +u\cos\alpha,\ a_x = 0$$

$$S_x = u_x t + \frac{1}{2}a_x t^2$$

$$\Rightarrow\quad PQ = u\cos\alpha \times \frac{2u\sin\alpha}{g\cos\theta} = \frac{u^2\sin 2\alpha}{g\cos\theta}$$

(b) For the observer (on ground) to see the horizontal displacement to be zero, the distance travelled by the box in time $\dfrac{2u\sin\alpha}{g\cos\theta}$ should be equal to the range PQ of the particle. Let the speed of the box at the time of projection of particle be u_1. Then for the motion of box with respect to ground (fig.).

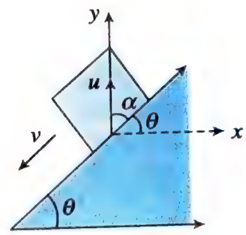


$$PQ = u_1 t + \frac{1}{2}g\sin\theta t^2.$$

Put the values of PQ and t and solve to get

$$u_1 = \frac{u\cos(\alpha+\theta)}{\cos\theta}$$

Method 2: The above condition can be met if the box covers exactly the same distance as the range of particle.



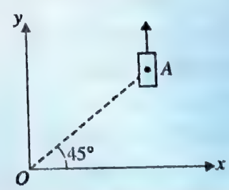
$$\left(\frac{u^2\sin 2\alpha}{g\cos\theta}\right) = v\left(\frac{2u\sin\alpha}{g\cos\theta}\right) + \frac{1}{2}g\sin\theta\left(\frac{2u\sin\alpha}{g\cos\theta}\right)^2$$

or $$u\cos\alpha = v + \frac{u\sin\theta\sin\alpha}{\cos\theta}$$

or $$v = u\left(\frac{\cos\alpha\cos\theta - \sin\alpha\cos\theta}{\cos\theta}\right) = \frac{u\cos(\alpha+\theta)}{\cos\theta}$$

EXAMPLE 5.14

On a frictionless horizontal surface, assumed to be the x-y plane, a small trolley A is moving along a straight line parallel to the y-axis with a constant velocity of $(\sqrt{3}-1)\text{ m s}^{-1}$. At a particular instant, when the line OA makes an angle of 45° with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley.

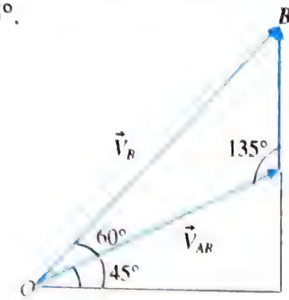


(a) The motion of the ball is observed from the frame of the trolley. Calculate the angle θ made by the velocity vector of the ball with the x-axis in this frame.

(b) Find the speed of the ball with respect to the surface, if $\phi = 4\theta/3$.

Sol. Method 1:

(a) Since the ball hits the trolley, relative to trolley, the velocity of ball should be directed towards the trolley. Hence, in the frame of trolley, the ball will appear to be moving towards OA, or in the frame of trolley, ball's velocity will make an angle of 45°.



(b) $\phi = \dfrac{4\theta}{3} = \dfrac{4\times 45^\circ}{3} = 60^\circ$

Using sine rule $\dfrac{V_B}{\sin 135^\circ} = \dfrac{V_A}{\sin 15^\circ}$

$\Rightarrow \quad V_B = 2\ \text{m s}^{-1}$

Method 2:

(a) Let A stands for trolley and B for ball.

Relative velocity of B with respect to $A(\vec{v}_{BA})$ should be along OA for the ball to hit the trolley.

Hence, $\vec{v}_{BA}$ will make an angle of 45° with positive x-axis.

(b) $\tan\theta = \dfrac{v_{BAy}}{v_{BAx}} = \tan 45^\circ$ or $v_{BAy} = v_{BAx}$ ...(i)

Further $v_{BAy} = v_{By} - v_{Ay}$ or $v_{BAx} = v_{Bx} - 0$ ...(ii)

$v_{BAy} = v_{By} - (\sqrt{3}-1)$ ...(iii)

$\tan\theta = \dfrac{v_{By}}{v_{Bx}}$ or $v_{By} = v_{Bx}\tan\phi$ ...(iv)

From (i), (ii), (iii), and (iv), we get

$$v_{Bx} = \frac{(\sqrt{3}-1)}{\tan\phi - 1} \quad \text{and} \quad v_{By} = \frac{(\sqrt{3}-1)}{\tan\theta - 1}\cdot\tan\phi$$

$$\phi = \frac{4\theta}{3} = \frac{4}{3}(45^\circ)$$

Speed of ball w.r.t. surface,

$$v_B = \sqrt{v_{Bx}^2 + v_{By}^2} = \frac{\sqrt{3}-1}{\tan\phi - 1}\sec\phi$$

Substituting $\phi = 60^\circ$, we get $v_B = 2\ \text{m s}^{-1}$

EXAMPLE 5.15

An aircraft is flying horizontally with a constant velocity 200 m s^{-1}, at a height 1 km above the ground. At the moment shown, a bomb is released from the aircraft and the cannon-gun below fires a shell with initial speed 200 m s^{-1}, at some angle θ.

For what value of θ will the projectile shell destroy the bomb in mid-air? If the value of θ is 53°, find the minimum distance between the bomb and the shell as they fly past each other. Take sin 53° = 4/5.

Sol.

(a) Suppose the shell destroys the bomb at time t. Then for horizontal motion,

$$t(200 + 200\cos\theta) = \sqrt{3}\times 1000$$

$$\therefore \quad t(1+\cos\theta) = 5\sqrt{3} \qquad \text{...(i)}$$

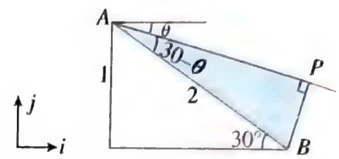
For vertical motion,

$$\frac{1}{2}gt^2 + (200\sin\theta)t - \frac{1}{2}gt^2 = 1000 \sin\theta t = 5 \qquad \text{...(ii)}$$

From (i) and (ii), $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1}{\sqrt{3}}$

On solving, we get $\theta = 60^\circ$.

(b) $\vec{v}_A = 200i$



$$\vec{v}_B = -200\cos 53^\circ\hat{i} + 200\sin 53^\circ\hat{j}$$

$$= -200\times\frac{3}{5}\hat{i} + 200\times\frac{4}{5}\hat{j} = -120\hat{i} + 160\hat{j}$$

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B = (200+120)\hat{i} - 160\hat{j}$$

$$\tan\theta = \left|-\frac{160}{320}\right| = \frac{1}{2}$$

$AB = 2$ km

BP = Minimum distance = $AB\sin(30^\circ - \theta)$

$BP = 2[\sin 30^\circ\cos\theta - \cos 30^\circ\sin\theta]$

$$= 2\left[\frac{1}{2}\times\frac{2}{\sqrt{5}} - \frac{\sqrt{5}}{2}\times\frac{1}{\sqrt{5}}\right] = \frac{2-\sqrt{3}}{\sqrt{5}}\ \text{km}$$

EXAMPLE 5.16

A river has a width d. A fisherman in a boat crosses the river twice. During the first crossing, his goal is to minimize the crossing time. During the second crossing, his goal is to minimize the distance that the boat is carried downstream. In the first case, the crossing time is T_0. In the second case, the crossing time is $3T_0$. What is the speed of the river flow? Find all possible answers.

Sol.

Case I: If $v_R < v_B$, the boat can cross river along a path perpendicular to flow.

Case II: If $v_B < v_R$, drift cannot be zero, apply calculus in this case.

Case I: If $v_R < v_B$

Shortest path: $\dfrac{d}{v_B\sin\theta} = 3T_0$ ...(i)

Quickest path: $\dfrac{d}{v_B} = T_0$ ...(ii)

Also $v_R - v_B \cos\theta = 0$ for shortest path ...(iii)

Thus, $\sin\theta = \frac{1}{3}$ from (i) and (ii) or $v_R = v_B \cos\theta$

$$= \frac{d}{T_0}\sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}d}{3T_0}$$

Case II: If $v_B < v_R$

$$x = \left(\frac{d}{v_B \sin\theta}\right)(v_R - v_B \cos\theta)$$

$$= \frac{d}{v_B}(v_R \operatorname{cosec}\theta - v_B \cot\theta)$$

For minimum x, $\frac{dx}{d\theta} = -0$

$$v_B(-\operatorname{cosec}^2\theta) + v_R \operatorname{cosec}\theta \cot\theta = 0$$

$$\cos\theta = \frac{v_B}{v_R}$$

Time taken in this case is given by $3T_0 = \frac{v_R d}{v_B\sqrt{v_R^2 - v_B^2}}$

Also $v_B = \frac{d}{T_0}$

On solving, we get $v_R = \sqrt{\frac{3}{2}}\frac{d}{T_0}$

Exercises

Single Correct Answer Type

1. Two bullets are fired horizontally with different velocities from the same height. Which will reach the ground first?
(1) Slower one
(2) Faster one
(3) Both will reach simultaneously
(4) Cannot be predicted

2. A projectile can have same range R for two angles of projection. It t_1 and t_2 are the times of flight in the two cases, then what is the product of two times of flight?
(1) $t_1t_2 \propto R^2$ (2) $t_1t_2 \propto R$
(3) $t_1t_2 \propto \frac{1}{R}$ (4) $t_1t_2 \propto \frac{1}{R^2}$

3. A ball is thrown at different angles with the same speed u and from the same point and it has the same range in both the cases. If y_1 and y_2 are the heights attained in the two cases, then $y_1 + y_2$ is equal to
(1) $\frac{u^2}{g}$ (2) $\frac{2u^2}{g}$
(3) $\frac{u^2}{2g}$ (4) $\frac{u^2}{4g}$

4. The range R of projectile is same when its maximum heights are h_1 and h_2. What is the relation between R, h_1, and h_2?
(1) $R=\sqrt{h_1h_2}$ (2) $R=\sqrt{2h_1h_2}$
(3) $R=2\sqrt{h_1h_2}$ (4) $R=4\sqrt{h_1h_2}$

5. A particle is projected with a velocity v so that its range on a horizontal plane is twice the greatest height attained. If g is acceleration due to gravity, then its range is
(1) $\frac{4v^2}{5g}$ (2) $\frac{4g}{5v^2}$
(3) $\frac{4v^3}{5g^2}$ (4) $\frac{4v}{5g^2}$

6. During a projectile motion, if the maximum height equals the horizontal range, then the angle of projection with the horizontal is
(1) $\tan^{-1}(1)$ (2) $\tan^{-1}(2)$
(3) $\tan^{-1}(3)$ (4) $\tan^{-1}(4)$

7. A particle is projected from ground at some angle with the horizontal. Let P be the point at maximum height H. At what height above the point P should the particle be aimed to have range equal to maximum height?
(1) H (2) 2H
(3) H/2 (4) 3H

8. The point from where a ball is projected is taken as the origin of the coordinate axes. The x and y components of its displacement are given by $x = 6t$ and $y = 8t - 5t^2$. What is the velocity of projection?
(1) 6 m s^{-1} (2) 8 m s^{-1}
(3) 10 m s^{-1} (4) 14 m s^{-1}

9. In the above problem, what is the angle of projection with horizontal?
(1) $\tan^{-1}(1/4)$ (2) $\tan^{-1}(4/3)$
(3) $\tan^{-1}(3/4)$ (4) $\tan^{-1}(1/2)$

10. A shot is fired from a point at a distance of 200 m from the foot of a tower 100 m high so that it just passes over it horizontally. The direction of shot with horizontal is
(1) 30° (2) 45°
(3) 60° (4) 70°

11. A projectile has a time of flight T and range R. If the time of flight is doubled, keeping the angle of projection same, what happens to the range?
(1) $R/4$ (2) R/2
(3) 2 R (4) 4 R

12. A ball is thrown from a point with a speed v_0 at an angle of projection θ. From the same point and at the same instant, a person starts running with a constant speed $v_0/2$ to catch the ball. Will the person be able to catch the ball? If yes, what should be the angle of projection?
(1) Yes, 60° (2) Yes, 30°
(3) No (4) Yes, 45°

13. A body is projected at an angle of 30° with the horizontal and with a speed of 30 m s^{-1}. What is the angle with the horizontal after 1.5 s? (g = 10 m s^{-2})
(1) 0° (2) 30°
(3) 60° (4) 90°

14. A body has an initial velocity of 3 m s^{-1} and has an acceleration of 1 m s^{-2} normal to the direction of the initial velocity. Then its velocity 4 s after the start is
(1) 7 m s^{-1} along the direction of initial velocity
(2) 7 m s^{-1} along the normal to the direction of initial velocity.
(3) 7 m s^{-1} midway between the two directions
(4) 5 m s^{-1} at an angle $\tan^{-1}(4/3)$ with the direction of initial velocity.

15. Two tall buildings are 30 m apart. The speed with which a ball must be thrown horizontally from a window 150 m above the ground in one building so that it enters a window 27.5 m from the ground in the other building is
(1) 2 m s^{-1} (2) 6 m s^{-1}
(3) 4 m s^{-1} (4) 8 m s^{-1}

16. A shell fired from the ground is just able to cross horizontally the top of a wall 90 m away and 45 m high. The direction of projection of the shell will be
(1) 25° (2) 30°
(3) 60° (4) 45°

17. The equation of motion of a projectile is $y = 12x - \frac{3}{4}x^2$. The horizontal component of velocity is 3 $m s^{-1}$. What is the range of the projectile?
(1) 18 m (2) 16 m
(3) 12 m (4) 21.6 m

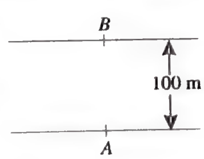
18. Two paper screens A and B are separated by 150 m. A bullet pierces A and B. The hole in B is 15 cm below the hole in A. If the bullet is travelling horizontally at the time of hitting A, then the velocity of the bullet at A is ($g = 10\ m s^{-2}$)
(1) $100\sqrt{3}\ m s^{-1}$ (2) $200\sqrt{3}\ m s^{-1}$
(3) $300\sqrt{3}\ m s^{-1}$ (4) $500\sqrt{3}\ m s^{-1}$

19. At what angle with the horizontal should a ball be thrown so that the range R is related to the time of flight as $R = 5T^2$? (Take $g = 10\ m s^{-2}$)
(1) 30° (2) 45°
(3) 60° (4) 90°

20. Rain is falling vertically downwards with a speed of 4 $km h^{-1}$. A girl moves on a straight road with a velocity of 3 $km h^{-1}$. The apparent velocity of rain with respect to the girl is
(1) 3 $km h^{-1}$ (2) 4 $km h^{-1}$
(3) 5 $km h^{-1}$ (4) 7 $km h^{-1}$

21. Ship A is travelling with a velocity of 5 $km h^{-1}$ due east. A second ship is heading 30° east of north. What should be the speed of second ship if it is to remain always due north with respect to the first ship?
(1) 10 $km h^{-1}$ (2) 9 $km h^{-1}$
(3) 8 $km h^{-1}$ (4) 7 $km h^{-1}$

22. A man swims from a point A on one bank of a river of width 100 m. When he swims perpendicular to the water current, he reaches the other bank 50 m downstream. The angle to the bank at which he should swim, to reach the directly opposite point B on the other bank is



(1) 10° upstream (2) 20° upstream
(3) 30° upstream (4) 60° upstream

23. Rain is falling vertically with a velocity of 25 $m s^{-1}$. A woman rides a bicycle with a speed of 10 $m s^{-1}$ in the north to south direction. What is the direction (angle with vertical) in which she should hold her umbrella to safe herself from rain?
(1) $\tan^{-1}(0.4)$ (2) $\tan^{-1}(1)$
(3) $\tan^{-1}(\sqrt{3})$ (4) $\tan^{-1}(2.6)$

24. A policeman moving on a highway with a speed of 30 $km h^{-1}$ fires a bullet at thief's car speeding away in the same direction with a speed of 192 $km h^{-1}$. If the muzzle speed of the bullet is 150 $m s^{-1}$, with what speed does the bullet hit the thief's car?
(1) 120 $m s^{-1}$ (2) 90 $m s^{-1}$
(3) 125 $m s^{-1}$ (4) 105 $m s^{-1}$

25. A bird is flying towards north with a velocity 40 $km h^{-1}$ and a train is moving with velocity 40 $km h^{-1}$ towards east. What is the velocity of the bird noted by a man in the train?
(1) $40\sqrt{2}\ km h^{-1}$ N-E (2) $40\sqrt{2}\ km h^{-1}$ S-E
(3) $40\sqrt{2}\ km h^{-1}$ N-W (4) $40\sqrt{2}\ km h^{-1}$ S-W

26. A river is flowing from west to east at a speed of 5 $m min^{-1}$. A man on the south bank of the river, capable of swimming at 10 $m min^{-1}$ in still water, wants to swim across the river in the shortest time. Finally, he will move in a direction
(1) $\tan^{-1}(2)$ E of N (2) $\tan^{-1}(2)$ N of E
(3) 30° E of N (4) 60° E of N

27. A car is moving towards east with a speed of 25 $km h^{-1}$. To the driver of the car, a bus appears to move towards north with a speed of $25\sqrt{3}\ km h^{-1}$. What is the actual velocity of the bus?
(1) 50 $km h^{-1}$, 30° E of N (2) 50 $km h^{-1}$, 30° N of E
(3) 25 $km h^{-1}$, 30° E of N (4) 25 $km h^{-1}$, 30° N of E

28. A swimmer wishes to cross a 500-m river flowing at 5 $km h^{-1}$. His speed with respect to water is 3 $km h^{-1}$. The shortest possible time to cross the river is
(1) 10 min (2) 20 min
(3) 6 min (4) 7.5 min

29. A train of 150 m length is going toward north direction at a speed of 10 $m s^{-1}$. A parrot flies at a speed of 5 $m s^{-1}$ toward south direction parallel to the railway track. The time taken by the parrot to cross the train is equal to
(1) 12 s (2) 8 s
(3) 15 s (4) 10 s

30. A man can swim in still water with a speed of 2 $m s^{-1}$. If he wants to cross a river of water current speed $\sqrt{3}\ m s^{-1}$ along the shortest possible path, then in which direction should he swim?
(1) At an angle 120° to the water current
(2) At an angle 150° to the water current
(3) At an angle 90° to the water current
(4) None of these

31. A truck is moving with a constant velocity of 54 $km h^{-1}$. In which direction (angle with the direction of motion of truck) should a stone be projected up with a velocity of 20 $m s^{-1}$, from the floor of the truck, so as to appear at right angles to the truck, for a person standing on earth?
(1) $\cos^{-1}\left(-\frac{3}{4}\right)$ (2) $\cos^{-1}\left(-\frac{1}{4}\right)$
(3) $\cos^{-1}\left(\frac{2}{3}\right)$ (4) $\cos^{-1}\left(\frac{3}{4}\right)$

32. A river flows with a speed more than the maximum speed with which a person can swim in still water. He intends to cross the river by the shortest possible path (i.e., he wants to reach the point on the opposite bank which directly opposite to the starting point). Which of the following is correct?
(1) He should start normal to the river bank.
(2) He should start in such a way that he moves normal to the bank, relative to the bank.

(3) He should start in a particular (calculated) direction making an obtuse angle with the direction of water current.

(4) The man cannot cross the river in that way.

33. Rain, driven by the wind, falls on a railway compartment with a velocity of 20 m s^{-1}, at an angle of 30° to the vertical. The train moves, along the direction of wind flow, at a speed of 108 km h^{-1}. Determine the apparent velocity of rain for a person sitting in the train.

(1) $20\sqrt{7}$ m s^{-1} (2) $10\sqrt{7}$ m s^{-1}

(3) $15\sqrt{7}$ m s^{-1} (4) $10\sqrt{7}$ km h^{-1}

34. The ratio of the distance carried away by the water current, downstream, in crossing a river, by a person, making same angle with downstream and upstream is 2 : 1. The ratio of the speed of person to the water current cannot be less than

(1) 1/3 (2) 4/5

(3) 2/5 (4) 4/3

35. A person sitting in the rear end of a compartment throws a ball towards the front end. The ball follows a parabolic path. The train is moving with the uniform velocity of 20 m s^{-1}. A person standing outside on the ground also observes the ball. How will the maximum heights (h_m) attained and the ranges (R) seen by the thrower and the outside observer compare each other?

(1) Same h_m, different R (2) same h_m, and R

(3) Different h_m, same R (4) different h_m, and R

36. Two stones are projected with the same speed but making different angles with the horizontal. Their ranges are equal. If the angle of projection of one is $\pi/3$ and its maximum height is h_1 then the maximum height of the other will be

(1) $3h_1$ (2) $2h_1$

(3) $h_1/2$ (4) $h_1/3$

37. A ball is projected from the ground at angle θ with the horizontal. After 1 s, it is moving at angle 45° with the horizontal and after 2 s it is moving horizontally. What is the velocity of projection of the ball?

(1) $10\sqrt{3}$ m s^{-1} (2) $20\sqrt{3}$ m s^{-1}

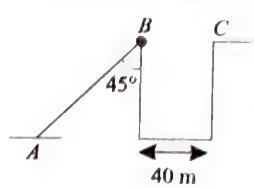
(3) $10\sqrt{5}$ m s^{-1} (4) $20\sqrt{2}$ m s^{-1}

38. A body is projected horizontally from the top of a tower with initial velocity 18 m s^{-1}. It hits the ground at angle 45°. What is the vertical component of velocity when strikes the ground?

(1) 9 m s^{-1} (2) $9\sqrt{2}$ m s^{-1}

(3) 18 m s^{-1} (4) $18\sqrt{2}$ m s^{-1}

39. A body is projected up along a smooth inclined plane with velocity u from the point A as shown in figure. The angle of inclination is 45° and the top is connected to a well of diameter 40 m. If the body just manages to cross the well, what is the value of u? The length of inclined plane is $20\sqrt{2}$ m.



(1) 40 m s^{-1} (2) $40\sqrt{2}$ m s^{-1}

(3) 20 m s^{-1} (4) $20\sqrt{2}$ m s^{-1}

40. A rifle shoots a bullet with a muzzle velocity of 400 m s^{-1} at a small target 400 m away. The height above the target at which the bullet must be aimed to hit the target is (g = 10 m s^{-2}).

(1) 1 m (2) 5 m

(3) 10 m (4) 0.5 m

41. A projectile is fired from level ground at an angle θ above the horizontal. The elevation angle ϕ of the highest point as seen from the launch point is related to θ by the relation

(1) $\tan\phi = 2\tan\theta$ (2) $\tan\phi = \tan\theta$

(3) $\tan\phi = \frac{1}{2}\tan\theta$ (4) $\tan\phi = \frac{1}{4}\tan\theta$

42. A projectile has initially the same horizontal velocity as it would acquire if it had moved from rest with uniform acceleration of 3 m s^{-2} for 0.5 min. If the maximum height reached by it is 80 m, then the angle of projection is (g = 10 m s^{-2})

(1) $\tan^{-1}3$ (2) $\tan^{-1}(3/2)$

(3) $\tan^{-1}(4/9)$ (4) $\sin^{-1}(4/9)$

43. A particle is projected from the ground with an initial speed of v at an angle θ with horizontal. The average velocity of the particle between its point of projection and highest point of trajectory is

(1) $\frac{v}{2}\sqrt{1+2\cos^2\theta}$ (2) $\frac{v}{2}\sqrt{1+2\cos^2\theta}$

(3) $\frac{v}{2}\sqrt{1+3\cos^2\theta}$ (4) $v\cos\theta$

44. Two balls A and B are thrown with speeds u and $u/2$ respectively. Both the balls cover the same horizontal distance before returning to the plane of projection. If the angle of projection of ball B is 15° with the horizontal, then the angle of projection of A is

(1) $\sin^{-1}\left(\frac{1}{8}\right)$ (2) $\frac{1}{2}\sin^{-1}\left(\frac{1}{8}\right)$

(3) $\frac{1}{3}\sin^{-1}\left(\frac{1}{8}\right)$ (4) $\frac{1}{4}\sin^{-1}\left(\frac{1}{8}\right)$

45. A car is moving horizontally along a straight line with a uniform velocity of 25 m s^{-1}. A projectile is to be fired from this car in such a way that it will return to it after it has moved 100 m. The speed of the projection must be

(1) 10 m s^{-1} (2) 20 m s^{-1}

(3) 15 m s^{-1} (4) 25 m s^{-1}

46. The horizontal range and maximum height attained by a projectile are R and H, respectively. If a constant horizontal acceleration $a = g/4$ is imparted to the projectile due to wind, then its horizontal range and maximum height will be

(1) $(R+H), \frac{H}{2}$ (2) $\left(R+\frac{H}{2}\right), 2H$

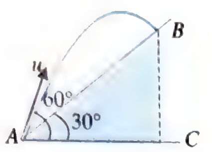
(3) $(R + 2H)$, H (4) $(R + H)$, H

47. A particle is projected with a certain velocity at an angle α above the horizontal from the foot of an inclined plane of inclination 30°. If the particle strikes the plane normally, then α is equal to

(1) $30° + \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (2) 45°

(3) 60° (4) $30° + \tan^{-1}(2\sqrt{3})$

48. In the figure, the time taken by the projectile to reach from A to B is t. Then the distance AB is equal to



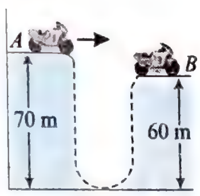
(1) $\frac{ut}{\sqrt{3}}$

(2) $\frac{\sqrt{3}ut}{2}$

(3) $\sqrt{3}\,ut$

(4) 2ut

49. A motor cyclist is trying to jump across a path as shown in figure by driving horizontally off a cliff A at a speed of 5 $m\,s^{-1}$. Ignore air resistance and take g = 10 $m\,s^{-2}$. The speed with which he touches peak B is



(1) 20 $m\,s^{-1}$ (2) 12 $m\,s^{-1}$

(3) 25 $m\,s^{-1}$ (4) 15 $m\,s^{-1}$

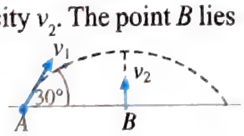
50. The height y and the distance x along the horizontal plane of a projectile on a certain planet (with no surrounding atmosphere) are given by $y = (8t - 5t^2)$ m and $x = 6t$ m, where t is in seconds. The velocity with which the projectile is projected at $t = 0$ is

(1) 8 $m\,s^{-1}$ (2) 6 $m\,s^{-1}$

(3) 10 $m\,s^{-1}$

(4) Not obtainable from the data

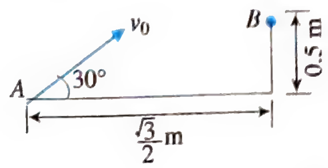
51. A body is projected with velocity v_1 from the point A as shown in figure. At the same time, another body is projected vertically upwards from B with velocity v_2. The point B lies vertically below the highest point of first particle. For both the bodies to collide, v_2/v_1 should be



(1) 2 (2) $\sqrt{\frac{3}{2}}$

(3) 0.5 (4) 1

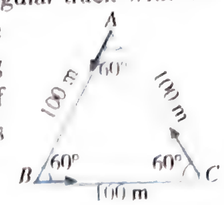
52. A ball is projected from a point A with some velocity at an angle 30° with the horizontal as shown in the figure. Consider a target at point B. The ball will hit the target if it is thrown with a velocity v_0 equal to



(1) 5 $m\,s^{-1}$ (2) 6 $m\,s^{-1}$

(3) 7 $m\,s^{-1}$ (4) None of these

53. Three boys are running on a equitriangular track with the same speed 5 $m\,s^{-1}$. At start, they were at the three corners with velocity along indicated directions. The velocity of approach of any one of them towards another at t = 10 s equals



(1) 7.5 $m\,s^{-1}$ (2) 10 $m\,s^{-1}$

(3) 5 $m\,s^{-1}$ (4) 0 $m\,s^{-1}$

54. Two boys P and Q are playing on a river bank. P plans to swim across the river directly and come back. Q plans to swim downstream by a length equal to the width of the river and then come back. Both of them bet each other, claiming that the boy succeeding in less time will win. Assuming the swimming rate of both P and Q to the same, it can be concluded that

(1) P wins

(2) Q wins

(3) A draw takes place

(4) Nothing certain can be stated.

55. A man holds an umbrella at 30° with the vertical to keep himself dry. He, then, runs at a speed of 10 $m\,s^{-1}$, and find the rain drops to be hitting vertically. Study the following statements and find the correct options.

i. Velocity of rain w.r.t. Earth is $20\,m\,s^{-1}$

ii. Velocity of rain w.r.t. man is $10\sqrt{3}\,m\,s^{-1}$

iii. Velocity of rain w.r.t. Earth is 30 $m\,s^{-1}$

iv. Velocity of rain w.r.t. man is $10\sqrt{2}\,m\,s^{-1}$

(1) Statements (i) and (ii) are correct.

(2) Statements (i) and (iii) are correct.

(3) Statements (iii) and (iv) are correct.

(4) Statements (ii) and (iv) are correct.

56. Rain appears to fall vertically to a man walking at 3 $km\,h^{-1}$, but when he changes his speed to double, the rain appears to fall at 45° with vertical. Study the following statements and find which of them are correct.

i. Velocity of rain is $2\sqrt{3}\,km\,h^{-1}$

ii. The angle of fall of rain (with vertical) is

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

iii. The angle of fall of rain (with vertical) is

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right).$$

iv. Velocity of rain is $3\sqrt{2}\,km\,h^{-1}$.

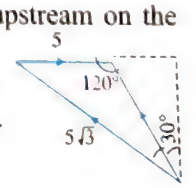
(1) Statements (i) and (ii) are correct.

(2) Statements (i) and (iii) are correct.

(3) Statements (iii) and (iv) are correct.

(4) Statements (ii) and (iv) are correct.

57. A motor boat is to reach at a point 30° upstream on the other side of a river flowing with velocity 5 $m\,s^{-1}$. The velocity of motor boat with respect to water is $5\sqrt{3}\,m\,s^{-1}$. The driver should steer the boat at an angle



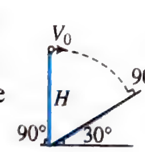
(1) 30° w.r.t. the line of destination from the starting point
(2) 60° w.r.t. normal to the bank
(3) 120° w.r.t. stream direction
(4) None of these

58. Raindrops are hitting the back of a man walking at a speed of 5 km h^{-1}. If he now starts running in the same direction with a constant acceleration, the magnitude of the velocity of the rain with respect to him will
(1) gradually increase
(2) gradually decrease
(3) first decrease then increase
(4) first increase then decrease

59. A particle has been projected with a speed of 20 m s^{-1} at an angle of 30° with the horizontal. The time taken when the velocity vector becomes perpendicular to the initial velocity vector is
(1) 4 s (2) 2 s
(3) 3 s (4) Not possible in this case

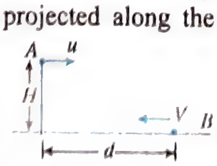
60. Two particles are projected simultaneously from the same point, with the same speed, in the same vertical plane, and at different angles with the horizontal in a uniform gravitational field acting vertically downwards. A frame of reference is fixed to one particle. The position vector of the other particle, as observed from this frame, is $\vec{r}$. Which of the following statements is correct?
(1) $\vec{r}$ is a constant vector.
(2) $\vec{r}$ changes in magnitude as well as direction with time.
(3) The magnitude of $\vec{r}$ increases linearly with time; its direction does not change.
(4) The direction of $\vec{r}$ changes with time; its magnitude may or may not change, depending on the angles of projection.

61. In the figure, the angle of inclination of the inclined plane is 30°. Find the horizontal velocity V_0 so that the particle hits the inclined plane perpendicularly.



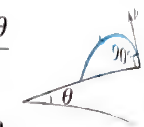
(1) $V_0 = \sqrt{\frac{2gH}{5}}$ (2) $V_0 = \sqrt{\frac{2gH}{7}}$
(3) $V_0 = \sqrt{\frac{gH}{5}}$ (4) $V_0 = \sqrt{\frac{gH}{7}}$

62. Two particles A and B are placed as shown in the figure. The particle A, on the top of tower, is projected horizontally with a velocity u and particle B is projected along the surface towards the tower, simultaneously. If both particles meet each other, then the speed of projection of particle B is [ignore any friction]



(1) $d\sqrt{\frac{g}{2H}} - u$ (2) $d\sqrt{\frac{g}{2H}}$
(3) $d\sqrt{\frac{g}{2H}} + u$ (4) u

63. A projectile is fired with a velocity v at right angle to the slope inclined at an angle θ with the horizontal. The range of the projectile along the inclined plane is

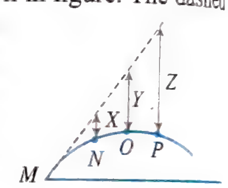


(1) $\frac{2v^2 \tan\theta}{g}$ (2) $\frac{v^2 \sec\theta}{g}$
(3) $\frac{2v^2 \tan\theta \sec\theta}{g}$ (4) $\frac{v^2 \sin\theta}{g}$

64. A ball rolls off the top of a stairway horizontally with a velocity of 4.5 m s^{-1}. Each step is 0.2 m high and 0.3 m wide. If g is 10 m s^{-2}, and the ball strikes the edge of nth step, then n is equal to
(1) 9 (2) 10
(3) 11 (4) 12.

65. The maximum range of a projectile is 500 m. If the particle is thrown up a plane, which is inclined at an angle of 30° with the same speed, the distance covered by it along the inclined plane will be
(1) 250 m (2) 500 m
(3) 750 m (4) 1000 m

66. A cannon fires a projectile as shown in figure. The dashed line shows the trajectory in the absence of gravity. The points M, N, O, and P correspond to time at $t = 0$, 1 s, 2 s and 3 s, respectively. The lengths of X, Y, and Z are, respectively,

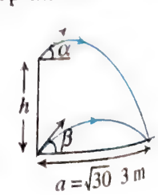


(1) 5 m, 10 m, 15 m (2) 10 m, 40 m, 90 m
(3) 5 m, 20 m, 45 m (4) 10 m, 20 m, 30 m

67. The speed of a projectile at its highest point is v_1 and at the point half the maximum height is v_2. If $\frac{v_1}{v_2} = \sqrt{\frac{2}{5}}$, then find the angle of projection.
(1) 45° (2) 30°
(3) 37° (4) 60°

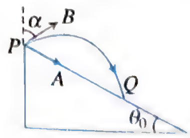
68. A particle is projected at an angle of elevation α and after t second, it appears to have an angle of elevation β as seen from the point of projection. The initial velocity will be
(1) $\frac{gt}{2\sin(\alpha-\beta)}$ (2) $\frac{gt\cos\beta}{2\sin(\alpha-\beta)}$
(3) $\frac{\sin(\alpha-\beta)}{2gt}$ (4) $\frac{2\sin(\alpha-\beta)}{gt\cos\beta}$

69. Shots are fired simultaneously from the top and bottom of a vertical cliff with the elevation $\alpha = 30°$, $\beta = 60°$, respectively (figure). The shots strike an object simultaneously at the same point. If $a = 30\sqrt{3}$ m is the horizontal distance of the object from the cliff, then the height h of the cliff is



(1) 30 m (2) 45 m
(3) 60 m (4) 90 m

70. The figure shows that particle A is projected from point P with velocity u along the plane and simultaneously another particle B with velocity v at an angle α with vertical. The particles collide at point Q on the plane. Then

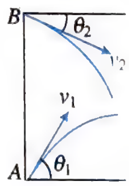


(1) $v \sin(\alpha - \theta_0) = u$ (2) $v \cos(\alpha - \theta_0) = u$
(3) $v = u$ (4) None of these

71. A platform is moving upwards with an acceleration of $5\,\text{m s}^{-2}$. At the moment when its velocity is $u = 3\,\text{m s}^{-1}$, a ball is thrown from it with a speed of $30\,\text{m s}^{-1}$ w.r.t. platform at an angle of $\theta = 30°$ with horizontal. The time taken by the ball to return to the platform is

(1) 2 s (2) 3 s
(3) 1 s (4) 2.5 s

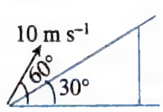
72. Two balls are projected from points A and B in vertical plane as shown in figure. AB is a straight vertical line. The balls can collide in mid air if v_1/v_2 is equal to



(1) $\dfrac{\sin\theta_1}{\sin\theta_2}$ (2) $\dfrac{\sin\theta_2}{\sin\theta_1}$

(3) $\dfrac{\cos\theta_1}{\cos\theta_2}$ (4) $\dfrac{\cos\theta_2}{\cos\theta_1}$

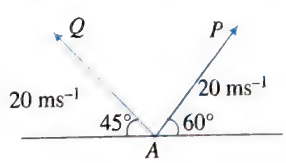
73. A particle is thrown at time $t = 0$ with a velocity of $10\,\text{m s}^{-1}$ at an angle 60° with the horizontal from a point on an inclined plane, making an angle of 30° with the horizontal. The time when the velocity of the projectile becomes parallel to the incline is



(1) $\dfrac{2}{\sqrt{3}}$ s (2) $\dfrac{1}{\sqrt{3}}$ s

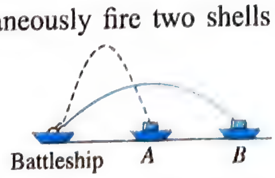
(3) $\sqrt{3}$ s (4) $\dfrac{1}{2\sqrt{3}}$ s

74. Two particles P and Q are projected simultaneously away from each other from a point A as shown in figure. The velocity of P relative to Q in m s^{-1} at the instant when the motion of P is horizontal is



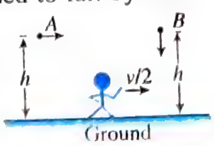
(1) $10\sqrt{4-\sqrt{3}}$ (2) $20\sqrt{4-\sqrt{3}}$
(3) $10\sqrt{4+\sqrt{3}}$ (4) $20\sqrt{4+\sqrt{3}}$

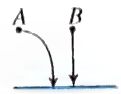
75. Two guns on a battleship simultaneously fire two shells with same speed at enemy ships. If the shells follow the parabolic trajectories as shown in figure, which ship will get hit first?

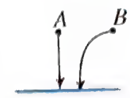


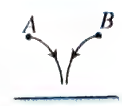
(1) A (2) B
(3) both at same time (4) need more information

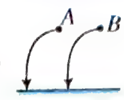
76. Two identical balls are set into motion simultaneously from equal heights h. While the ball A is thrown horizontally with velocity v, the ball B is just released to fall by itself. Choose the alternative that best represents the motion of A and B with respect to an observer who moves with velocity $v/2$ with respect to the ground as shown in figure.



(1)


(2)


(3)


(4)


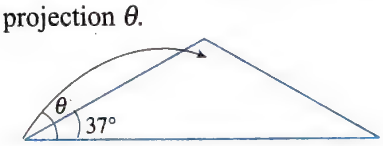
77. An aeroplane is flying vertically upwards. When it is at a height of 1000 m above the ground and moving at a speed of 367 m/s, a shot is fired at it with a speed of $567\,\text{m s}^{-1}$ from a point directly below it. What should be the acceleration of aeroplane so that it may escape from being hit?

(1) $>5\,\text{m s}^{-2}$ (2) $>10\,\text{m s}^{-2}$
(3) $<10\,\text{m s}^{-2}$ (4) Not possible

78. Jai is standing on the top of a building of height 25 m he wants to throw his gun to Veeru who stands on top of another building of height 20 m at distance 15 m from first building. For which horizontal speed of projection, it is possible?

(1) $5\,\text{m s}^{-1}$ (2) $10\,\text{m s}^{-1}$
(3) $15\,\text{m s}^{-1}$ (4) $20\,\text{m s}^{-1}$

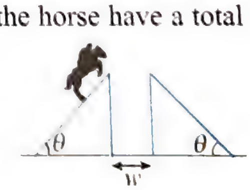
79. A shot is fired at an angle θ to the horizontal such that it strikes the hill while moving horizontally. Find the initial angle of projection θ.



(1) $\tan\theta = \dfrac{2}{5}$ (2) $\tan\theta = \dfrac{3}{8}$

(3) $\tan\theta = \dfrac{3}{2}$ (4) None of these

80. A man is riding on a horse. He is trying to jump the gap between two symmetrical ramps of snow separated by a distance W as shown in fi -e. He launches off the first ramp with a speed V_L. The ı ın and the horse have a total mass m. and their size is small as compared to W The value of initial launch speed V_L which will put the horse exactly at the peak of the second ramp is



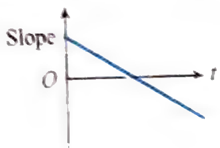
(1) $\sqrt{\dfrac{Wg}{\sin\theta \times \cos\theta}}$ (2) $\sqrt{\dfrac{Wg}{\sin(\theta/2) \times \cos(\theta/2)}}$

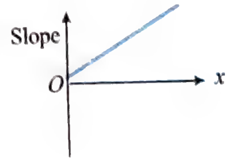
(3) $\sqrt{\dfrac{Wg}{2\sin\theta\cos\theta}}$ (4) $\sqrt{\dfrac{2Wg}{\sin\theta\cos\theta}}$

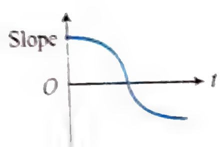
Multiple Correct Answers Type

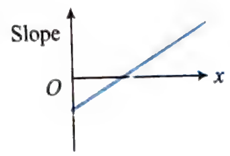
1. A particle is moving in xy-plane with $y = x/2$ and $v_x = 4 - 2t$. Choose the correct options.
   (1) Initial velocities in x and y directions are negative.
   (2) Initial velocities in x and y directions are positive.
   (3) Motion is first retarded, then accelerated.
   (4) Motion is first accelerated, then retarded.

2. A heavy particle is projected with a velocity at an angle with the horizontal into a uniform gravitational field. The slope of the trajectory of the particle varies not according to which of the following curves?

   (1) 

   (2) 

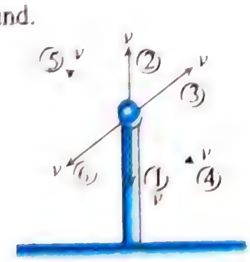
   (3) 

   (4) 

3. A body is projected with velocity u at an angle of projection θ with the horizontal. The direction of velocity of the body makes angle 30° with the horizontal at $t = 2$ s and then after 1 s it reaches the maximum height. Then
   (1) $u = 20\sqrt{3}\ \mathrm{m\,s^{-1}}$ (2) $\theta = 60°$
   (3) $\theta = 30°$ (4) $u = 10\sqrt{3}\ \mathrm{m\,s^{-1}}$

4. A river is flowing towards with a velocity of 5 m s^{-1}. The boat velocity is 10 m s^{-1}. The boat crosses the river by shortest path. Hence,
   (1) The direction of boat's velocity is 30° west of north.
   (2) The direction of boat's velocity is north-west.
   (3) Resultant velocity is $5\sqrt{3}\ \mathrm{m\,s^{-1}}$.
   (4) Resultant velocity of boat is $5\sqrt{2}\ \mathrm{m\,s^{-1}}$.

5. All the particles thrown with same initial velocity would strike the ground.



   (1) with same speed
   (2) simultaneously
   (3) time would be least for the particle thrown with velocity v downward i.e., particle 1
   (4) time would be maximum for the particle 2

6. Two cities A and B are connected by a regular bus service with buses plying in either direction every T seconds. The speed of each bus is uniform and equal to V_b. A cyclist cycles from A to B with a uniform speed of V_c. A bus goes past the cyclist in T_1 second in the direction A to B and every T_2 second in the direction B to A. Then
   (1) $T_1 = \dfrac{V_b T}{V_b + V_c}$ (2) $T_2 = \dfrac{V_b T}{V_b - V_c}$
   (3) $T_1 = \dfrac{V_b T}{V_b - V_c}$ (4) $T_2 = \dfrac{V_b T}{V_b + V_c}$

7. Ship A is located 4 km north and 3 km east of ship B. Ship A has a velocity of 20 km h^{-1} towards the south and ship B is moving at 40 km h^{-1} in a direction 37° north of east. Take x- and y-axes along east and north directions, respectively.
   (1) Velocity of A relative to B is $-32\hat{i} - 44\hat{j}$.
   (2) Position of A relative to B as a function of time is given by $\vec{r}_{AB} = (3 - 32t)\hat{i} + (4 - 44t)\hat{j}$ where $t = 0$ when the ships are in position described above.
   (3) Velocity of B relative to A is $-32\hat{i} - 44\hat{j}$.
   (4) At some moment A will be west of B.

8. A particle is projected at an angle $\theta = 30°$ with the horizontal, with a velocity of 10 m s^{-1}. Then
   (1) After 2 s, the velocity of particle makes an angle of 60° with initial velocity vector.
   (2) After 1 s, the velocity of particle makes an angle of 60° with initial velocity vector.
   (3) The magnitude of velocity of particle after 1 s is 10 m s^{-1}
   (4) The magnitude of velocity of particle after 1 s is 5 m s^{-1}

9. A particle moves along positive branch of the curve $y = \dfrac{x}{2}$, where $x = \dfrac{t^3}{3}$, x and y are measured in meters and t in seconds, then
   (1) The velocity of particle at $t = 1$ s is $\hat{i} + \dfrac{1}{2}\hat{j}$.
   (2) The velocity of particle at $t = 1$ s is $\dfrac{1}{2}\hat{i} + \hat{j}$.
   (3) The acceleration of particle at $t = 2$ s is $2\hat{i} + \hat{j}$.
   (4) The acceleration of particle at $t = 2$ s is $\hat{i} + 2\hat{j}$.

10. If T is the total time of flight, h is the maximum height and R is the range for horizontal motion, the x and y co-ordinates of projectile motion and time t are related as
    (1) $y = 4h\left(\dfrac{t}{T}\right)\left(1 - \dfrac{t}{T}\right)$ (2) $y = 4h\left(\dfrac{x}{R}\right)\left(1 - \dfrac{x}{R}\right)$
    (3) $y = 4h\left(\dfrac{T}{t}\right)\left(1 - \dfrac{T}{t}\right)$ (4) $y = 4h\left(\dfrac{R}{x}\right)\left(1 - \dfrac{R}{x}\right)$

11. Consider a shell that has a muzzle velocity of 45 m s^{-1} fired from the tail gun of an airplane moving horizontally with a velocity of 215 m s^{-1}. The tail gun can be directed at any angle with the vertical in the plane of motion of the airplane. The shell is fired when the plane is above point A on ground, and the plane is above point B on ground when the shell hits the ground. (Assume for simplicity that the Earth is flat)

(1) Shell may hit the ground at point A.

(2) Shell may hit the ground at point B.

(3) Shell may hit a point on earth which is behind point A.

(4) Shell may hit a point on earth which is ahead of point B.

12. A boat is traveling due east at 12 m s^{-1}. A flag on the boat flaps at 53°N of W. Another flag on the shore flaps due north.

(1) Speed of wind with respect to ground is 16 m s^{-1}

(2) Speed of wind with respect to ground is 20 m s^{-1}

(3) Speed of wind with respect to boat is 20 m s^{-1}

(4) Speed of wind with respect to boat is 16 m s^{-1}

13. A particle has initial velocity $4\hat{i} + 4\hat{j}$ m s^{-1} and an acceleration $-0.4\hat{i}$ m s^{-2}, at what time will its speed be 5 m s^{-1}?

(1) 2.5 s (2) 17.5 s

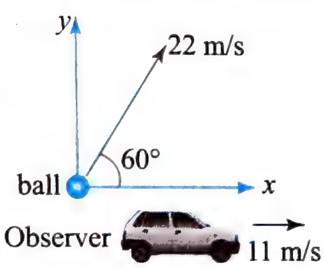
(3) s (4) 8.5 s

14. A cubical box dimension $L = 5/4$ m starts moving with an acceleration $\vec{a} = 0.5$ m s$^{-2}\hat{i}$ from the state of rest. At the same time, a stone is thrown from the origin with velocity $\vec{v} = v_1\hat{i} + v_2\hat{j} - v_3\hat{k}$ with respect to earth. Acceleration due to gravity $\vec{g} = 10$ m s$^{-2}(-\hat{j})$. The stone just touches the roof of box and finally falls at the diagonally opposite point. then:

(1) $v_1 = \dfrac{3}{2}$ (2) $v_2 = 5$

(3) $v_3 = \dfrac{5}{4}$ (4) $v_3 = \dfrac{5}{2}$

15. A football is kicked with a speed of 22 m/s at an angle of 60° to the positive x direction taken along horizontal. At that instant, an observer moves past the football in a car that moves with a constant speed of 11 m/s in the positive x direction. Take +ve y direction vertically upwards. ($g = 10$ m/s^2)



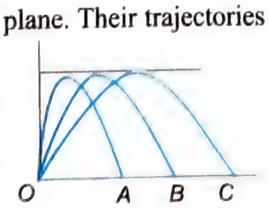
(1) The initial velocity of the ball relative to the observer in the car is $11\sqrt{3}$ m/s in the $+y$ direction

(2) The initial velocity of the ball relative to the observer in the car is 17 m/s at 60° to the $+x$ direction.

(3) According to the observer in the car, the ball will follow a path that is straight up and down in the y direction.

(4) According to the observer in the car, the ball will follow a straight line that is angled (less than 90°) with respect to the observer.

16. Three projectiles A, B and C are thrown simultaneously from the same point in the same vertical plane. Their trajectories are shown in the figure. Then which of the following statement(s) is/are correct.



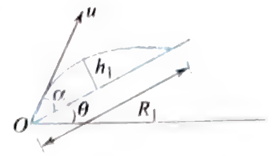
(1) The time of flight is the same for all the three.

(2) The launch speed is greatest for particle C

(3) The vertical velocity component for particle C is greater than that for the other particles

(4) Y-coordinate of all particles is always same

17. Two balls are thrown from an inclined plane at angle of projection α with the plane one up the incline plane and other down the incline as shown in the figure. If R_1 and R_2 be their respective ranges, then:



(1) $h_1 = h_2$

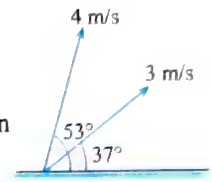
(2) $R_2 - R_1 = T_1^2$

(3) $R_2 - R_1 = g \sin \theta T_2^2$

(4) $R_2 - R_1 = g \sin \theta T_1^2$

[here T_1 and T_2 are times of flight in the two cases respectively]

18. Two particles are projected with speed 4 m/s and 3 m/s simultaneously from same point as shown in the figure. Then:



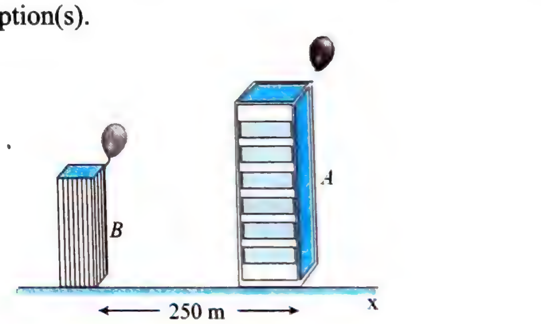
(1) Their relative velocity is along vertical direction

(2) Their relative acceleration is non-zero and it is along vertical direction

(3) They will hit the surface simultaneously

(4) Their relative velocity is constant and has magnitude 1.4 m/s

19. Two balloons are simultaneously released from two buildings A and B. Balloon from A rises with constant velocity 10 ms^{-1}, While the other one rises with constant velocity of 20 ms^{-1}. Due to wind the balloons gather horizontal velocity $V_x = 0.5\,y$, where y is the height from the point of release. The buildings are at a distance of 250 m and after some time t the balloons collide. Choose the correct option(s).



(1) $t = 5$ sec

(2) difference in height of buildings is 100 m

(3) difference in height of buildings is 50 m

(4) $t = 10$ sec

20. A person initially at rest throws a ball upward with speed 10 m/s at angle 37° with horizontal. He tries to catch the ball. For this, he accelerates just after he throws the ball, with constant acceleration for 1 sec and then continues to run at a constant speed and catches the ball exactly at the same height he throws the ball. Choose the correct option(s). (Use $g = 10$m/s^2)

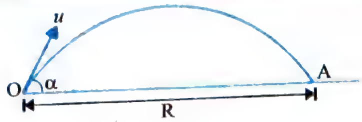
(1) Constant speed of person is approx. 13.7 m/s

(2) Acceleration of person is 15.2 m/s^2

(3) Acceleration of person is approx. 13.2 m/s^2

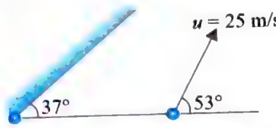
(4) Speed of person is 23 m/s

21. A particle projected from O and moving freely under gravity strikes the horizontal plane passing through O at a distance R from starting point O as shown in the figure. Then:



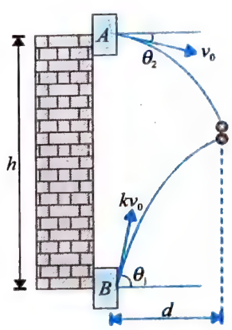
(1) there will be two angles of projection if $Rg < u^2$
(2) there will be more than two angles of projection if $Rg < u^2$
(3) the two possible angles of projection are complementary
(4) the product of the possible times of flight from O to A is $2R/g$

22. A particle is projected with speed 25 m/s at angle 53° from horizontal in front of an inclined plane mirror as shown in figure. Choose the correct option(s) ($g = 10$ m/s²).



(1) Speed of image w.r.t. object will be maximum after time 4 sec of projection.
(2) Minimum speed of image w.r.t. object is zero.
(3) Speed of image w.r.t. object will be minimum after time 7/8 sec of projection.
(4) Maximum speed of image w.r.t. object is 50 m/s.

23. At the same instant, two boys throw balls A and B from a window with speeds v_0 and kv_0, respectively, where k is constant. They collide in air at time t. Which of the following option(s) is/are correct?



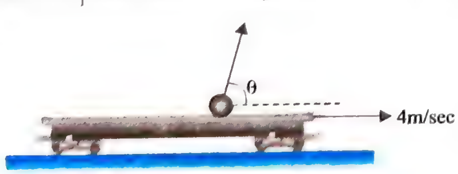
(1) $k = \dfrac{\sin\theta_2}{\sin\theta_1}$

(2) $k = \dfrac{\cos\theta_2}{\cos\theta_1}$

(3) $t = \dfrac{h}{(kv_0 \sin\theta_1 + v_0 \sin\theta_2)}$

(4) $t = \dfrac{h}{(kv_0 \cos\theta_1 + v_0 \cos\theta_2)}$

24. A cart is moved horizontally with a constant velocity of 4 m/sec. A ball is thrown from it with a velocity of 4 m/sec and at an angle θ with the horizontal with respect to the cart. Assume the height of the cart is very small, so that the motion of the ball is assumed to be a ground-to-ground projectile. Horizontal range of the ball with respect to the ground is R_1 and that with respect to the cart is R_2. Then



(1) R_1 will be maximum for $\theta = 60°$
(2) R_1 will be maximum for $\theta = 90°$
(3) R_2 will be maximum for $\theta = 60°$
(4) R_2 will be maximum for $\theta = 45°$

25. Two particles are projected from the same point with the same speed at different angles θ_1 and θ_2 to the horizontal. They have the same range. Their times of flight are t_1 and t_2, respectively. Then choose the correct option(s).

(1) $\theta_1 = 90 - \theta_2$
(2) $\dfrac{t_1}{t_2} = \tan\theta_2$
(3) $\dfrac{t_1}{\sin\theta_1} = \dfrac{t_2}{\sin\theta_2}$
(4) $\dfrac{t_1}{t_2} = \tan\theta_1$

Linked Comprehension Type

For Problems 1–3

Projectile motion is a combination of two one-dimensional motions: one in horizontal and other in vertical direction. Motion in 2D means in a plane. Necessary condition for 2D motion is that the velocity vector is coplanar to the acceleration vector. In case of projectile motion, the angle between velocity and acceleration will be $0° < \theta < 180°$. During the projectile motion, the horizontal component of velocity remains unchanged but vertical component of velocity is time dependent. Now answer the following questions:

1. A particle is projected from the origin in the x-y plane. The acceleration of particle in negative y-direction is α. If equation of path of the particle is $y = ax - bx^2$, then initial velocity of the particle is

(1) $\sqrt{\dfrac{\alpha}{2b}}$
(2) $\sqrt{\dfrac{\alpha(1+a^2)}{2b}}$
(3) $\sqrt{\dfrac{\alpha}{a^2}}$
(4) $\sqrt{\dfrac{\alpha b}{a^2}}$

2. An object is projected from origin in x-y plane in which velocity changes according to relation $\vec{v} = a\hat{i} + bx\hat{j}$. Path of particle is

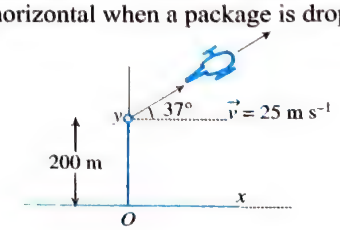
(1) Hyperbolic
(2) Circular
(3) Elliptical
(4) Parabolic

3. A body is projected at angle of 30° and 60° with the same velocity. Their horizontal ranges are R_1 and R_2 and maximum heights are H_1 and H_2, respectively, then

(1) $\dfrac{R_1}{R_2} > 1$
(2) $\dfrac{H_1}{H_2} > 1$
(3) $\dfrac{R_1}{R_2} < 1$
(4) $\dfrac{H_1}{H_2} < 1$

For Problems 4 and 5

A helicopter is flying at 200 m and flying at 25 m s^{-1} at an angle 37° above the horizontal when a package is dropped from it.

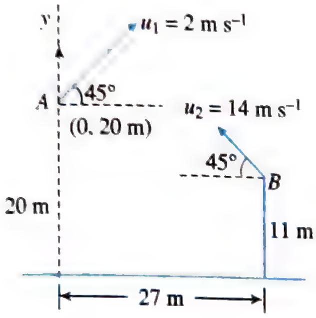


4. The distance of the point from point O where the package lands is
(1) 80 m (2) 100 m
(3) 200 m (4) 160 m

5. If the helicopter flies at constant velocity, find the x and y coordinates of the location of the helicopter when the package lands.
(1) 160 m, 320 m (2) 100 m, 200 m
(3) 200 m, 400 m (4) 50 m, 100 m

For Problems 6–8

Two particles are thrown simultaneously from points A and B with velocities $u_1 = 2\ \text{m s}^{-1}$ and $u_2 = 14\ \text{m s}^{-1}$, respectively, as shown in figure.



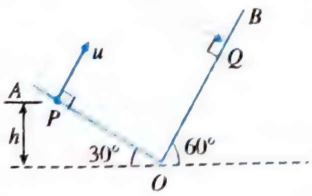
6. The relative velocity of B as seen from A in
(1) $-8\sqrt{2}\,\hat{i} + 6\sqrt{2}\,\hat{j}$ (2) $4\sqrt{2}\hat{i} + 3\sqrt{3}\hat{j}$
(3) $3\sqrt{5}\hat{i} + 2\sqrt{3}\hat{j}$ (4) $3\sqrt{2}\hat{i} + 4\sqrt{3}\hat{j}$

7. The direction (angle) with horizontal at which B will appear to move as seen from A is
(1) 37° (2) 53°
(3) 15° (4) 90°

8. Minimum separation between A and B is
(1) 3 m (2) 6 m
(3) 12 m (4) 9 m

For Problems 9–13

Two inclined planes OA and OB having inclination (with horizontal) 30° and 60°, respectively, intersect each other at O as shown in figure. A particle is projected from point P with velocity $u = 10\sqrt{3}\ \text{m s}^{-1}$ along a direction perpendicular to plane OA. If the particle strikes plane OB perpendicularly at Q, calculate



9. The velocity with which particle strikes the plane OB,
(1) $15\ \text{m s}^{-1}$ (2) $30\ \text{m s}^{-1}$
(3) $20\ \text{m s}^{-1}$ (4) $10\ \text{m s}^{-1}$

10. Time of flight of the particle
(1) 8 s (2) 6 s
(3) 4 s (4) 2 s

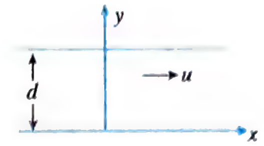
11. The vertical height h of P from O,
(1) 10 m (2) 5 m
(3) 15 m (4) 20 m

12. The maximum height attained by the particle (from the line O)
(1) 20.5 m (2) 5 m
(3) 16.25 m (4) 11.25 m

13. The distance PQ,
(1) 20 m (2) 10 m
(3) 5 m (4) 2.5 m

For Problems 14–16

We know that when a boat travels in water, its net velocity w.r.t. ground is the vector sum of two velocities. First is the velocity of boat itself in river and other is the velocity of water w.r.t. ground. Mathematically:



$$\vec{v}_{\text{boat}} = \vec{v}_{\text{boat, water}} + \vec{v}_{\text{water}}$$

Now given that velocity of water w.r.t. ground in a river is u. Width of the river is d. A boat starting from rest aims perpendicular to the river with an acceleration of $a = 5t$, where t is time. The boat starts from point (1, 0) of the coordinate system as shown in figure. Assume SI units.

14. Obtain the total time taken to cross the river.
(1) $(3d/5)^{1/3}$ (2) $(6d/5)^{1/3}$
(3) $(6d/5)^{1/2}$ (4) $(2d/3)^{1/3}$

15. Find the equation of trajectory of the boat.
(1) $x - 1 = \left(\dfrac{3y}{5}\right)^{1/3}$ (2) $x = u\left(\dfrac{6y}{5}\right)^{1/3}$
(3) $x = 1 = u\left(\dfrac{6y}{5}\right)^{1/3}$ (4) None of these

16. Find the drift of the boat when it is in the middle of the river.
(1) $u\left(\dfrac{3d}{5}\right)^{1/3}$ (2) $u\left(\dfrac{3d}{5}\right)^{1/3} + 1$
(3) $u\left(\dfrac{6d}{5}\right)^{1/3}$ (4) None of these

For Problems 17–19

Ram and Shyam are walking on two perpendicular tracks with speed $3\ \text{m s}^{-1}$ and $4\ \text{m s}^{-1}$, respectively. At a certain moment (say t = 0 s), Ram and Shyam are at 20 m and 40 m away from the intersection of tracks, respectively, and moving towards the intersection of the tracks.

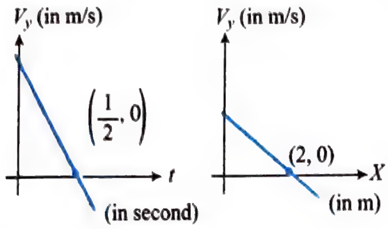
17. During the motion the magnitude of velocity of ram with respect to Shyam is
(1) $1\ \text{m s}^{-1}$ (2) $4\ \text{m s}^{-1}$
(3) $5\ \text{m s}^{-1}$ (4) $7\ \text{m s}^{-1}$

18. Shortest distance between them subsequently is
(1) 18 m (2) 15 m
(3) 25 m (4) 8m

19. The time t when they are at shortest distance from each other subsequently, is -
(1) 8.8 s (2) 12 s
(3) 15 s (4) 44 s

For Problems 20–22

Two graphs of the same projectile motion (in the xy-plane) projected from origin are shown. x-axis is along horizontal direction and y-axis is vertically upwards. Take $g = 10\ ms^{-2}$.



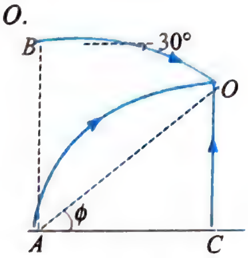
20. The projection speed is :
(1) $\sqrt{37}\ ms^{-1}$ (2) $\sqrt{41}\ ms^{-1}$
(3) $\sqrt{14}\ ms^{-1}$ (4) $\sqrt{40}\ ms^{-1}$

21. Projection angle with the horizontal is:
(1) $\tan^{-1}\left(\frac{4}{5}\right)$ (2) $\tan^{-1}\left(\frac{2}{3}\right)$
(3) $\tan^{-1}\left(\frac{5}{4}\right)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$

22. Maximum height attained from the point of projection is
(1) 1.25 m (2) 12.5 m
(3) 2.25 m (4) None of these

For Problems 23–25

Points A and C are on the horizontal ground and A and B are in same vertical plane at a distance of 1500 m. Simultaneously bullets are fired from A, B and C and they collide at O. The bullet at B is fired at an angle of 30° with horizontal towards the ground at velocity 100 m/s. The bullet at C is projected vertically upward at velocity of 100 m/s. The bullet projected from A reaches its maximum height at O.



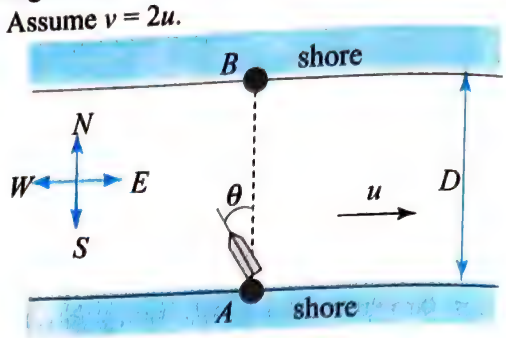
23. Find the time in which bullets will collide (seconds):
(1) 10 (2) 15
(3) 20 (4) 25

24. Find the elevation angle $\angle\theta = \angle OAC$:
(1) 60° (2) 45°
(3) 30° (4) 15°

25. Find the velocity of bullet at A:
(1) 50 m/s (2) $50\sqrt{7}$ m/s
(3) $60\sqrt{7}$ m/s (4) 60 m/s

For Problems 26–28

Two ports, A and B, on a North-South line are separated by a river of width D. The river flows east with speed u. A boat crosses the river starting from port A. The speed of the boat relative to the river is v. Assume $v = 2u$.



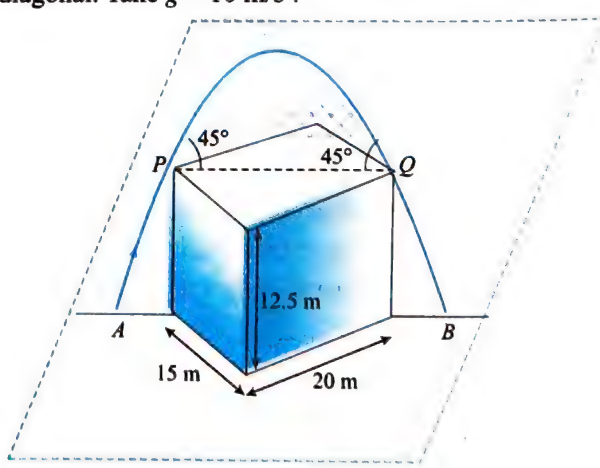
26. What is the direction of the velocity of boat relative to river, θ, so that it crosses directly on a line from A to B?
(1) 30° west of north (2) 30° east of north
(3) 60° west of north (4) 60° east of north

27. Suppose the boat wants to cross the river from A to the other side in the shortest possible time. Then what should be the direction of the velocity of boat relative to river?
(1) 30° west of north (2) 30° east of north
(3) 60° west of north (4) along north

28. The boat crosses the river from A to the other side in shortest possible time, then how far is the boat from the port B after crossing the river
(1) $D/\sqrt{2}$ (2) $\sqrt{2}D$
(3) $2D$ (4) $D/2$

For Problems 29–33

A particle is fired from A in the diagonal plane of a building of dimension 20 m (length) × 15 m (breadth) × 12.5 (height). just clears the roof diagonally and falls on the other side of the building at B. It is observed that the particle is travelling at an angle 45° with the horizontal when it clears the edges P and Q of the diagonal. Take $g = 10\ m/s^2$.



29. The speed of the particle at point P will be:
(1) $5\sqrt{10}$ m/s (2) $10\sqrt{5}$ m/s
(3) $5\sqrt{15}$ m/s (4) $5\sqrt{5}$ m/s

30. The angle of projection at A will be:
(1) 30° (2) 45°
(3) 60° (4) 75°

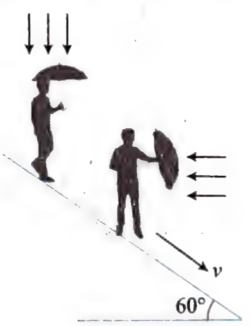
31. The speed of projection of the particle at A will be:
(1) $5\sqrt{10}$ m/s (2) $10\sqrt{5}$ m/s
(3) $5\sqrt{15}$ m/s (4) $5\sqrt{5}$ m/s

32. The speed of the particle at the top of the trajectory:
(1) $5\sqrt{10}$ m/s (2) $10\sqrt{5}$ m/s
(3) $5\sqrt{15}$ m/s (4) $5\sqrt{5}$ m/s

33. The range that is AB will be:
(1) $5\sqrt{10}$ m/s (2) $25\sqrt{3}$ m/s
(3) $5\sqrt{15}$ m/s (4) $25\sqrt{5}$ m/s

For Problems 34 and 35

A man standing on an inclined plain observes rain is falling vertically. When he starts moving down the inclined plain with velocity $v = 6$ m/s observes rain hitting him horizontally.



34. The actual velocity of rain is
(1) 3 m/s (2) $3\sqrt{3}$ m/s
(3) 4 m/s (4) none of these

35. The velocity of rain with respect to man when he is moving down
(1) 3 m/s (2) $3\sqrt{3}$ m/s
(3) 4 m/s (4) none of these

Matrix Match Type

1. If $\vec{v}_{mw}$ is the velocity of a man relative to water, $\vec{v}_w$ is the velocity of water, and $\vec{v}_m$ is the velocity of man relative to ground, match the following:

Column I	Column II
i. Minimum distance for $v_{m\omega} > v_w$	a. $\theta = \sin^{-1}\left(\frac{v_{mw}}{v_w}\right)$
ii. Minimum time for $v_{m\omega} \geq v_w$	b. $\vec{v}_m \perp \vec{v}_w$
iii. Minimum distance for $v_{m\omega} < v_w$	c. $\vec{v}_{mw} \perp \vec{v}_w$
iv. Minimum time for $v_{m\omega} < v_w$	d. $\theta = \sin^{-1}\frac{v_w}{v_{mw}}$

where θ is the angle between $\vec{v}_{mw}$ and the width of the river.

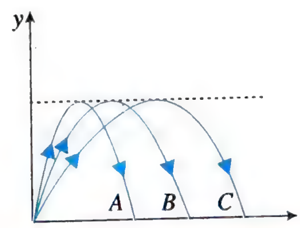
2. A ball is projected from the ground with velocity v such that its range is maximum.

Column I	Column II
i. Velocity at half of the maximum height	a. $\frac{\sqrt{3}\,v}{2}$
ii. Velocity at the maximum height	b. $\frac{v}{\sqrt{2}}$
iii. Change in its velocity when it returns to the ground	c. $v\sqrt{2}$
iv. Average velocity when it reaches the maximum height	d. $\frac{v}{2}\sqrt{\frac{5}{2}}$

3. Match the entries of Column I with that of Column II

Column I	Column II
i. For a particle moving in a circle	a. The acceleration may be perpendicular to its velocity.
ii. For a particle moving in a straight line	b. The acceleration may be in the direction of velocity
iii. For a particle undergoing projectile motion with the angle of projection α; $0 \leq \alpha \leq \frac{\pi}{2}$	c. The acceleration may be at some angle $\theta \left(0 < \theta < \frac{\pi}{2}\right)$ with the velocity.
iv. For a particle moving in space	d. The acceleration may be opposite to its velocity.

4. The trajectories of the motion of three particles are shown in figure. Match the entries of Column I with the entries of Column II.



Column I	Column II
i. Time of flight is least for	a. A
ii. Vertical component of the velocity is greatest for	b. B
iii. Horizontal component of the velocity is greatest for	c. C
iv. Launch speed is least for	d. No appropriate match given

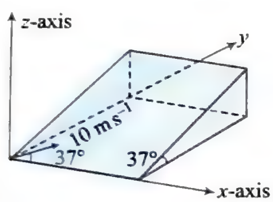
5. The path of projectile is represented by $y = Px - Qx^2$.

Column I		Column II	
i.	Range	**a.**	P/Q
ii.	Maximum height	**b.**	P
iii.	Time of flight	**c.**	$P^2/4Q$
iv.	Tangent of angle of projection is	**d.**	$\sqrt{\dfrac{2}{Qg}}P$

6. A body is projected with a velocity of 60 m s^{-1} at 30° to horizontal.

Column I		Column II	
i.	Initial velocity vector	**a.**	$60\sqrt{3}\hat{i} + 40\hat{j}$
ii.	Velocity after 3 s	**b.**	$30\sqrt{3}\hat{i} + 10\hat{j}$
iii.	Displacement after 2 s	**c.**	$30\sqrt{3}\hat{i} + 30\hat{j}$
iv.	Velocity after 2 s	**d.**	$30\sqrt{3}\hat{i}$

7. A small ball is projected along the surface of a smooth inclined plane with speed 10 m s^{-1} along the direction shown at $t = 0$. The point of projection is origin, z-axis is along vertical. The acceleration due to gravity is 10 m s^{-2}.



Column I lists the values of certain parameters related to motion of ball and Column II lists different time instants. Match appropriately.

Column I		Column II	
i.	Distance from x-axis is 2.25 m	**a.**	0.5 s
ii.	Speed is minimum	**b.**	1.0 s
iii.	Velocity makes angle 37° with x-axis	**c.**	1.5 s
		d.	2.0 s

8. As shown in the figure there is a particle of mass $\sqrt{3}$ kg, is projected with speed 10 m/s at an angle 30° with horizontal then match the following:

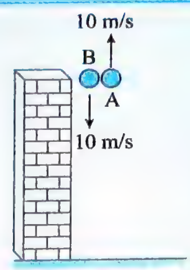
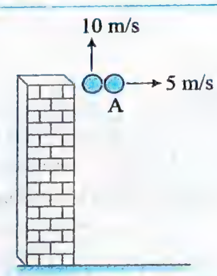
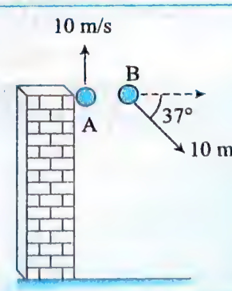
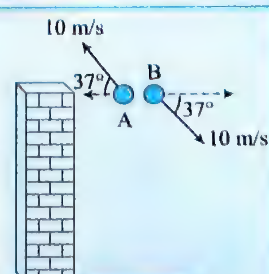
Column I		Column II	
i.	Average velocity (in m/s) during half of the time of flight is	**a.**	$\dfrac{1}{2}$
ii.	The time (in sec) after which the angle between velocity vector and acceleration vector becomes $\pi/2$	**b.**	$\dfrac{5}{2}\sqrt{13}$
iii.	Horizontal range (m)	**c.**	$5\sqrt{3}$
iv.	Change in linear momentum (N-s) when particle is at highest point	**d.**	at an angle of $\tan^{-1}\left(\dfrac{1}{2\sqrt{3}}\right)$ from horizontal

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	b,d	a	c	c
(2)	a, c	a	b, d	c
(3)	d	c	a, c	b
(4)	a, c	b	d	b

9. Both A and B are thrown simultaneously as shown from a very high tower.

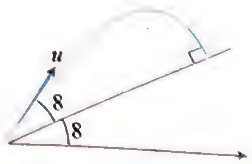
	Column I		Column II
i.		**p.**	distance between the two balls after two seconds is $8\sqrt{5}$ m
ii.		**q.**	distance between two balls after 2 seconds is 40 m
iii.		**r.**	magnitude of relative velocity of B with respect to A is $5\sqrt{5}$ m/s
iv.		**s.**	magnitude of relative velocity of B w.r.t. A is $5\sqrt{2}$ m/s.

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	b	c	b	d
(2)	b	c	a	b
(3)	d	c	a	b
(4)	c	b	d	b

10. The projectile collides perpendicularly with the inclined plane. (Refer the figure)



	Column I		Column II
i.	Maximum height attained by the projectile from the ground	a.	Zero
ii.	Maximum height attained by the projectile from inclined plane	b.	g
iii.	Acceleration of the projectile before striking the inclined plane	c.	$\frac{u^2\sin^2\beta}{2g\cos\alpha}$
iv.	Horizontal component of acceleration of the projectile.	d.	$\frac{u^2\sin^2(\alpha+\beta)}{2g}$

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	b	a	c	d
(2)	b	c	a	d
(3)	d	c	b	a
(4)	c	b	d	b

11. Two particles A and B moving in x-y plane are at origin at $t = 0$ sec. The initial velocity vectors of A and B are $\vec{u}_A = 8\hat{i}$ m/s and $\vec{u}_B = 8\hat{j}$ m/s. The acceleration of A and B are constant and are $\vec{a}_A = -2\hat{i}$ m/s^2 and $\vec{a}_B = -2\hat{j}$ m/s^2. Column-I gives certain statements regarding particle A and B. Column-I gives corresponding results. Match the statements in Column-I with corresponding results in Column-II.

	Column I		Column II
i.	The time (in seconds) at which velocity of A relative to B is zero	a.	$16\sqrt{2}$
ii.	The distance (in metres) between A and B when their relative velocity is zero	b.	$8\sqrt{2}$
iii.	The time (in seconds) after $t = 0$ sec, at which A and B are at same position	c.	8
iv.	The magnitude of relative velocity of A and B	d.	4 at the instant they are at same position.

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	d	a	c	b
(2)	b	c	a	d
(3)	d	c	b	a
(4)	c	b	d	b

12. A particle is projected from level ground. Assuming projection point as origin, x-axis along horizontal and y-axis along vertically upwards. If particle moves in x-y plane and its path is given by $y = ax - bx^2$ where a, b are positive constants. Then match the physical quantities given in column-I with the values given in column-II (g in column-II is acceleration due to gravity).

	Column I		Column II
i	Horizontal component of velocity	a.	$\frac{a}{b}$
ii.	Time of flight	b.	$\frac{a^2}{4b}$
iii.	Maximum height	c.	$\sqrt{\frac{g}{2b}}$
iv.	Horizontal range	d.	$\sqrt{\frac{2a^2}{bg}}$

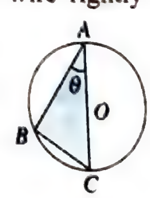
Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	d	a	c	b
(2)	c	d	b	a
(3)	a	b	c	d
(4)	b	c	b	a

Numerical Value Type

1. A particle is projected from the ground at an angle 30° with the horizontal with an initial speed 20 ms^{-1}. After how much time will the velocity vector of projectile be perpendicular to the initial velocity? [in second]
2. From the top of tower of height 80 m, two stones are projected horizontally with velocities 20 ms^{-1} and 30 ms^{-1} in opposite directions. Find the distance between both the stones on reaching the ground (in 10^2 m).
3. A bead is free to slide down on a smooth wire rightly stretched between points A and B on a vertical circle of radius 10 m. Find the time taken by the bead to reach point B, if the bead slides from rest from the highest point A on the circle.

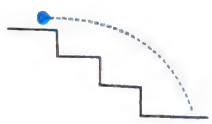


4. A golfer standing on level ground hits a ball with a velocity of 52 ms^{-1} at an angle θ above the horizontal. If $\tan\theta = 5/12$, then find the time for which the ball is atleast 15 m above the ground (take $g = 10\ ms^{-2}$).

5. A body is thrown with the velocity v_0 at an angle of θ to the horizon. Determine v_0 in ms^{-1} if the maximum height attained by the body is 5 m and at the highest point of its trajectory the radius of curvature is $r = 3$ m. Neglect air resistance. [Use $\sqrt{80}$ as 9]

6. A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of 1 ms^{-2} and the projection velocity in the vertical direction is 9.8 ms^{-1}. How far behind the boy will the ball fall on the car? (in meters)

7. A staircase contains three steps each 10 cm high and 20 cm wide. What should be the minimum horizontal velocity of the ball rolling off the uppermost plane so as to hit directly the lowest plane? (in ms^{-1})

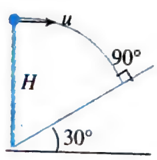


8. A particle is projected up an inclined plane of inclination β at an elevation α to the horizontal. Find the ratio between $\tan\alpha$ and $\tan\beta$, if the particle strikes the plane horizontally.

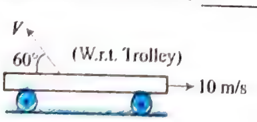
9. A particle is moving in a circle of radius R with constant speed. The time period of the particle is $T = 1$. In a time $t = T/6$, if the difference between average speed and average velocity of the particle is 2 ms^{-1}, find the radius R of the circle (in meters).

10. A ball is projected from the origin. The x- and y-coordinates of its displacement are given by $x = 3t$ and $y = 4t - 5t^2$. Find the velocity of projection (in ms^{-1}).

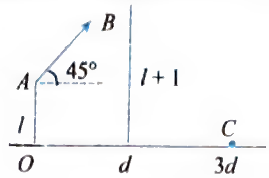
11. In figure, find the horizontal velocity u (in ms^{-1}) of a projectile so that it hits the inclined plane perpendicularly. Given $H = 6.25$ m.



12. A particle is projected from a stationary trolley. After projection, the trolley moves with a velocity $2\sqrt{15}$ m/s. For an observer on the trolley, the direction of the particle is as shown in the figure while for the observer on the ground, the ball rises vertically. The maximum height reached by the ball from the trolley is h meter. The value of h will be ________.

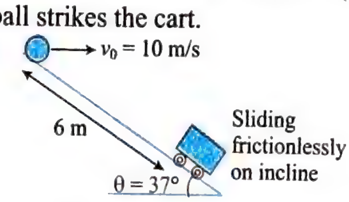


13. A projectile is launched at time $t = 0$ from point A which is at height 1 m above the floor with speed $v\ ms^{-1}$ and at an angle $\theta = 45°$ with the floor. It passes through a hoop at B which is 1 m above A and B is the highest point of the trajectory. The horizontal distance between A and B is d meters. The projectile then falls into a basket, hitting the floor at C a horizontal distance $3d$ meters from A. Find l (in m).



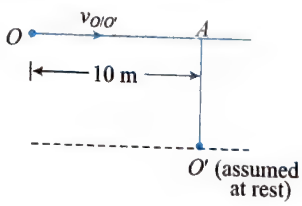
14. A student throws soft balls out of the window at different angles to the horizontal. All soft balls have the same initial speed $v = 10\sqrt{3}\ ms^{-1}$. It turns out that all soft balls' landing velocities make angles 30° or greater with the horizontal. Find the height h (in m) of the window above the ground.

15. A ball is projected horizontally from an incline so as to strike a cart sliding on the incline. Neglect height of cart and point of projection of ball above incline. At the instance the ball is thrown, the speed of cart is v (in m/s). Find v so that the ball strikes the cart.



16. A glass wind screen whose inclination with the vertical can be changed is mounted on a car. The car moves horizontally with a speed of 2m/s. Now the wind screen is placed at angle α with the vertical such that the rain drops falling vertically downwards with velocity 6 m/s strike the wind screen perpendicularly. What is the value of tanα?

17. Two particles are projected simultaneously from two points O and O' such that 10 m is the horizontal and 5 m is the vertical distance between them as shown in the figure. They are projected at the same inclination 60° to the horizontal with the same velocity 10 ms^{-1}. Find the time after which their separation becomes minimum.



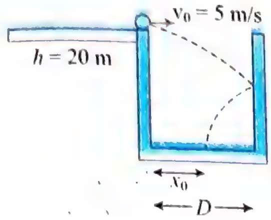
18. A particle is moving in x-y plane. At certain instant of time, the components of its velocity and acceleration are as follows:

$v_x = 3$ m/s, $v_y = 4$ m/s, $a_x = 2\ m/s^2$ and $a_y = 1\ m/s^2$. Find the rate of change of speed (in m/s^2) at this moment.

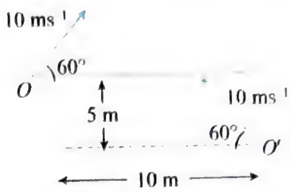
19. A ball is projected at an angle θ. As the ball flies through the air, the observer at the point of projection, follows it with his eyes. When it reaches maximum height Y, his eyes are directed at an angle ϕ with respect to the horizontal. What is the value of $\dfrac{\tan\theta}{\tan\varphi}$?

20. A ball leaves a horizontal table with velocity $v_0 = 5$ m/s. The ball bounces elastically from a vertical wall at a horizontal distance D (= 8 m) from the table, as shown in figure. The

ball then strikes the floor a distance x_0 from the table (g = 10 m/s^2). Find the value of x_0 (in m).



21. Two particles are projected simultaneously from two points O and O' such that 10 m is the horizontal and 5 m is the vertical distance between them as shown in the figure. They are projected at the same inclination 60° to the horizontal with the same velocity 10 ms^{-1}. Find the time (in sec) after which their separation becomes minimum.



22. The windscreens of two motorcars are having slopes 30° and 15° respectively. At what ratio v_1/v_2 of the velocities of cars will their drivers see the hailstones bounced by windscreen of their cars in the vertical direction? Assume hailstones are falling vertically.

Archives

JEE MAIN

Single Correct Answer Type

1. A water fountain on the ground sprinkles water all around it. If the speed of water coming out of the fountain is v, the total area around the fountain that gets wet is

(1) $\pi \frac{v^2}{g}$ (2) $\pi \frac{v^4}{g^2}$

(3) $\frac{\pi}{2}\frac{v^4}{g^2}$ (4) $\pi \frac{v^2}{g^2}$ **(AIEEE 2011)**

2. A projectile is given an initial velocity of $(\hat{i}+2\hat{j})$ m/s, where $\hat{i}$ is along the ground and $\hat{j}$ is along thevertical. If g = 10 m/s^2, the equation of its trajectory is

(1) $y = 2x - 5x^2$ (2) $y = x - 5x^2$

(3) $4y = 2x - 5x^2$ (4) $y = 2x + 5x^2$

(JEE Main 2013)

3. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground, is n times that taken by it to reach the highest point of its path. The relation between H, u and n is:

(1) $2gH = nu^2(n-2)$ (2) $gH = (n-2)u^2$

(3) $2gH = n^2u^2$ (4) $gH = (n-2)2u^2$

(JEE Main 2014)

4. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground, is n times that taken by it to reach the highest point of its path. The relation between H, u and n is

(1) $2gH = nu^2(n-2)$ (2) $gH = (n-2)u^2$

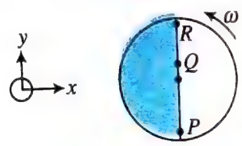
(3) $2gH = n^2u^2$ (4) $gH = (n-2)2u^2$

(JEE Main 2014)

JEE ADVANCED

Single Correct Answer Type

1. Consider a disc rotating in the horizontal plane with a constant angular speed ω about its center O. The disc has a shaded region on one side of the diameter and an unshaded region on the other side as shown in the figure. When the disc is in the orientation as shown, two pebbles P and Q are simultaneously projected at an angle towards R. The velocity of projection in the y–z plane is the same for both pebbles with respect to the disc. Assume that (i) they land back on the disc before the disc has completed 1/8 rotation, (ii) their range is less than half the disc radius, and (iii) ω remains constant throughout. Then



(1) P lands in the shaded region and Q in the unshaded region.

(2) P lands in the unshaded region and Q in the shaded region.

(3) Both P and Q land in the unshaded region.

(4) Both P and Q land in the shaded region.

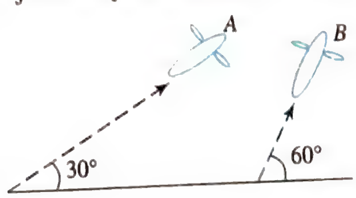
(IIT-JEE 2012)

Numerical Value Type

1. A train is moving along a straight line with a constant acceleration a. A boy standing in the train throws a ball forward with a speed of 10 m s^{-1}, at an angle of 60° to the horizontal. The body has to move forward by 1.15 m inside the train to catch the ball back to the initial height. The acceleration of the train, in m s^{-2}, is

(IIT-JEE 2011)

2. Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in the figure. The speed of A is $100\sqrt{3}$ ms^{-1}. At time t = 0 s, an observer in A finds B at a distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at $t = t_0$, A just escapes being hit by B, t_0 in seconds is



(JEE Advanced 2014)

3. A ball is projected from the ground at an angle of 45° with the horizontal surface. It reaches a maximum height of 120 *m* and returns to the ground. Upon hitting the ground for the first time, it loses half of its kinetic energy. Immediately after the bounce, the velocity of the ball makes an angle of 30° with the horizontal surface. The maximum height it reaches after the bounce, in metres, is................ .

(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	**2.** (2)	**3.** (3)	**4.** (4)	**5.** (1)
6. (4)	**7.** (1)	**8.** (3)	**9.** (2)	**10.** (2)
11. (4)	**12.** (1)	**13.** (1)	**14.** (4)	**15.** (2)
16. (4)	**17.** (2)	**18.** (4)	**19.** (2)	**20.** (3)
21. (1)	**22.** (4)	**23.** (1)	**24.** (4)	**25.** (3)
26. (2)	**27.** (1)	**28.** (1)	**29.** (4)	**30.** (2)
31. (1)	**32.** (4)	**33.** (2)	**34.** (1)	**35.** (1)
36. (4)	**37.** (3)	**38.** (3)	**39.** (4)	**40.** (2)
41. (3)	**42.** (3)	**43.** (3)	**44.** (2)	**45.** (2)
46. (4)	**47.** (1)	**48.** (1)	**49.** (4)	**50.** (3)
51. (3)	**52.** (4)	**53.** (4)	**54.** (1)	**55.** (1)
56. (3)	**57.** (3)	**58.** (3)	**59.** (4)	**60.** (3)
61. (1)	**62.** (1)	**63.** (3)	**64.** (1)	**65.** (2)
66. (3)	**67.** (4)	**68.** (2)	**69.** (3)	**70.** (1)
71. (1)	**72.** (4)	**73.** (2)	**74.** (2)	**75.** (2)
76. (3)	**77.** (2)	**78.** (3)	**79.** (3)	**80.** (3)

Multiple Correct Answers Type

1. (2),(3)	**2.** (2),(3),(4)	**3.** (1),(2)
4. (1),(3)	**5.** (1),(3),(4)	**6.** (3),(4)
7. (1),(2)	**8.** (2),(3)	**9.** (1),(3)
10. (1),(2)	**11.** (2),(4)	**12.** (1),(3)
13. (1),(2)	**14.** (1),(2),(3)	**15.** (1),(3)
16. (1),(2),(4)	**17.** (1),(3),(4)	**18.** (1),(4)
19. (2),(4)	**20.** (1),(3)	**21.** (1),(3),(4)
22. (1),(2),(3),(4)	**23.** (2),(3)	**24.** (1),(4)
25. (1),(3),(4)		

Linked Comprehension Type

1. (2)	**2.** (4)	**3.** (4)	**4.** (4)	**5.** (1)
6. (1)	**7.** (2)	**8.** (1)	**9.** (4)	**10.** (4)
11. (2)	**12.** (3)	**13.** (1)	**14.** (2)	**15.** (3)
16. (1)	**17.** (3)	**18.** (4)	**19.** (1)	**20.** (2)

21. (3)	**22.** (1)	**23.** (1)	**24.** (3)	**25.** (2)
26. (2)	**27.** (2)	**28.** (4)	**29.** (1)	**30.** (3)
31. (2)	**32.** (4)	**33.** (2)	**34.** (1)	**35.** (2)

Matrix Match Type

1. i $\rightarrow$ b, d; ii $\rightarrow$ c; iii $\rightarrow$ a; iv $\rightarrow$ c
2. i $\rightarrow$ a; ii $\rightarrow$ b; iii $\rightarrow$ c; iv $\rightarrow$ d
3. i $\rightarrow$ a, c; ii $\rightarrow$ b, d; iii $\rightarrow$ a, c; iv $\rightarrow$ a, b, c, d
4. i $\rightarrow$ d; ii $\rightarrow$ d; iii $\rightarrow$ c; iv $\rightarrow$ a
5. i $\rightarrow$ a; ii $\rightarrow$ c; iii $\rightarrow$ d; iv $\rightarrow$ b
6. i $\rightarrow$ c; ii $\rightarrow$ d; iii $\rightarrow$ a; iv $\rightarrow$ b
7. i $\rightarrow$ a, c; ii $\rightarrow$ c; iii $\rightarrow$ d

8. (1)	**9.** (2)	**10.** (3)	**11.** (1)	**12.** (2)

Numerical Value Type

1. (4)	**2.** (2)	**3.** (2)	**4.** (2)	**5.** (9)
6. (2)	**7.** (2)	**8.** (2)	**9.** (7)	**10.** (5)
11. (5)	**12.** (9)	**13.** (3)	**14.** (5)	**15.** (4)
16. (3)	**17.** (1)	**18.** (2)	**19.** (2)	**20.** (6)
21. (1)	**22.** (3)			

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	**2.** (1)	**3.** (1)	**4.** (1)

JEE Advanced

Single Correct Answer Type

1. (3)

Numerical Value Type

1. (5)	**2.** (5)	**3.** (30)

6 Newton's Laws of Motion (Without Friction)

INTRODUCTION

In kinematics, we dealt with the motion of particles based on the definitions of position, velocity, and acceleration without analyzing its cause. We would like to be able to answer general questions related to the cause of motion such as "what mechanism causes change in motion?" and "why do some objects accelerate at higher rates than others?" In this section, we will discuss the causes of change in the motion of particles using the concepts of force and mass. We will discuss the three fundamental laws of motion, which are based on experimental observations and were formulated by Sir Isaac Newton.

NEWTONIAN MECHANICS

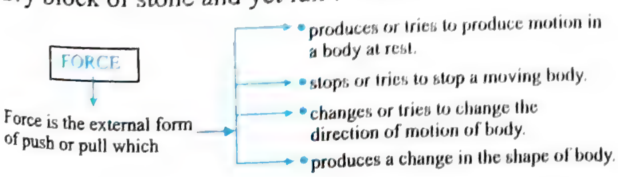
How are force and acceleration related?—This relationship was first understood by Isaac Newton. The study of that relation, as Newton presented it, is called *Newtonian mechanics*. Newtonian mechanics covers the discussion of the movement of objects under the influence of forces by making use of Newton's three laws.

If the speeds of the interacting objects are very large, comparable to the speed of light, Newtonian mechanics is not applicable. In that case we must replace Newtonian mechanics with Einstein's special theory of relativity, which holds at any speed, including those near the speed of light. If the interacting objects are on the scale of atomic structure, we must replace Newtonian mechanics with quantum mechanics.

CONCEPT OF FORCE

How a body moves? It is determined by the interaction of the body with its environment. This interaction is called force. The concept of force gives us a quantitative description of the interaction between two bodies or between a body and its environment. As a result of everyday experiences, everybody has a basic understanding of the concept of force. When you push or pull an object, you exert a force on it. You exert a force when you throw or kick a ball. In these examples, the word "force" is associated with the result of muscular activity and with some change in the state of motion of an object. However, force does not always cause an object to move.

For example, as you sit reading a book, the gravitational force acts on your body and yet you remain stationary. You can push on a heavy block of stone and yet fail to move it.



CLASSIFICATION OF FORCES

Based on the nature of the interaction between two bodies, forces may be broadly classified as under:

Contact forces: Tension, normal reaction, friction, etc. These forces act between bodies in contact.

Field forces (Non-contact forces): Weight, electrostatic forces, etc. Forces that act between bodies separated by a distance without any actual contact.

Since we will encounter these forces in our analysis, we will briefly discuss each force and how it acts between two bodies, its nature, etc., and how we are going to take it into account. We will discuss some special forces.

NEWTON'S LAWS OF MOTION

The entire classical mechanics is based upon Newton's laws of motion. These, in fact, are simply known as laws of motion. These laws provide the basis for understanding the effect that forces have on an object.

NEWTON'S FIRST LAW OF MOTION

According to this law, a body continues to be in its state of rest or uniform motion along a straight line unless it is acted upon by some external force to change the state.

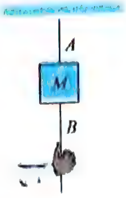
An object lying anywhere keeps on lying there unless someone moves it. For example, a chair, a table, a bed, etc., cannot change their position on their own, i.e., a body at rest cannot start moving on its own. The reverse is also true, though it is slightly difficult to perceive. If there were no forces of friction and air resistance, etc., a body moving uniformly along a straight line will never stop on its own.

This means a body, on its own, cannot change its state of rest or state of uniform motion along a straight line. This inability of a body to change by itself its state of rest or state of uniform motion along a straight line is called inertia of the body. Hence, Newton's first law defines inertia and is rightly called the *law of inertia*.

Quantitatively, the inertia of a body is measured by the mass of the body. The heavier the body, the greater the force required to change its state and hence, the greater is its inertia. The reverse is also true. Inertia are of three types: inertia of rest, inertia of motion, and inertia of direction.

Inertia of rest It is the inability of a body to change by itself, its state of rest. This means a body at rest remains at rest and cannot start moving by its own.

Examples



1. A person who is standing freely in bus is thrown backward when the bus starts suddenly.
2. In the arrangement shown in figure,
   (a) If string B is pulled with a sudden jerk, then it will experience tension while due to the inertia of rest of mass M, this force will not be transmitted to string A and so string B will break.
   (b) If string B is pulled steadily, the force applied to it will be transmitted from string B to A through mass M and as tension in A will be greater than that in B by Mg (weight of mass M), string A will break.
3. If we place a coin on a smooth piece of card board covering a glass and strike the card board piece suddenly with a finger, the cardboard slips away and the coin falls into the glass due to the inertia of rest.

Inertia of motion It is the inability of a body to change itself its state of uniform motion, i.e., a body in uniform motion can neither accelerate nor retard by its own.

Examples

1. A person jumping out of a moving train may fall forward.
2. An athlete runs a certain distance before taking a long jump. This is because the velocity acquired by running is added to the velocity of the athlete at the time of jump. Hence, he can jump over a longer distance.

Inertia of direction It is the inability of a body to change by itself the direction of motion.

Examples

1. The rotating wheel of any vehicle throw out mud, if any, tangentially due to directional inertia.
2. When a car goes round a curve suddenly, the person sitting inside is thrown outwards.

Important Points:

- According to Newton's first law, a state of rest (zero velocity) and a state of constant velocity are completely equivalent, in the sense that neither one requires the application of a net force to sustain it. An object continues in a state of rest or in a state of motion at a constant velocity (constant speed in a constant direction), unless compelled to change that state by a net force.
- Newton's first law says there are two possible states for an object with no net force on it. An object at rest is said to be in static equilibrium. An object moving with constant velocity is said to be in dynamic equilibrium.

Inertial Frame of Reference

We can observe a moving object from any number of reference frames. Newton's first law is not true in all reference frames, but we can always find reference frames in which it (as well as the rest of Newtonian mechanics) is true. Such special frames are referred to as **inertial reference frames,** or simply **inertial frames**. The acceleration of an inertial reference frame is zero, so it moves with a constant velocity. All of Newton's laws of motion are valid in inertial reference frames, and when we apply these laws, we will be assuming such a reference frame. In particular, the earth itself is a good approximation of an inertial reference frame. Any reference frame that moves with constant velocity relative to an inertial frame is itself an inertial frame. Clearly, Newton's first law does not hold for observers who use the accelerating frame of reference. As a result, such a reference frame is said to be *non-inertial*. All accelerating reference frames are *non-inertial*.

Linear Momentum, p Linear momentum is a vector quantity. It is the quantity of motion in a body. It is given by the product of mass and velocity. Thus,

$$p = mv$$

In vector form, $\vec{p} = m\vec{v}$. The direction of $\vec{p}$ is same as that of $\vec{v}$.

Newton's Second Law of Motion

According to this law, the rate of change of linear momentum of a body is directly proportional to the external force applied on the body and this change takes place always in the direction of the force applied.

When an unbalanced force is applied on a body, the momentum of the body changes. The rate of change of momentum with respect to time is defined as the net external force acting on the body, i.e., $\vec{F}_{ext} = d\vec{p}/dt$, where linear momentum $\vec{p} = m\vec{v}$, m is the mass of the body, and $\vec{v}$ is the instantaneous velocity.

$$\vec{F}_{ext} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

If m is constant, i.e., $\frac{dm}{dt} = 0 \quad \Rightarrow \quad \vec{F}_{ext} = m\vec{a}$

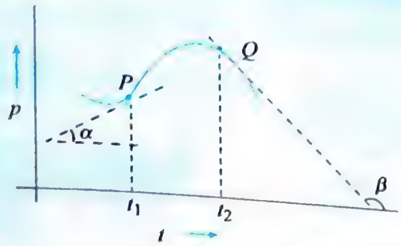
If the mass of the body is constant, the acceleration of the body is inversely proportional to its mass and directly proportional to the resultant force acting on it, i.e., $\sum \vec{F} = m\vec{a}$. This vector equation is equivalent to three algebraic equations:

$$\sum F_x = ma_x,\ \sum F_y = ma_y, \text{ and } \sum F_z = ma_z$$

Important Points:

- $\vec{F} = 0$, second law gives $\vec{a} = 0$; therefore, it is consistent with the first law.
- The second law of motion is a vector law. It means whatever be the direction of the instantaneous velocity of a particle, if any net external force is acting on it, this force will change only that component of the velocity which is in the direction of force.
- Strictly speaking, this law applies to particles, i.e., point masses only. However, with the introduction of the concept of the center of mass, this law can now be applied in case of extended bodies or a system of point masses also. You will study this in the chapter on systems of particles and rotational motion.

- The second law is a local law. This means that it applies to a particle at a particular instant without taking into consideration any history of the particle or its motion.
- As $\vec{F} = \frac{d\vec{p}}{dt}$, where $\vec{p}$ denotes momentum, the slope of momentum versus time graph gives force. In figure, $\tan\alpha$ gives the force at $t = t_1$ and $\tan\beta$ gives the force at $t = t_2$. The force $\vec{F}$ in the law stands for the net external force. Any internal forces in the system are not included in $\vec{F}$.



PRINCIPLE OF CONSERVATION OF LINEAR MOMENTUM

The second law suggests: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$;

If no external force acts on the system $\vec{F}_{net} = 0$, then $\vec{p}$ = constant. It means if no external force acts on a system, the linear momentum of the system will remain constant. This statement is called **principle of conservation of momentum.**

Practical Applications of the Principle of Conservation of Linear Momentum

- When a bullet is fired from a gun, the gun recoils or gives a kick in backward direction. Let M be the mass of gun and m, the mass of the bullet. Initially, both the gun and the bullet are at rest. On firing the gun, suppose that the bullet moves with a velocity $\vec{v}$ and the gun moves with velocity $\vec{V}$.

  According to the principle of conservation of momentum, total momentum of gun and bullet before firing = total momentum of gun and bullet after firing.

  i.e. $0 = M\vec{V} + m\vec{v}$

  or, $\vec{V} = -\frac{m}{M}\vec{v}$

  The negative sign shows that $\vec{V}$ and $\vec{v}$ are in opposite direction, i.e. as the bullet moves forward, the gun will move in backward direction. The backward motion of the gun is called recoil of the gun.
- When a man jumps from a boat to the shore, the boat slightly moves away from the shore. Initially, the total momentum of the boat and the man is zero. When the man jumps from the boat to the shore, total momentum of man and the boat will be zero only if the boat moves in opposite direction.
- A rocket works on the principle of conservation of momentum. As the fuel in the rocket undergoes combustion, the burnt gases leave the body of the rocket with a large velocity in downward direction and thus provide upthrust to the rocket. If we assume that the fuel is burnt at a constant rate, then the rate of change of momentum of the rocket will be constant. As more and more fuel gets burnt, the mass of the rocket goes on decreasing and it leads to increase in the velocity of the rocket more and more rapidly.

  It may be pointed out that rocket propulsion is an application of the principle of conservation of momentum to a situation, in which the mass of the system goes on changing.
- If an astronaut in open space, away from space ship, wants to return to his space ship, he can do so by throwing something in a direction opposite to that in which the space ship is moving. When the astronaut throws some object away from the space ship, he himself will recoil i.e. will move in the opposite direction. Due to this, the astronaut will move towards the space ship.

IMPULSE

According to Newton's second law, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some problems. To build a better understanding of this important concept, let us assume that a net force $\Sigma\vec{F}$ acts on a particle and that this force may vary with time. According to Newton's second law,

$$\Sigma\vec{F} = \frac{d\vec{p}}{dt} \text{ or } d\vec{p} = \Sigma\vec{F}\,dt \quad \text{...(i)}$$

We can integrate this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle changes from $\vec{p}_i$, at time t_i, to $\vec{p}_f$, at time t_f, integrating Eq. (i) gives

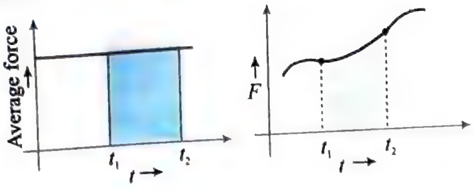
$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \sum \vec{F}\,dt \quad \text{...(ii)}$$

To evaluate the integral, we need to know how the net force varies with time. The quantity on the right side of this equation is a vector called the impulse of the net force $\Sigma\vec{F}$ acting on a particle over the time interval $\Delta t = t_f - t_i$.

$$\vec{J} = \int_{t_i}^{t_f} \sum \vec{F}\,dt \quad \text{...(iii)}$$

From its definition, we see that impulse $\vec{J}$ is a vector quantity having a magnitude equal to the area under the force–time curve as described in figure. It is assumed the force varies with time in the general manner shown in figure and is non-zero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is same as the direction of the change in momentum. Impulse has the dimensions of momentum, that is, ML/T. Impulse is not a property of a particle; rather, it is a measure of the degree to which an external force changes the particle's momentum.

Equation (ii) is an important statement known as the impulse–momentum theorem: if we plot a graph between average force and time [Figure], the area under the curve will give the impulse imparted during the time interval under consideration.

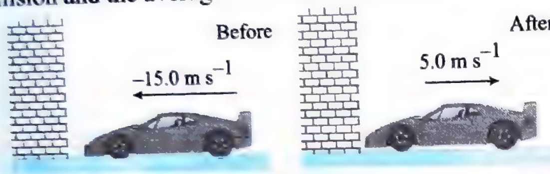


(a) (b)

$$J = \int_{t_1}^{t_2} F\,dt = \int_{t_1}^{t_2} dp = \text{Total change in linear momentum} = \text{Impulse}$$

ILLUSTRATION 6.1

In a particular crash test, a car of mass 1500 kg collides with a wall as shown in figure. The initial and final velocities of the car are $\vec{v}_i = -15.0\hat{i}\ \text{m s}^{-1}$ and $\vec{v}_f = 5.00\hat{i}\ \text{m s}^{-1}$, respectively. If the collision lasts 0.150 s, find the impulse caused by the collision and the average force exerted on the car.



Sol. The collision time is short, so we can imagine the car being brought to rest very rapidly and then moving back in the opposite direction with a reduced speed.

Let us assume that the force exerted by the wall on the car is large compared with other forces on the car (such as friction and air resistance). Furthermore, the gravitational force and the normal force exerted by the road on the car are perpendicular to the motion and, therefore, do not affect the horizontal momentum. Therefore, we categorize the problem as one in which we can apply the impulse approximation in the horizontal direction.

Evaluating the initial and final momenta of the car, we get

$$\vec{p}_i = m\vec{v}_i = (1500\,\text{kg})(-15.0\hat{i}\,\text{m s}^{-1}) = -2.25\times10^4\,\hat{i}\ \text{kg m s}^{-1}$$

$$\vec{p}_f = m\vec{v}_f = (1500\,\text{kg})(5.0\hat{i}\,\text{m s}^{-1}) = 0.75\times10^4\,\hat{i}\ \text{kg m s}^{-1}$$

Impulse on the car:

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$= 0.75\times10^4\,\hat{i}\ \text{kg m s}^{-1} - (-2.25\times10^4\,\hat{i}\,\text{kg m s}^{-1})$$

$$\Rightarrow \quad \vec{J} = 300\times10^4\,\hat{i}\,\text{kg m s}^{-1}$$

The average force exerted by the wall on the car

$$\vec{F}_{avg} = \frac{\Delta\vec{p}}{\Delta t} = \frac{3.00\times10^4\,\hat{i}\,\text{kg m s}^{-1}}{0.150\ \text{s}} = 2.00\times10^5\,\hat{i}\ \text{N}$$

ILLUSTRATION 6.2

A body of mass $m = 1$ kg falls from a height $h = 20$ m from the ground level.

(a) What is the magnitude of total change in momentum of the body before it strikes the ground?

(b) What is the corresponding average force experienced by it? (g = 10m s^{-2})

Sol.

(a) The body falls from rest ($u = 0$) through a distance h before striking the ground, the speed v of the body is given by kinematical equation: $v^2 = u^2 + 2as$;

Putting $a = g$ and $s = h$, we obtain $v = \sqrt{2gh}$

Thus, the magnitude of the total change in momentum of the body just before it strikes the ground $= \Delta p = |mv - 0| = mv$

where $v = \sqrt{2gh} \Rightarrow \Delta p = m\sqrt{2gh}$

or $\Delta p = (1)\sqrt{(2\times10\times20)}\ \text{kg m s}^{-1} = 20\ \text{kg m s}^{-1}$. ...(i)

(b) We define average force experienced by the body,

$$\vec{F}_{av} = \frac{\Delta\vec{p}}{\Delta t};\ \Delta t = \text{time of motion of the body} = t\text{(say)}.$$

We know $\Delta p = 20$ kg m s^{-1} from (i). Now, let us find t using the facts given in problem. From kinematics, we know

$$s = ut + \frac{1}{2}at^2 \qquad \text{(here } u = 0,\ s = h, \text{ and } a = g)$$

$$\Rightarrow \quad h = \frac{1}{2}gt^2 \quad \text{or} \quad t = \sqrt{\frac{2h}{g}}$$

$$\therefore \quad F_{av} = \frac{\Delta p}{\Delta t} = \frac{\Delta p}{t}$$

Putting the general values of Δp and t, we get

$$F_{av} = \frac{m\sqrt{2gh}}{\sqrt{2h/g}} = \text{mg} \Rightarrow \vec{F}_{av} = m\vec{g}$$

where mg is the weight of the body and $\vec{g}$ is directed vertically downward. Therefore, the body experiences a constant vertically downward force of magnitude mg.

ILLUSTRATION 6.3

An iron ball of mass $m = 50$ g falls from a height of $h_1 = 5$m and rises upto $h_2 = 3.2$m after colliding with the horizontal surface. If the time of contact of the glass half is $\Delta t = 0.02$s, find the average contact force exerted on the ball by the horizontal surface.

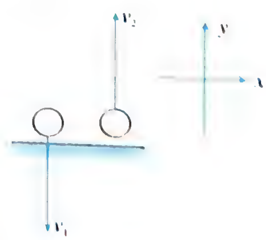
Sol. The change in linear momentum during collision is

$$\Delta\vec{p} = m\vec{v}_2 - m\vec{v}_1 \quad = m(\vec{v}_2 - \vec{v}_1)$$

$$= m[v_2\hat{j} - (-v_1\hat{j})] = m[(v_1 + v_2)]\hat{j}$$

The average force during the collision is

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \frac{m(v_1+v_2)\hat{j}}{\Delta t}$$



We can calculate v_1 and v_2 by using kinematics.

Using $v^2 = u^2 + 2as$

Putting $a = g$ and $s = h$, we get

$$v_1 = \sqrt{2gh_1},\ v_2 = \sqrt{2gh_2}$$

$$\vec{F} = \frac{m(\sqrt{2gh_1} + \sqrt{2gh_2})\hat{j}}{\Delta t}$$

or

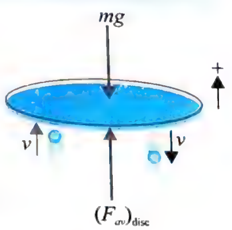
$$= \left(\frac{50}{1000}\right)\frac{(\sqrt{2\times10\times5} + \sqrt{2\times10\times3.2})}{0.02}\hat{j} = 45\hat{j}\text{ N}$$

During collision, the horizontal surface pushes the ball up with an average force of 45 N.

ILLUSTRATION 6.4

A disc of mass 10 g is kept floating horizontally by throwing 10 marbles s^{-1} against it from below. If the mass of each marble is 5 g, calculate the velocity with which the marbles are striking the disc. Assume that the marbles strike the disc normally and rebound downward with the same speed.

Sol. As 10 marbles strike against the disc per second, the above change in momentum of 10 marbles takes place in 1 s. Therefore, rate of change of momentum of the marbles or the force exerted by the marbles on the disc will balance the weight of the disc.



Let v be the speed with which the marbles strike the disc and taking upward direction as positive.

Then, initial momentum of 10 marbles (striking s^{-1}), $p_1 = 10(mv)$

Final momentum of 10 marbles (striking s^{-1}),

$$p_2 = 10(-mv) = -10mv$$

The change in the momentum of the 10 marbles per second,

$$\Delta p = p_2 - p_1 = -10mv - (10mv) = -20mv$$

Therefore, rate of change of momentum of the marbles or the average force exerted by the disc on the marbles,

$$\left(\vec{F}_{av}\right)_{mar} = \frac{\Delta\vec{p}}{\Delta t} = \frac{-20mv}{1} = -20mv \text{ (Downward direction)}$$

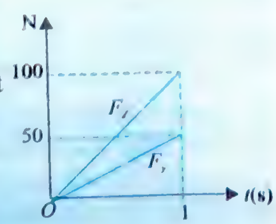
Hence force applied by marbles on the disc will be in upward direction. Hence, $\left(\vec{F}_{av}\right)_{disc} = 20mv$

Since the disc remains floating, $\left(\vec{F}_{av}\right)_{disc}$ = weight of the disc = Mg

or $\quad 20 \times 5 \times 10^{-3} v = 10 \times 10^{-3} \times 10 \Rightarrow v = 1\text{ m/s}$

ILLUSTRATION 6.5

A particle is acted upon by a force for 1 second whose x-component remains constant at $F_x = 5\sqrt{19}$ N but y and z components vary with time as shown in the graph. Calculate the magnitude of change in momentum of the particle in 1 sec. What angle does the impulse make with x-axis?



Sol. We know impulse $(\vec{J})$ = Change in linear momentum $(\Delta\vec{P})$

or $\quad \vec{J} = \Delta\vec{P} = \int\vec{F}dt = \hat{i}\int F_x dt + \hat{j}\int F_y dt + \hat{k}\int F_z dt \quad$ ...(i)

Here, the area under F_y–t graph gives $\int F_y dt$ and the area under F_z–t graph gives $\int F_z dt$.

$$\vec{J} = \Delta\vec{P} = \hat{i}\left(\int_0^1 5\sqrt{19}\,dt\right) + \hat{j}\left(\frac{1}{2}\times50\times1\right) + \hat{k}\left(\frac{1}{2}\times100\times1\right)$$

$$\vec{J} = \left(5\sqrt{19}\hat{i} + 25\hat{j} + 50\hat{k}\right)\text{Ns}$$

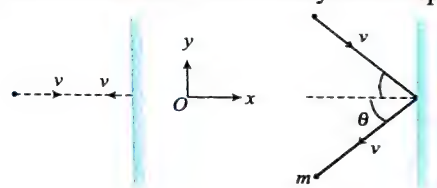
$$\therefore \quad |\vec{J}| = \sqrt{\left(5\sqrt{19}\right)^2 + 25^2 + 50^2} = 60\text{ Ns}$$

If angle made by $\vec{J}$ with x axis is θ, then

$$\cos\theta = \frac{5\sqrt{19}}{60} = \frac{\sqrt{19}}{12} \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{19}}{12}\right)$$

CONCEPT APPLICATION EXERCISE 6.1

1. As shown in figure, two identical balls strike a rigid wall with equal speeds but at different angles of incidence. They are reflected back without any loss in speed.

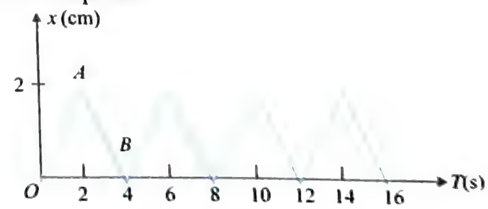


(a) (b)

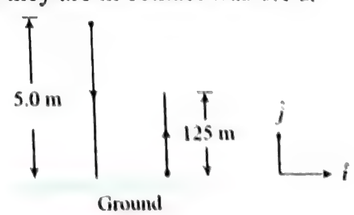
(a) Determine the direction of force exerted by each ball on the wall.

(b) Determine the ratio of impulse imparted by the two balls on the wall in both cases.

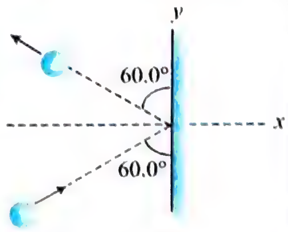
2. Figure shows the position–time graph of a particle of mass 0.04 kg. Suggest a suitable physical context for this motion. What is the time between two consecutive impulses received by the particle? What is the magnitude of each impulse?



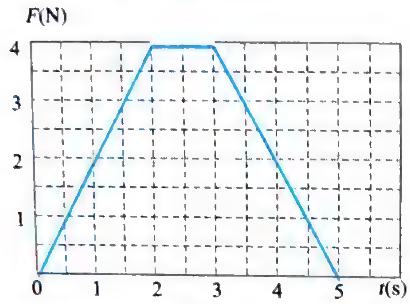
3. A rubber ball of mass 50 g falls from a height of 5 m and rebounds to a height of 1.25 m. Find the impulse and the average force between the ball and the ground if the time for which they are in contact was 0.1 s.



4. A 3-kg steel ball strikes a wall with a speed of 10.0 ms^{-1} at an angle of 60.0° with the surface of the wall. The ball bounces off with the same speed and same angle. If the ball was in contact with the wall for 0.2 s, find the average force exerted by the wall on the ball.

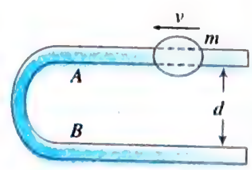


5. The magnitude of the net force exerted in the x direction on a 2.50-kg particle varies with time as shown in figure. Find (a) the impulse of the force, (b) the final velocity the particle attains if it is originally at rest, (c) its final velocity if its original velocity is –2.0 ms^{-1}, and (d) the average force exerted on the particle for the time interval between 0 and 5 s.



6. During a heavy rain, hailstones of average size 1.0 cm in diameter fall with an average speed of 20 $m s^{-1}$. Suppose 2000 hailstones strike every square meter of a 10 m × 10 m roof perpendicularly in one second and assume that the hailstones do not rebound. Calculate the average force exerted by the falling hailstones on the roof. Density of the hailstones is 900 $kg m^{-3}$.

7. A U-shaped smooth wire has a semi-circular bending between A and B as shown in figure. A bead of mass m moving with uniform speed v through the wire enters the semicircular bend at A and leaves at B. Find the average force exerted by the bead on the part AB of the wire.



8. Wind with a velocity 100 km h^{-1} blows normally against one of the walls of a house with an area of 108 m^2. Calculate the force exerted on the wall if the air moves parallel to the wall after striking it and has a density of 1.2 kg m^{-3}.

ANSWERS

1. (a) Force will be along x-direction on the wall.

(b) $\dfrac{2mv}{2mv\cos\theta}, \dfrac{1}{\cos\theta}$

2. 8×10^{-4} Ns, 2 s **3.** $0.75\hat{j}$ Ns, 7.5 N **4.** $150\sqrt{3}$ N

5. (a) 12 Ns **(b)** 4.8 ms^{-1} **(c)** 2.8 ms^{-1} **(d)** 2.40 N

6. 1900 N **7.** $\dfrac{4mv^2}{\pi d}$

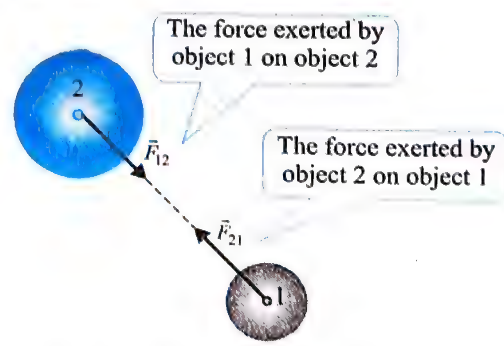
8. $1.2 \approx 10^5$ N

NEWTON'S THIRD LAW OF MOTION

Force is always a two-body interaction. The first law describes qualitatively and the second law describes quantitatively. What happens to a body if a force acts on it? Both these laws do not reveal anything about what happens to the other body participating in the interaction responsible for the force. For understanding the force during two-body interaction, let us do one activity. If you press against a corner of a book with your fingertip, the book pushes back and makes a small dent in your skin. If you push harder, the book does the same and the dent in your skin is a little larger. This simple activity illustrates that forces are interactions between two objects: when your finger pushes on the book, the book pushes back on your finger. This important principle is known as Newton's third law, which states that

If two objects interact, the force $\vec{F}_{12}$ exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\vec{F}_{21}$ exerted by object 2 on object 1:

$\vec{F}_{12} = -\vec{F}_{21}$, where the minus sign means that these two forces are in opposite directions. The force that object 1 exerts on object 2 is popularly called the *action force*, and the force of object 2 on object 1 is called the *reaction force*. Note that these two forces do not act on the same body but are the forces with which two bodies act on each other.



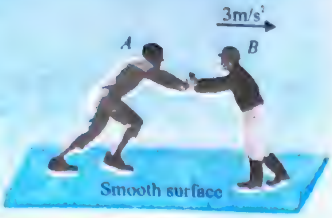
To every action, there is always an opposed equal reaction. It doesnot matter which force we call action and which we call reaction. The important thing is that they are co pairs of a single interaction, and neither force exists without the other.

- When you walk, you interact with the floor. You push against the floor, and the floor pushes against you. The pair of forces occurs at the same time (they are simultaneous).
- Likewise, the tires of a car push against the road while the road pushes back on the tires– the tires and road simultaneously push against each other.
- In swimming, you interact with water, pushing water backwards, while water simultaneously pushes you forward—you and water push against each other.

The reaction forces are what account for our motion in these examples. These forces depend on friction; a person or car on ice, for example, may be unable to exert the action force to produce the needed reaction force. Neither force exists without the other.

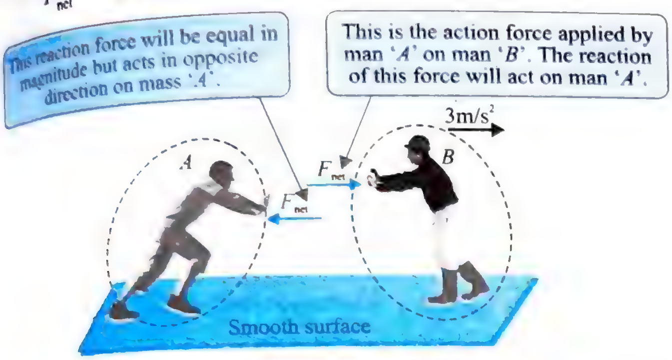
ILLUSTRATION 6.6

Man 'A' of mass 60 kg pushes the other man 'B' of mass 75 kg due to which man 'B' starts moving with acceleration 3 m/s^2. Calculate the acceleration of man 'A' at that instant.



If man 'B' accelerates with acceleration 3 m/s^2, the net force on man 'B' should be in the direction of acceleration and it should be

$$F_{net} = m_B a_B = 75 \times 3 = 225 \text{ N}$$



According to Newton's third law of motion, if two objects interact, the force $\vec{F}_{AB}$ exerted by object A on object B is equal in magnitude and opposite in direction to the force $\vec{F}_{BA}$ exerted by object B on object A,

From F.B.D of man 'A'

$$a_A = \frac{225}{60} = \frac{15}{4} \text{ m/s}^2$$

FREE-BODY DIAGRAMS

In free-body diagrams (FBDs), the object of interest is isolated from its surroundings, and the interactions between the object and the surroundings are represented in terms of forces.

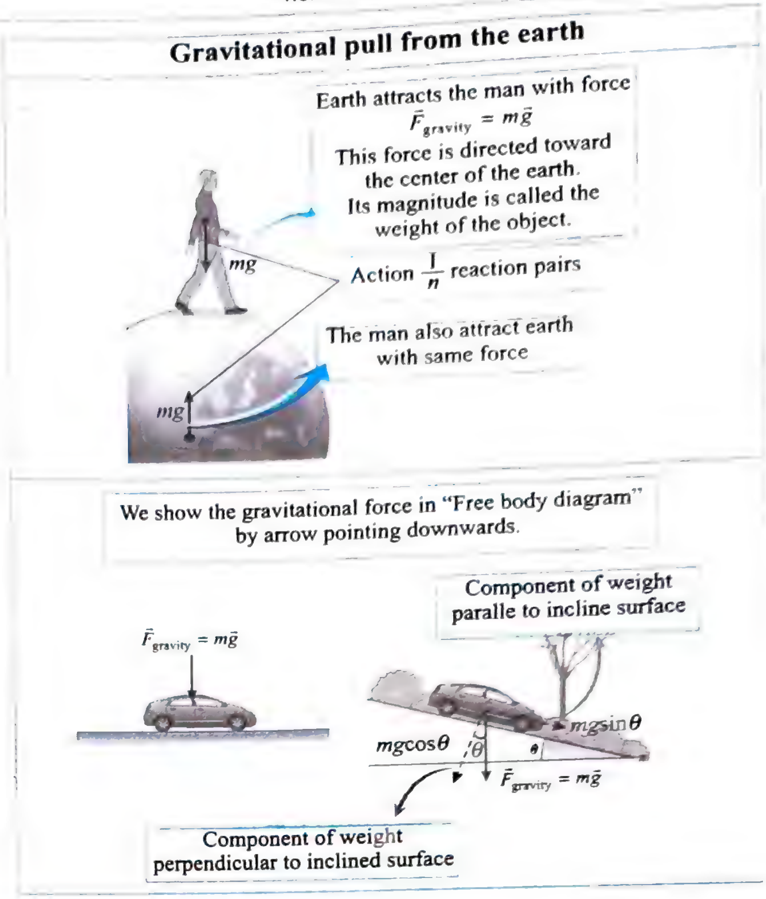
After knowing the nature of different forces, let us draw a "free-body diagram." The phrase itself reveals that we must mentally free (isolate) the bodies (or particles) in the system. Then consider all the forces acting on all the particles of the system.

The common forces encountered in mechanics and representing these forces through free-body diagrams are

1. Weight
2. Normal force
3. Tension
4. Frictional force
5. Elastic spring force

WEIGHT

Weight is the gravitational force with which the Earth pulls an object. The weight of an object can be written as the acceleration due to gravity, which is always directed towards the center of the Earth. For a small stretch of the Earth's surface, which can be considered to be flat, the acceleration due to gravity g can be taken to be uniform, pointing vertically downwards, and in this situation, the weight of the object is mg, directed vertically downwards.

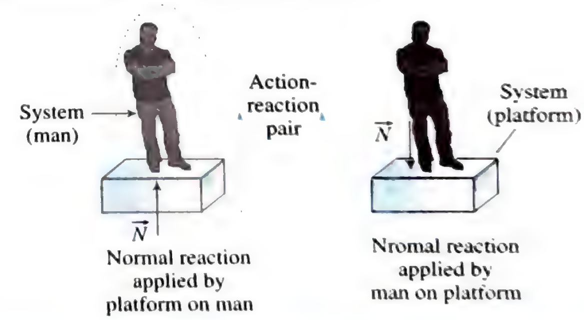


NORMAL FORCE

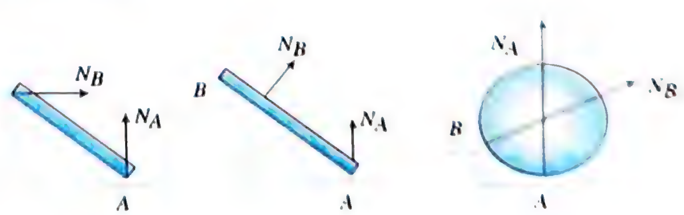
Whenever two surfaces are in contact, they press (or push) each other by a force called contact force. The component of the contact force perpendicular to the surface is called *normal reaction* and along the surface of contact is called *frictional force*.

The normal reaction to the contact surface can be computed by using $\sum \vec{F} = m\vec{a}$. Figure shows action–reaction pairs of normal forces for two physical situations.

Normal forces on a block are drawn normal to the contact surface directed towards the block, and $N \geq 0$.



REPRESENTING NORMAL REACTION IN DIFFERENT SITUATIONS

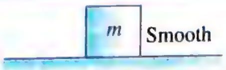
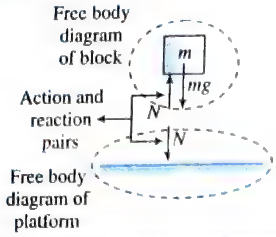
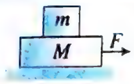
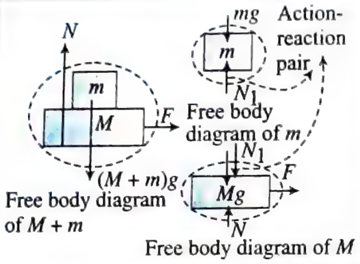
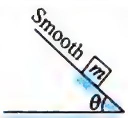
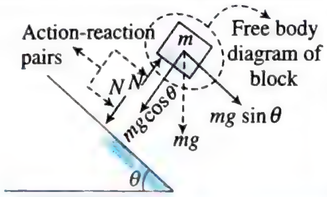


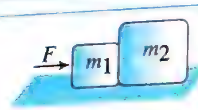
If the direction of contact force cannot be determined, it should be shown as two components (as shown in figure).

Important Points:

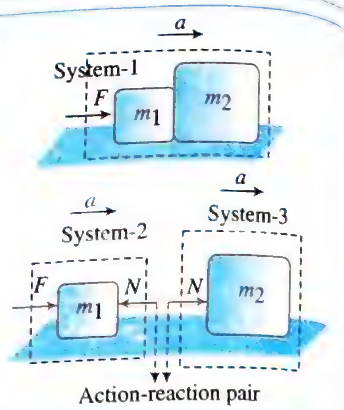
- Normal force will be perpendicular to the surface of contact.
- Normal reaction is a pushing force.
- If perpendicular to the surface of contact cannot be drawn, the normal force will act perpendicular to the surface of the body.
- If neither can be done, normal force has to be drawn as two components—one in the x-direction and one in the y-direction. Remember there is no relation between the forces acting along the x- and the y-directions. They are independent of each other.
- The number of normal forces acting on a body depends on the number of points or surfaces of contact.

Representing Normal Reactions and Weight in Free-Body Diagrams

Situation	FBD
 A block of mass m is placed on smooth surface.	
 Let us consider two blocks of masses M and m placed as shown in figure and a force of magnitude F acting on M.	
 A block is released from a fixed smooth inclined plane.	



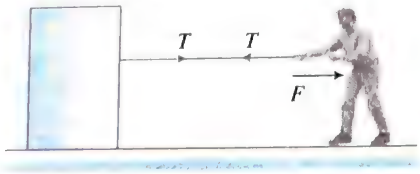
Two blocks are placed side by side on a smooth horizontal surface and a horizontal force applied on a block.



Tension

It is quite practical that we can pull objects by a string, but we cannot push objects by the string. This gives us an idea that a string can pull but cannot push. Tension force is an inter molecular force between the atoms of a string, which acts or reacts when the string is stretched. There are some important points to remember about the tension in a string, which are helpful in drawing free-body diagram of the bodies in a system.

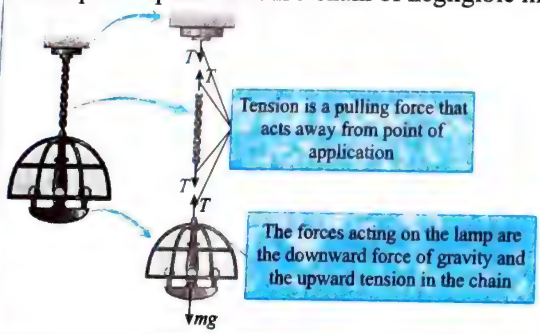
- The force of tension acts on a body in the direction away from the point of contact or tied ends of the string. For example consider figure. A man pulls a box with a string. The tension in string acts on the box towards right or in the direction away from the tied point and on the man it is again away from it. The way of showing the direction of tension is shown in figure.

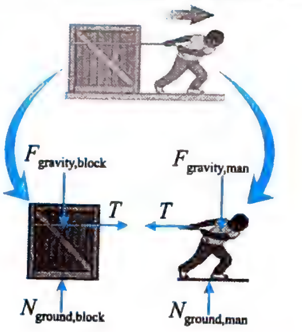


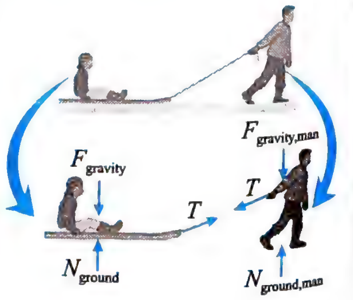
- If string is massless and frictionless, the tension throughout the string remains constant as shown in figure. But if the string is massless and not frictionless, at every contact in the length of the string, tension changes and if it is not light, tension at each point will be different depending on the acceleration of the string.
- If a force is directly applied on a string, as say a child is pulling a tied string from the other end with some force, the tension in the sting will be equal to the applied force. Irrespective of the motion of the pulling agent in figure, the man is applying a force F on string. Thus, the tension in string will be equal to this force, irrespective of whether the box will move or not, man will move or not.

Tension force of strings

A lamp is suspended from a chain of negligible mass.







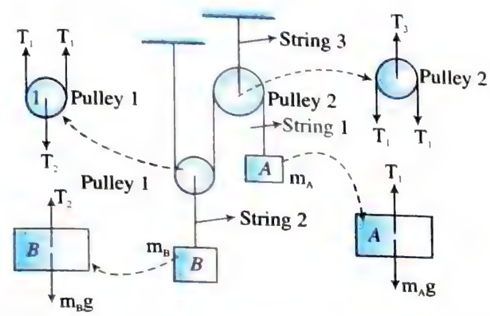
Important Points:

- String is assumed to be inextensible (perfectly elastic) unless stated. This is why the magnitude of acceleration of any number of masses connected through string is always same.
- String is assumed to be massless unless stated. This is why the tension in it everywhere remains the same and equal to applied force. However, if a string has a mass, the tension at different points will be different being maximum (= applied force) at the end through which force is applied and minimum at the other end connected to a body.

REPRESENTATION OF TENSION FORCE IN DIFFERENT SITUATIONS

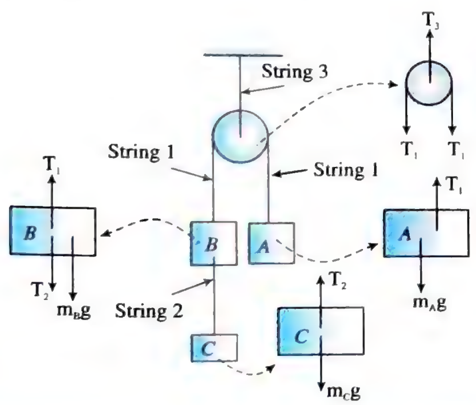
Two blocks of masses m_A and m_B are arranged in the diagram as shown. the free-body diagrams of

1. Block m_A
2. Block m_B
3. Pulley 1
4. Pulley 2



Three blocks of masses m_A, m_B, and m_C are arranged in the diagram as shown. The free-body diagrams of

1. Block m_A
2. Block m_B
3. Block m_C
4. Pulley



Important Points:

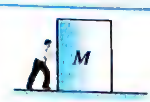
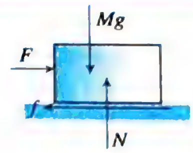
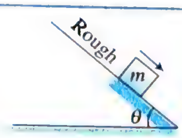
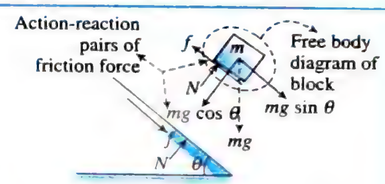
- Tension force always pulls a body.
- Tension can never push a body.
- Tension across massless pulley or frictionless pulley remains constant.
- Ropes become slack when tension force becomes zero.

FRICTION

When a surface of a solid body slides (or has a tendency to slide if it does not actually slide) on a surface of another solid body, its (surface's) motion (or the tendency of motion) is opposed. The force which opposes the relative motion (or tendency of the relative motion) between the contact surfaces is called *frictional force*.

The origin of frictional force is a complicated matter. Friction is a consequence of molecular interaction originating in the realm of molecules and atoms because of electrical reasons. On an atomic level, both surfaces of contact are irregular. There are many points of contact where atoms seem to cling together, and when the surface is pulled along, the atoms snap apart and vibration ensues.

If we push a block on the table, the table starts pushing the block back along the surface of contact (tangentially). Hence, we can call this contact force *tangential contact*.

Situation	FBD
 A man is pushing a block placed on a rough horizontal surface.	
 A block is sliding down on a rough inclined plane.	

EQUILIBRIUM OF A PARTICLE

The equilibrium of a particle in mechanics refers to the situation when the net external force acting on the particle is zero.

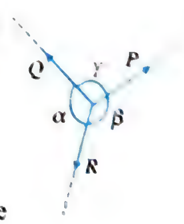
The above condition is correct and complete as far as the equilibrium of a particle (i.e., a point mass) is concerned. In case of rigid bodies (i.e., extended bodies), there are two conditions to be satisfied for such bodies to have equilibrium. First, the net external force acting on the body should be zero. Second, the net external torque acting on the body should be zero.

CONCURRENT FORCES

If two or more forces act on the same particle, we call them concurrent forces.

LAMI'S THEOREM

If three concurrent forces P, Q, and R acting on a particle keep the particle in equilibrium, then Lami's theorem states



$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

Here α is the angle opposite to $\vec{P}$, β is the angle opposite to $\vec{Q}$, and γ is the angle opposite to $\vec{R}$.

Note: If concurrent forces are coplanar but more than three, then it is generally convenient to resolve all of them along two mutually perpendicular directions and then the resultant of each set of these resolved components will be zero.

$$\Sigma F_x = 0;\ \Sigma F_y = 0$$

If the forces are not coplanar, then they can be resolved along any three mutually perpendicular directions and then the following conditions apply:

$$\Sigma F_x = 0;\ \Sigma F_y = 0;\ \Sigma F_z = 0$$

PROBLEM-SOLVING BY APPLYING NEWTON'S LAWS

The following procedure is recommended when dealing with problems involving Newton's law.

Step 1: Draw a simple, neat diagram of the system to help establish the mental representation. Establish convenient coordinate axes for each object in the system.

Step 2: If an acceleration component for an object is zero, it is modeled as a particle in equilibrium in this direction and $\Sigma \vec{F} = 0$. If not, the object is modeled as a particle under a net force in this direction and $\Sigma \vec{F} = m\vec{a}$.

Step 3: Isolate the object whose motion is being analyzed. Draw arrows on your sketch to show the direction of each force acting on the object. Arrows are drawn to represent direction of forces acting on the body. This diagram is called direction of forces acting on the body. This diagram is called a free-body diagram. Draw a free-body diagram for this object. For systems containing more than one object, draw separate free-body diagrams for each object. Do not include in the free-body diagram forces exerted by the object on its surroundings.
Find the components of the forces along the coordinate axes. Apply Newton's second law, $\Sigma \vec{F} = m\vec{a}$, in component form. Check your dimensions to make sure all terms have units of force.

Step 4: Solve the component equations for the unknowns. Remember that to obtain a complete solution, you must have as many independent equations as you have unknowns.

Step 5: Make sure your results are consistent with the free-body diagram. Also check the predictions of your solutions for extreme values of the variables. By doing so, you can often detect errors in your results.

ILLUSTRATION 6.7

A string is connected between surface and a block of mass 1 kg which is pulled by another string by applying force $F = 10$ N as shown in figure. Calculate

(i) tension is string (1)

(ii) tension in string (2)

Sol. For calculating tension force in strings, we need to draw the F.B.D of the block. The force acting on block are tension force at top and bottom side of the block and weight of the block. Tension is a puling force, it will act away from the block as shown in figure.

If we apply a force at the end of a light string, the force applied is equal to tension developed. Hence, the tension in string (2) $T_2 = F = 10$ N

Now we apply Newton's second law, $\sum F_y = ma_y$

As $a_y = 0$, $\sum F_y = 0$.

From free body diagram of block,

$$T_2 = T_1 + mg \Rightarrow 10 = T_1 + 10 \quad \Rightarrow T_1 = 0$$

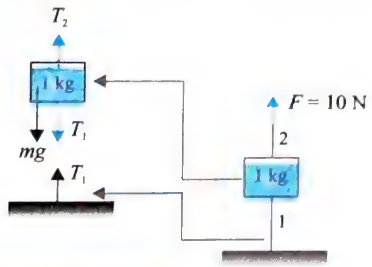
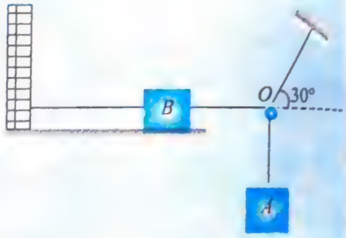


ILLUSTRATION 6.8

The breaking strength of the string connecting wall and block B is 175 N, Find the magnitude of weight of block A for which the system will be stationary. Block B weighs 700 N.



Sol. The breaking strength of the string means maximum possible tension in the string.

As system is at rest from Newton's second law of motion, $\sum F = ma$. As $a = 0$, $\sum F = 0$

From F.B.D of A, $\sum F_y = 0$

$$T_A = W_A \quad \text{...(i)}$$

From F.B.D of B, $\sum F_x = 0$

$$T_B = T_1 = 175 \text{ N} \quad \text{...(ii)}$$

From F.B.D of junction O,

Balancing forces in horizontal direction, $T_2 \cos 30° = T_B$

$$T_2\left(\frac{\sqrt{3}}{2}\right) = 175 \Rightarrow T_2 = \frac{350}{\sqrt{3}} \text{ N}$$

Balancing forces in vertical direction, $T_2 \sin 30° = T_A = W_A$

$$\left(\frac{350}{\sqrt{3}}\right) \times \left(\frac{1}{2}\right) = W_A \Rightarrow W_A = \frac{175}{\sqrt{3}} \text{ N}$$

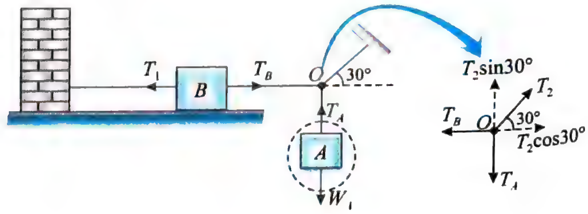
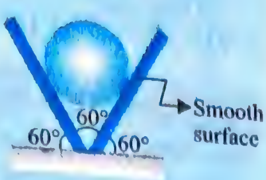


ILLUSTRATION 6.9

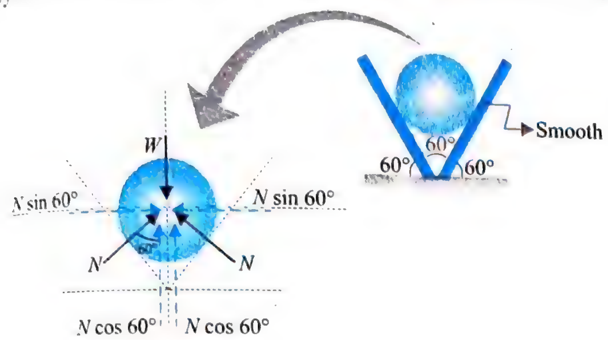
A cylinder of weight W is resting on two inclined planes forming a V-groove as shown in figure. Ignore friction everywhere.



(a) Draw its free body diagram of the sphere.

(b) Calculate normal reactions between the cylinder and two inclined walls.

Sol. Consider the cylinder as our system. On cylinder, there will be two contact forces and one field force, contact force in the form of normal reactions exerted by inclined planes and field force is the weight of the cylinder. As we know normal reaction is a pushing force, it will act towards and in normal to cylinder or in radial direction and weight will act vertically downwards through center of cylinder as shown in free body diagram of cylinder. As the cylinder is at rest, net force on the cylinder should be zero.



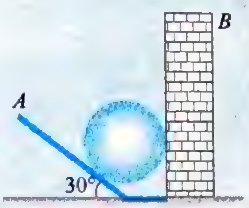
In horizontal direction, $\sum F_x = 0$

and in vertical direction $\sum F_y = 0$ or $2N\cos 60° = W$

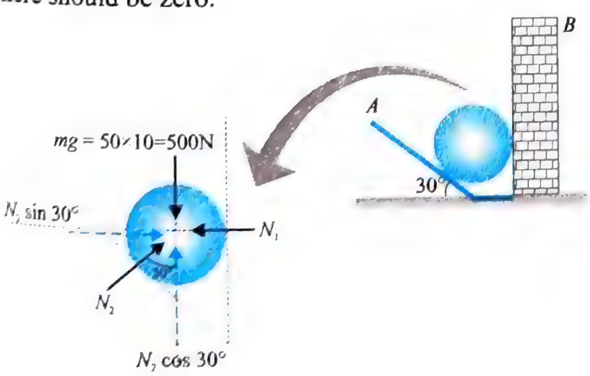
$$\Rightarrow \quad 2N \times \left(\frac{1}{2}\right) = W \Rightarrow N = W$$

ILLUSTRATION 6.10

A 50-kg homogeneous smooth sphere rests on the 30° incline A and bears against the smooth vertical wall B. Calculate the contact forces at A and B.



Sol. Consider the sphere as our system. On sphere there will be two contact forces and one field force, contact force in the form of normal reactions exerted by inclined planes and vertical wall also field force is the weight of the sphere as shown in free body diagram of cylinder. Here the sphere is at rest, net force on the sphere should be zero.



In vertical direction : $\sum F_y = 0$ or $N_2 \cos 30° = 500$

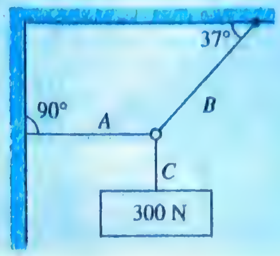
$$\Rightarrow \quad N_2 \times \left(\frac{\sqrt{3}}{2}\right) = 500 \Rightarrow N_2 = \frac{1000}{\sqrt{3}}\,\text{N}$$

In horizontal direction, $\sum F_x = 0$ or $N_1 = N_2 \sin 30°$

$$\Rightarrow \quad N_1 = \left(\frac{1000}{\sqrt{3}}\right) \times \left(\frac{1}{2}\right) = \frac{500}{\sqrt{3}}\,\text{N}$$

ILLUSTRATION 6.11

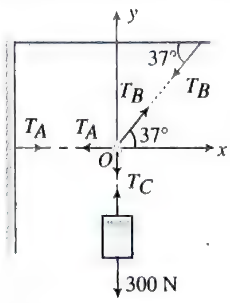
A block of mass 30 kg is suspended by three strings as shown in figure. Find the tension in each string.



Sol.

Method I: Considering equilibrium of each part of system

The whole system is in equilibrium; therefore, for each part $\Sigma \vec{F} = 0$. From the free-body diagram of block C, $T_C = 300$ N.



Now consider the equilibrium of point O,

$$\Sigma F_x = 0 \text{ or } T_B \cos 37° - T_A = 0$$

$$\therefore \quad T_A = T_B \cos 37° = T_B \cdot \frac{4}{5} \qquad \text{...(i)}$$

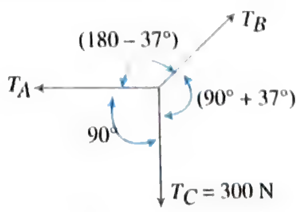
$$\text{and} \quad \Sigma F_y = 0 \text{ or } T_B \sin 37° - T_C = 0 \qquad \text{...(ii)}$$

$$\therefore \quad T_B = \frac{T_C}{\sin 37°} = \frac{300}{3/5} = 500\,\text{N}$$

From Eq. (i), we get

$$T_A = \frac{4}{5} T_B = \frac{4}{5} \times 500 = 400\text{ N}$$

Method II: Using Lami's theorem



By Lami's theorem, we have

$$\frac{T_A}{\sin(90 + 37°)} = \frac{T_B}{\sin 90°} = \frac{T_C}{\sin(180 - 37°)}$$

But $TC = 300$ N

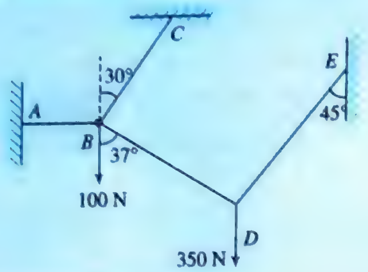
and $T_B = \dfrac{T_C}{\sin 37°} = \dfrac{300}{3/5} = 500\,\text{N}$

$$T_A = T_C\left(\frac{\sin(90+37°)}{\sin(180-37°)}\right) = 300\left(\frac{\cos 37°}{\sin 37°}\right) = 400\,\text{N}$$

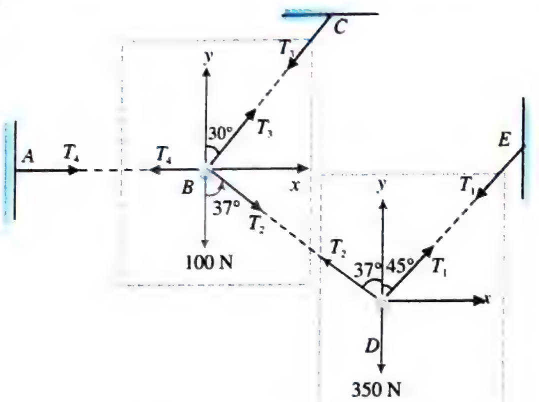
ILLUSTRATION 6.12

Two particles of masses 10 kg and 35 kg are connected with four strings at points B and D as shown in figure.

Determine the tensions in various segments of the string.



Sol. The free-body diagram of the whole system is shown in figure.



Analyzing the equilibrium of point D

$\Sigma F_x = 0$ or $T_1 \sin 45° - T_2 \sin 37° = 0$ ...(i)

and $\Sigma F_y = 0$ or $T_1 \cos 45° + T_2 \cos 37° = 350$ ...(ii)

From (i), we have $T_2 = \dfrac{T_1 \sin 45°}{\sin 37°}$

Now from (ii), $T_1 \cos 45° + \dfrac{T_1 \sin 45°}{\sin 37°} \times \cos 37° = 350$

or $\dfrac{T_1}{\sqrt{2}} + \dfrac{T_1}{\sqrt{2}} \times \dfrac{4}{3} = 350$

or $\dfrac{T_1}{\sqrt{2}}\left[1 + \dfrac{4}{3}\right] = 350 \Rightarrow T_1 = 150\sqrt{2}$ N

and $T_2 = \dfrac{T_1 \sin 45°}{\sin 37°} = \dfrac{150\sqrt{2} \times \left(\dfrac{1}{\sqrt{2}}\right)}{3/5} = 250$ N

Analyzing the equilibrium of point B

$\Sigma F_x = 0$ or $T_2 \sin 37° + T_3 \sin 30° - T_4 = 0$ ...(i)

and $\Sigma F_y = 0$ or $T_3 \cos 30° - T_2 \cos 37° - 100 = 0$ ...(ii)

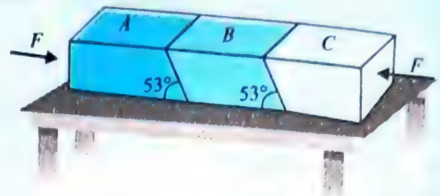
From (ii), $T_3 \cos 30° - 250 \times \dfrac{4}{5} - 100 = 0$

$\Rightarrow T_3 = 200\sqrt{3}$ N

From (i), $T_4 = 150 + 100\sqrt{3} = 50(3 + 2\sqrt{3})$ N

ILLUSTRATION 6.13

Three blocks A, B and C each of mass $m = 10$ kg are placed on a smooth horizontal table. There is no friction between the contact surfaces of the blocks as well. Horizontal force $F = 40$ N is applied on each of A and B as shown. Find the

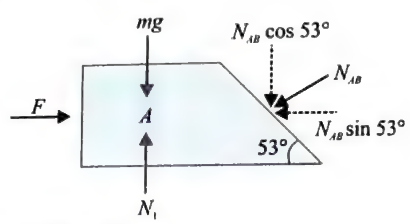


(i) magnitude of normal reaction between blocks $A-B$ and $B-C$.

(ii) normal reaction applied by ground on the blocks A, B and C.

Sol. As equal and opposite forces are acting on the system of the blocks, it means no net external force is acting on the system. Hence, the system will be in equilibrium. For finding normal reactions between the blocks, we need to draw F.B.D. of the blocks.

Block A:



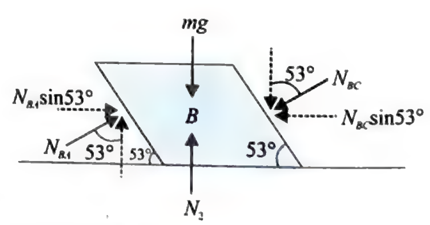
For equilibrium of block A,

In horizontal direction: $N_{AB} \sin 53° = F \Rightarrow N_{AB} = \dfrac{5}{4}F = 50\text{N}$

In vertical direction:

$$N_1 = N_{AB} \cos 53° + mg = 50 \times \frac{3}{5} + 10 \times 10 = 130\,\text{N}$$

Block B:



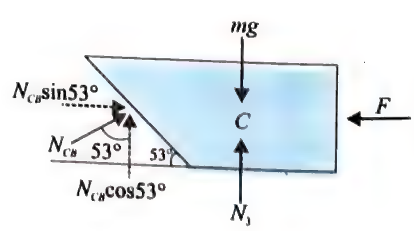
For equilibrium of block B,

In horizontal direction:

$$N_{BC} \sin 53° = N_{BA} \sin 53° \Rightarrow N_{BC} = N_{BA} = N_{AB} = 50\text{N}$$

In vertical direction: $N_2 = mg = 10 \times 10 = 100\,\text{N}$

Block C:



For equilibrium of block C,

In horizontal direction: $N_{CB}\sin 53° = F$

or $N_{CB} = \frac{5}{4}F = \frac{5}{4}\times 40 = 50\text{ N}$

In vertical direction: $N_3 + N_{CB}\cos 53° = mg$

$$N_3 + 50\times\frac{3}{5} = 10\times 10 \Rightarrow N_3 = 70\text{ N}$$

ILLUSTRATION 6.14

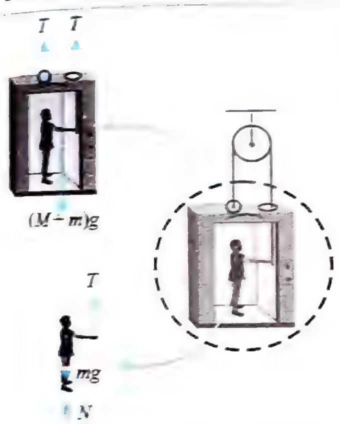
A man of mass m in hanging cabin of mass M is able to keep himself and cabin at equilibrium.

(i) Draw the forces acting on man and cabin.

(ii) Calculate force applied by the man on rope and normal reaction of the floor acting on man.

Sol. The force applied by man at the end of a light string is equal to tension in string. As the cabin is at equilibrium, it means net force on the system should be zero i.e., $\Sigma F = 0$.

Free body diagrams



Equation of motion

From F.B.D. of man and cabin $2T = (M + m)g$

$$\Rightarrow T = \left(\frac{m+M}{2}\right)g \quad \text{...(i)}$$

From F.B.D. of man:

$$T + N = mg \quad \text{...(ii)}$$

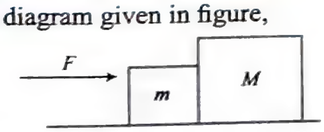
Solving equations (i) and (ii), we have

$$\left(\frac{m+M}{2}\right)g + N = mg$$

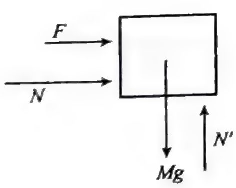
or $N = \left(\frac{m-M}{2}\right)g$

CONCEPT APPLICATION EXERCISE 6.2

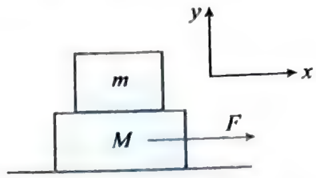
1. As per the diagram given in figure,



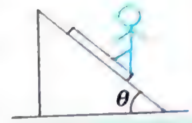
The free-body diagram of M is the figure shown below. Is it correct or incorrect? Assume all surfaces are frictionless.



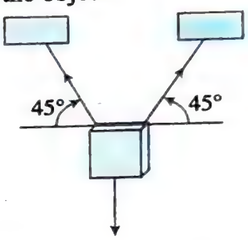
2. A mass m is placed on a body of mass M. There is no friction anywhere. Force F is applied on M and it moves with acceleration a. Find the force (along x-axis) on the top body.



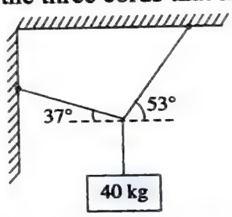
3. In figure, the mass of the man is M. Calculate the mass of the man as registered by weighing machine. Assume weighing machine, man, and wedge all are stationary.



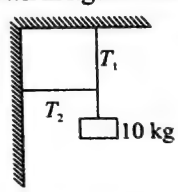
4. A small object is suspended at rest from two strings as shown in figure. The magnitude of the force exerted by each string on the object is $10\sqrt{2}$ N. find the magnitude of the mass of the object.



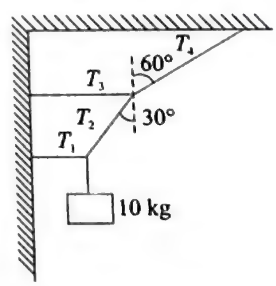
5. The object in figure weighs 40 kg and hangs at rest. Find the tensions in the three cords that hold it.



6. A block of mass $m = 10$ kg is suspended with the help of three strings as shown in figure. Find the tensions T_1 and T_2.

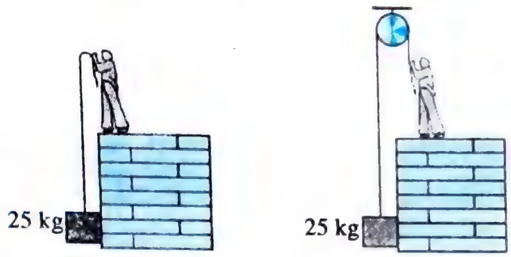


7. Determine tension T_4 in figure



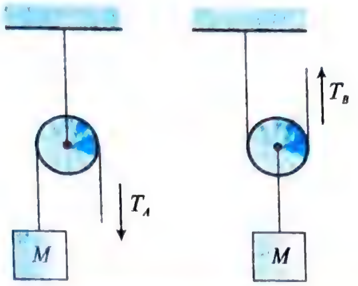
8. A block of mass 25 kg is raised by a 50-kg man in two different ways as shown in figure. What is the action on the floor by the man in the two cases? If the floor yields

to a normal force of 700 N, which mode should the man adopt to lift the block without the floor yielding?



(a) (b)

9. Consider the two configurations shown in equilibrium. Find the ratio of T_A/T_B. (Ignore the mass of the rope and the pulley.)



ANSWERS

1. Incorrect
2. No force is acting in horizontal direction on m
3. $M \cos \theta$
4. 2 kg
5. $T_1 = 240$ N and $T_2 = 320$ N
6. $T_1 = 100$ N, $T_2 = 0$
7. 200 N
8. Case (b)
9. 2

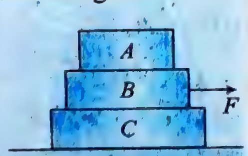
DYNAMICS OF PARTICLES: TRANSLATIONAL MOTION OF ACCELERATED BODIES

We can better understand the application of Newton's laws of motion through its application in different situations.

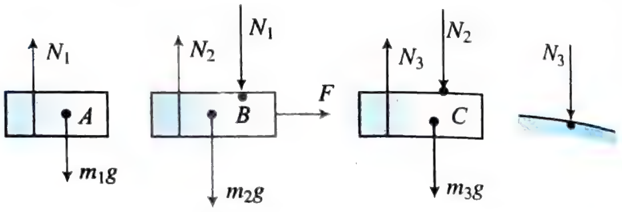
ILLUSTRATION 6.15

Three blocks A, B, and C of masses m_1, m_2, and m_3, respectively, are resting one on top of the other as shown in figure.

A horizontal force F is applied on block B. Assuming all the surfaces are frictionless, calculate (1) acceleration of block A, block B, and block C; (2) normal reactions between A and B, B and C, and between C and ground.



Sol. This system cannot be in equilibrium because in the horizontal direction, the system has a net external force F. As far as the vertical direction is concerned, all the forces are internal action and reaction forces. These forces are equal and opposite; hence, they will cancel each other out.



Body A: No external force is acting on it; hence, it will remain stationary in equilibrium. So acceleration at block A, $a_A = 0$ and

$$N_1 = m_1 g \quad \text{...(i)}$$

Body B: There is no external force acting on it vertically; hence, it will not have any acceleration in the vertical direction.

$$N_2 = N_1 + m_2 g = (m_1 + m_2)g \quad \text{...(ii)}$$

However, there is one external force F acting on it. So it will have some acceleration a in the horizontal direction such that

$$F = m_2 a_B \quad \text{...(iii)}$$

which gives acceleration of block B, $a_B = \dfrac{F}{m_2}$

Body C and earth: Same comments as in the case of body A above.

$$N_3 = N_2 + m_3 g \quad \text{...(iv)}$$

and $\quad N_3 = (m_1 + m_2 + m_3)g \quad \text{...(v)}$

No external force acts on block C in the horizontal direction; hence, a_C is equal to 0.

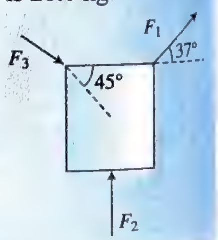
ILLUSTRATION 6.16

Three boys, each of mass 45 kg, pull simultaneously a block on a smooth surface. The mass of block is 20.0 kg.

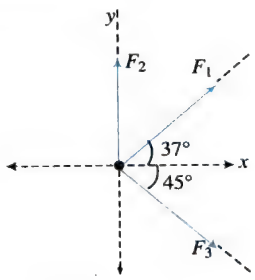
(a) Find the acceleration of the block.

(b) Find the acceleration of the boy exerting force F_1. Assume no friction between the boy and the surface.

[Given $F_1 = 90$ N, $F_2 = 114$ N, and $F_3 = 128\sqrt{2}$ N]



Sol. First we will resolve all the forces acting on the block into x and y components.



$$\Sigma F_x = F_1 \cos 37° + F_3 \cos 45°$$

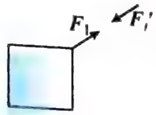
$$\Sigma F_y = F_1 \sin 37° + F_2 - F_3 \sin 45°$$

Now, $a_x = \dfrac{\Sigma F_x}{m}$ and $a_y = \dfrac{\Sigma F_y}{m}$

$$a_x = \frac{(90.0)\cos 37° + 128\sqrt{2}\cos 45.0°}{20.0} = 10 \text{ m s}^{-2}$$

$$a_y = \frac{(90.0)(\sin 37°) + 114 - (128\sqrt{2}\sin 45.0°)}{20.0} = 2 \text{ m s}^{-2}$$

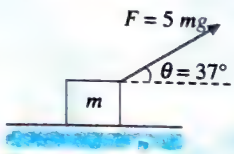
According to Newton's third law, the force exerted on the block by the boy must be equal to the force exerted on the boy by the block. Therefore,



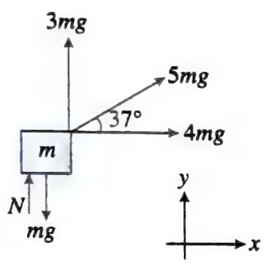
$$a_1 = \frac{F_1'}{m} = \frac{90.0}{45} = 2 \text{ m s}^{-2}$$

ILLUSTRATION 6.17

A block of mass m is placed on a horizontal surface. If the block is pulled by applying a force of magnitude $F = 5mg$ at an angle $\theta = 37°$ with horizontal as shown in figure, find the acceleration of the block at the given instant.



Sol. Three forces act on the block (a) $mg \downarrow$ (b) $F = 5mg \nearrow$ (c) $N \uparrow$



Resolving the forces along x- and y-axis, we have equation of motion

$$\Sigma F_x = ma_x$$

$$4\,mg = ma_x$$

This gives $a_x = 4g$ ...(i)

$$\Sigma F_y = ma_y$$

$$3mg - mg + N = ma_y$$

$$2mg + N = ma_y \quad ...(iii)$$

Let $a_y = 0$. Then $N = -2mg$

The negative result signifies that N will be directed down (opposite to the assumed direction), but the ground cannot pull the block down. Hence, the block will lose contact with the ground.

or $\quad N = 0$

Hence, from (ii), $a_y = 2g \uparrow$

Hence, the net acceleration, $|\vec{a}| = \sqrt{a_x^2 + a_y^2}$

where $a_x = 4g$ from Eq. (i) and $a_y = 2g$ from Eq. (ii).

This gives $a = \sqrt{a_x^2 + a_y^2} = \sqrt{(4g)^2 + (2g)^2} = 2\sqrt{5}\,g$

ILLUSTRATION 6.18

A smooth box of mass m is kept on an inclined plane of angle of inclination θ. A force F is applied on the box

(i) in horizontal direction as in figure (i).

(ii) at an angle ϕ with the inclined plane as in figure (ii).

If the box does not lose contact, find

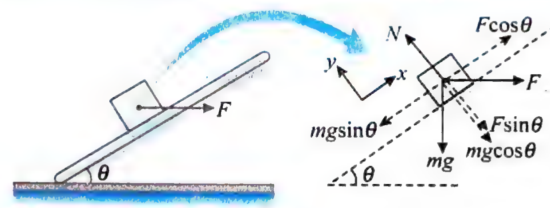
(a) acceleration of the block.

(b) normal reaction offered by the box on the inclined plane.

Case (i) Case (ii)

Sol. The motion of the block will be parallel to inclined surface, it means we should make the components of forces acting on the block parallel and perpendicular to inclined plane.

Case (i): Free body diagram



Now applying Newton's second law of motion

Force equation, $\sum F_x = ma_x$

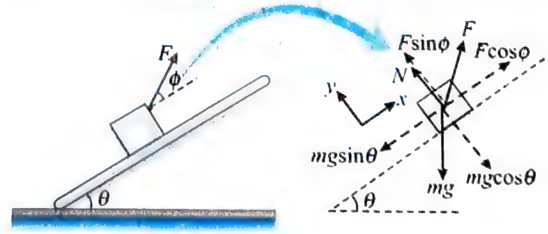
$$F\cos\theta - mg\sin\theta = ma_x$$

$$a_x = \frac{F\cos\theta - mg\sin\theta}{m} \quad ...(i)$$

$\sum F_y = ma_y$. As $a_y = 0$, hence $\sum F_y = 0$

$$\Rightarrow \quad N = (F\sin\theta + mg\cos\theta) \quad ...(ii)$$

Case (ii): Free body diagram



Now applying Newton's second law of motion

Force equation, $\sum F_x = ma_x$

$$F\cos\phi - mg\sin\theta = ma_x$$

$$a_x = \frac{F\cos\phi - mg\sin\theta}{m} \quad ...(i)$$

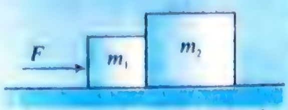
$\sum F_y = ma_y$. As $a_y = 0$, hence $\sum F_y = 0$

$$\Rightarrow \quad N = mg\cos\theta - F\sin\phi \quad ...(ii)$$

ANALYSIS OF NEWTON'S LAWS OF MOTION IN CONNECTED BODIES: PROBLEMS BASED ON NORMAL REACTION

ILLUSTRATION 6.19

Two blocks of masses m_1 and m_2 are placed side by side on a smooth horizontal surface as shown in figure. A horizontal force F is applied on the block.

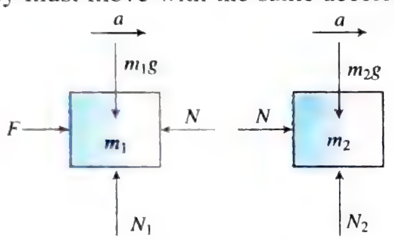


(a) Find the acceleration of each block.

(b) Find the normal reaction between the two blocks.

Sol.

Method 1: Since the two blocks always remain in contact with each other, they must move with the same acceleration.



FBD of m_1 FBD of m_2

Using Newton's second law, we get

For block m_1: $F - N = m_1a$ ...(i)

For block m_2: $N = m_2a$ ...(ii)

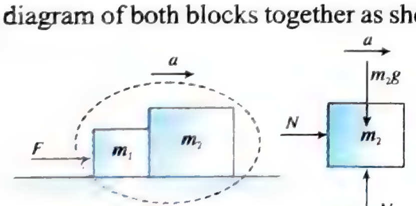
On adding the two equations, we get

$$F = (m_1 + m_2)\, a \Rightarrow a = \frac{F}{m_1 + m_2}$$

Substituting the value of a in (ii), we get

$$N = m_2a = \frac{Fm_2}{m_1 + m_2}$$

Method 2: The situation may be considered as follows: Instead of drawing the free-body diagrams of each block, we can draw the free-body diagram of both blocks together as shown in figure.



FBD of both the blocks considered as one FBD of m_2

The net force acting on the system is F, and the total mass of the system is $m_1 + m_2$. Thus, $a = \dfrac{F}{m_1 + m_2}$

To find out the normal reaction N between the two blocks, we can imagine the following: Block m_2 is moving with an acceleration a; therefore, the net force acting on it must be m_2a, which is nothing but the normal reaction applied by the block m_1. Thus,

$$N = m_2a = \frac{m_2F}{m_1 + m_2}$$

ILLUSTRATION 6.20

Two blocks of masses 1 kg and 2 kg are placed in contact on a smooth horizontal surface as shown in figures. A horizontal force of 6 N is applied (a) on a 1-kg block (b) a 2-kg block. Find the force of interaction of the blocks.

Sol.

Since both the blocks are in contact, therefore, they will move together with an acceleration

$$a = \frac{F_{\text{net}}}{M_{\text{total}}} = \frac{6}{2+1} = 2\,\text{m s}^{-2}$$

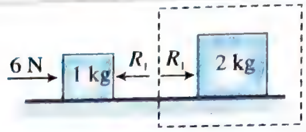
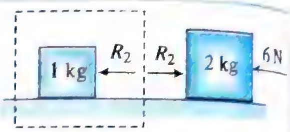
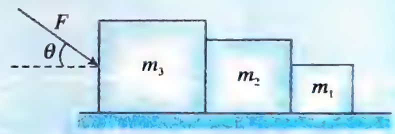
	
Let the force of interaction between them is R_1. By Newton's second law for 2-kg block, Normal reaction on 2-kg block, $R_1 = 2a = 2 \times 2 = 4$ N	Let the force of interaction between them is R_2. By Newton's second law for 1-kg block, normal reaction on 2-kg block, $R_2 = 1a = 1 \times 2 = 2$ N

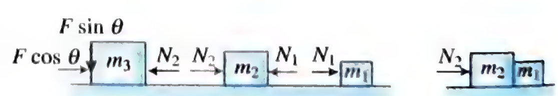
ILLUSTRATION 6.21

Find the force of interaction between the bodies as shown in figure. Blocks are in contact.



Sol. All the bodies move together along horizontal direction with an acceleration,

$$a = \frac{[F_{\text{horizontal}}]_{\text{net}}}{m_{\text{total}}} = \frac{F\cos\theta}{m_1 + m_2 + m_3}$$



(a) (b)

By Newton's second law for block m_1,

$$N_1 = m_1a = m_1\left(\frac{F\cos\theta}{m_1 + m_2 + m_3}\right)$$

and for block m_2, we have $N_2 - N_1 = m_2a$

or $$N_2 = N_1 + m_2a = N_1 + m_2\left(\frac{F\cos\theta}{m_1 + m_2 + m_3}\right)$$

Here the force of interaction is of compressive nature.

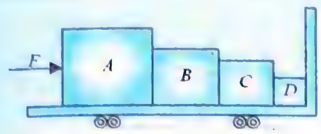
Hence, $$N_2 = \frac{(m_1 + m_2)\,F\cos\theta}{(m_1 + m_2 + m_3)}$$

We can calculate the value of N_2 by taking m_1 and m_2 together as system. From FBD of '$m_1 + m_2$' as Fig. (b)

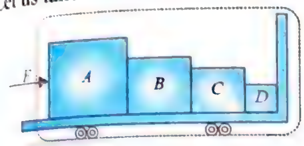
or $N_2 = (m_1 + m_2)a = \dfrac{(m_1 + m_2)\, F \cos\theta}{(m_1 + m_2 + m_3)}$

ILLUSTRATION 6.22

A toy cart has mass of 4 kg and is kept on a smooth horizontal surface. Four blocks A, B, C and D of masses 2 kg, 2 kg, 1 kg and 1 kg, respectively, have been placed on the cart. A horizontal force of $F = 60$ N is applied to the block A (see figure). Find the contact force between block D and the front vertical wall of the cart.



Sol. Let us take the cart and the blocks together as system.



System: 'cart+blocks'

F.B.D. of system: 'cart'

Acceleration of the system,

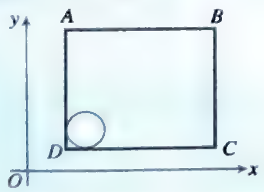
$$a = \frac{F}{m_A + m_B + m_C + m_D + m_{cart}} = \frac{60}{2+2+1+1+4} = 6\,\text{m/s}^2$$

The cart moves because of the normal reaction applied by the block 'D' on the cart. Now considering the F.B.D. of the cart (considering horizontal forces only)

$\therefore \quad N_{D,cart} = m_{cart} \cdot a = 4 \times 6 = 24\,\text{N}$

ILLUSTRATION 6.23

A solid sphere of mass 2 kg is resting inside a cube as shown in figure The cube is moving with a velocity $\vec{v} = (5t\hat{i} + 2t\hat{j})\,\text{ms}^{-1}$. Here t is time in seconds. All surfaces are smooth. The sphere is at rest with respect to the cube. What is the total force exerted by the sphere on the cube?



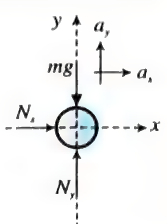
Sol. The velocity of the sphere is same as that of the cube, which is given as $\vec{v} = 5t\hat{i} + 2t\hat{j}$.

Hence, acceleration of the sphere: $\vec{a} = \dfrac{d\vec{v}}{dt}$

or $\vec{a} = (5\hat{i} + 2\hat{j})\,\text{m s}^{-2}$

Hence, $a_x = 5\,\text{m s}^{-2}$ and $a_y = 2\,\text{m s}^{-2}$

From FBD of sphere,



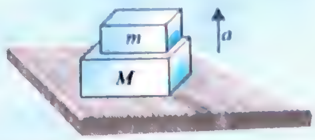
$N_x = ma_x = 2 \times 5 = 10$ N

$N_y - mg = ma_y \Rightarrow N_y = 2 \times 10 + 2 \times 2 = 24$ N

Total force $= \sqrt{N_x^2 + N_y^2} = \sqrt{(10)^2 + (24)^2} = 26\,\text{N}$

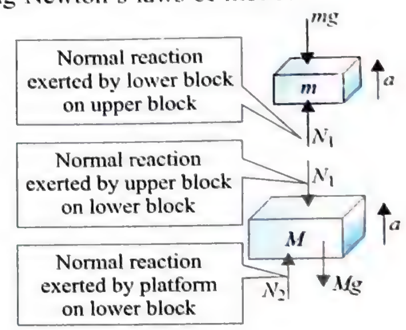
ILLUSTRATION 6.24

Two blocks m and M are placed on a platform which moves up with an upward acceleration a. Find the normal reaction between the blocks and normal reaction on block exerted by platform.



Sol. For finding normal reaction between the blocks we need to draw the F.B.D of the blocks and for calculating the normal reactions between the blocks and normal reaction exerted by platform, we should analyze the F.B.D of the blocks.

Now applying Newton's laws of motion



For block m, $\quad \sum F_y = ma \Rightarrow N_1 - mg = ma \quad$ ...(i)

$N_1 = m(a + g)$

For block M, $\quad \sum F_y = Ma \Rightarrow N_2 - Mg = Ma \quad$ ...(ii)

$N_2 = (m + M)(a + g)$

APPARENT WEIGHT

Usually, the weight of an object can be determined with the aid of a weighing machine. However, even though a weighing machine is working properly, there are situations in which it does not give the correct weight. In such situations, the reading on the weighing machine gives only the "apparent" weight, rather than the gravitational force or "true" weight. The apparent weight is the force that the object exerts on the weighing machine with which it is in contact.

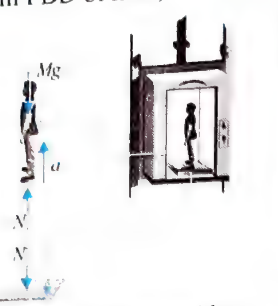
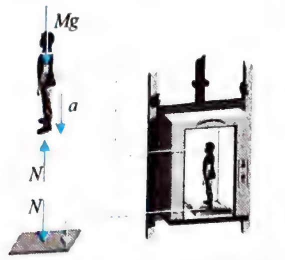
ILLUSTRATION 6.25

A man of mass M stands on a weighing machine in an elevator. Calculate the reading of the weighing machine, if the lift is

(i) accelerating upwards with an acceleration a.

(ii) accelerating downwards with an acceleration a.

Sol. The weighing machine measures the normal reaction.

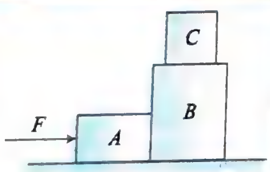
The lift accelerating upwards	The lift accelerating downwards
From FBD of man,  using Newton's second law, $N - Mg = Ma \Rightarrow N = M(g + a)$ N is the normal reaction on weighing machine. Hence, the reading of weighing machine will be $M(g + a)$.	From FBD of man,  using Newton's second law, $Mg - N = Ma \Rightarrow N = M(g - a)$ N is the normal reaction on weighing machine. Hence, the reading of weighing machine will be $M(g - a)$.

Important Points:

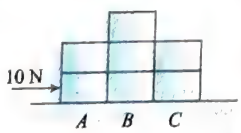
If the elevator is accelerating, the apparent weight and the true weight are not equal. When the elevator accelerates upward, the apparent weight is greater than the true weight. Conversely, if the elevator accelerates downward, the apparent weight is less than the true weight. In fact, if the elevator falls freely, so its acceleration is equal to the acceleration due to gravity, the apparent weight becomes zero. In a situation such as this, where the apparent weight is zero, the person is said to be "weightless." The apparent weight, then, does not equal the true weight if the scale and the person on it are accelerating.

CONCEPT APPLICATION EXERCISE 6.3

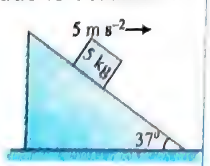
1. The masses of blocks A, B, and C are 1 kg, 2 kg, and 0.5 kg, respectively. All surfaces are smooth. If force F = 50 N acts as shown in figure at the instant shown, find the force which A exerts on B and the acceleration of C.



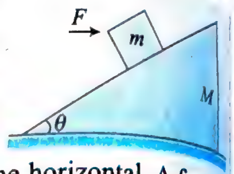
2. Seven identical dominoes (i.e., blocks) each of mass m = 1 kg are to be stacked in three columns (figure gives an example) and pushed across a frictionless ice rink by a horizontal 10-N force. Assume dominoes do not slip w.r.t. each other. How many dominoes should be in each column, with a minimum of one, (a) to maximize the acceleration of the dominoes, (b) to maximize the force on column C due to column B, (c) to maximize the net force on column B due to columns A and C, and (d) to maximize the force on column B due to column A?



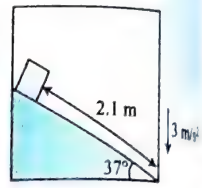
3. An inclined plane is moved toward right with an acceleration of 5 m s^{-2} as shown in figure. Find force in newton which block of mass 5 kg exerts on the incline plane. (All surfaces are smooth.)



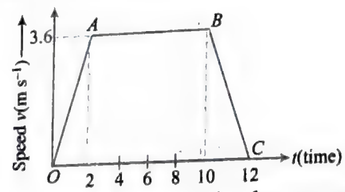
4. A small cubical block is placed on a triangular block M so that they touch each other along a smooth inclined contact plane as shown in figure. The inclined surface makes an angle θ with the horizontal. A force F is to be applied on the block m in horizontal direction so that the two bodies move without slipping against each other assuming the floor to be smooth also. Determine the
   (a) normal force with which m and M press against each other.
   (b) magnitude of external force F. Express your answers in terms of m, M, θ, and g.



5. A block of mass 1 kg is kept on the tilted floor of a lift moving down with 3 m s^{-1}. If the block is released from rest as shown, what will be the time taken by block to reach the bottom?

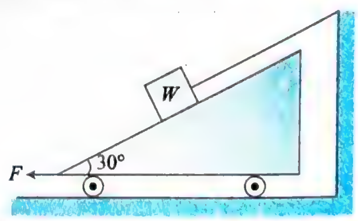


6. A lift is going up. The total mass of the lift and the passengers is 1500 kg. The variation in the speed of the lift is given by the graph.



   (a) What will be the tension in the rope pulling the lift at time t equal to
   i. 1 s ii. 6 s iii. 11 s
   (b) What will be the average velocity and the average acceleration during the course of the entire motion?

7. A block of weight W is placed on a wedge and arranged as shown in figure. Find the force F needed to hold the cart equilibrium if there is no friction.



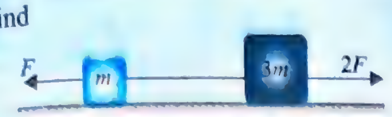
ANSWERS

1. $\frac{100}{3}$ N
2. (a) Does not depend upon the number of blocks
   (b) $A(1)$, $B(1)$, $C(5)$
   (c) $A(1)$, $B(5)$, $C(1)$
   (d) $A(1)$, remaining six can be placed in any arrangement.
3. 55 N
4. (a) $(M + m)a$ (b) $\frac{mg}{M}(m + M)\tan\theta$
5. 1s
6. (a) (i) 17700 N, (ii) 15000 N, (iii) 12300 N
   (b) 3 ms^{-1}, 0 ms^{-2}
7. $\frac{\sqrt{3}W}{4}$

PROBLEMS BASED ON BLOCKS CONNECTED WITH STRINGS

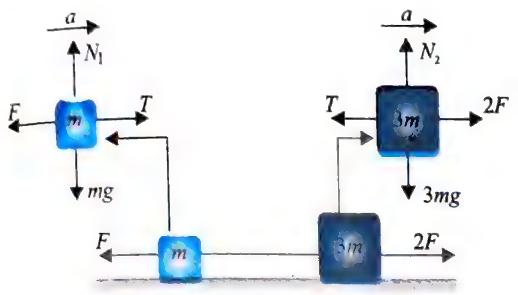
ILLUSTRATION 6.26

Two smooth blocks of masses m and $3m$ are connected by an inextensible light string. If the forces F and $2F$ act on the blocks, the find



(a) tension in the string, and
(b) acceleration of the blocks.

Sol. Both the blocks will move in rightwards direction with same acceleration.



The free body diagrams of both the blocks and

Now using Newton's second law of motion, $\sum F_x = ma_x$

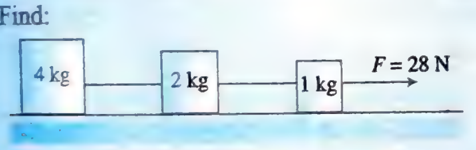
From F.B.D of m: $T - F = ma$ ...(i)

From F.B.D of $3m$: $2F - T = 3ma$ ...(ii)

Solving equations (i) and (ii), $T = \dfrac{5F}{4}$, $a = \dfrac{F}{4m}$

ILLUSTRATION 6.27

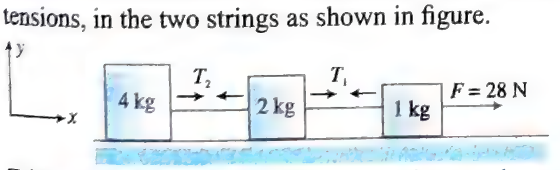
In the arrangement shown in figure, the strings are light and inextensible. The surface over which blocks are placed is smooth. Find:



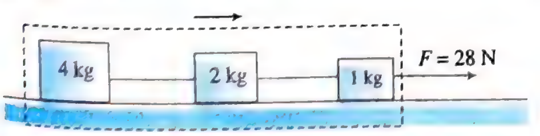
(a) the acceleration of each block
(b) the tension in each string

Sol.

(a) Let a be the acceleration of each block and T_1 and T_2 be the tensions, in the two strings as shown in figure.



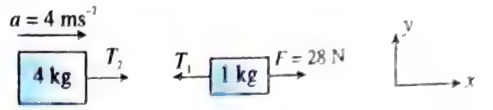
Taking the three blocks and the two strings as the system.



Using $\sum F_x = ma_x$

or $28 = (4 + 2 + 1)a$ or $a = \dfrac{28}{7} = 4\ \text{m s}^{-2}$

(b) Free-body diagrams (showing the forces in x-direction only) of 4-kg block and 1-kg block are shown in figure.



Using $\sum F_x = ma_x$

For 1-kg block: $F - T_1 = (1)(a)$

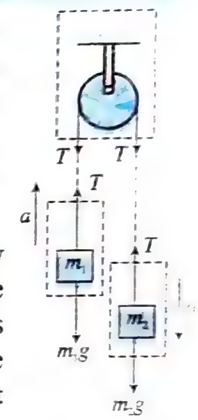
or $28 - T_1 = (1)(4) = 4 \Rightarrow T_1 = 28 - 4 = 24$ N

For 4-kg blocks: $T_2 = (4)(a)$

$\Rightarrow T_2 = (4)(4) = 16$ N

ILLUSTRATION 6.28

Two masses m_1 and m_2 are attached to a flexible inextensible massless rope, which passes over a frictionless and massless pulley. Find the accelerations of the masses and tension in the rope.



Sol. Let the tension in the rope be T (in fact, tension is the property of a point of the rope). In this case with ideal pulley (massless and frictionless) and ideal rope (inextensible, massless, and flexible), the tension will remain constant throughout the rope.

Let the acceleration of m_2 be vertically downwards, and acceleration of m_1 will also be a, vertically upwards. This is because the rope is inextensible; during motion, the length of the rope must not change, and the rope must not slacken either.

From the above statement, you must not conclude that the accelerations of the masses connected by a rope are always equal. The relationship between the accelerations of the masses depends on the configuration of the pulley–rope system, which can be obtained from the fact that the length of an ideal rope must not change and the rope must not slacken.

Using equation $\sum \vec{F} = m\vec{a}$ for the force diagrams of m_1 and m_2,

$$T - m_1 g = m_1 a \quad \text{...(i)}$$

$$m_2 g - T = m_2 a \quad \text{...(ii)}$$

Adding (i) and (ii), $m_2 g - m_1 g = m_1 a + m_2 a$

$$a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g \quad \text{...(iii)}$$

Substituting this value of a in (i), we get

$$T = m_1 g + m_1\left(\frac{m_2 - m_1}{m_2 + m_1}\right) g = \frac{2m_1 m_2}{(m_1 + m_2)} g \quad \text{...(iv)}$$

Important Points:

These type of problems in which all connected bodies have same acceleration magnitude, can be solved by the following method:

For calculating a use $a = \dfrac{F_{net}}{M_{total}} = \dfrac{\text{Driving force}}{\text{Total mass}}$

For driving force:

(a) If mass moves vertically, take mg

(b) If mass moves horizontally, take zero

If mass moves on inclined plane of inclination, take $mg \sin \theta$.

The external forces acting parallel to string taken as positive if acting in the direction of motion and negative if they are acting in the direction opposite to motion.

In above illustration:

Driving force, $F_{net} = m_2 g - m_1 g$

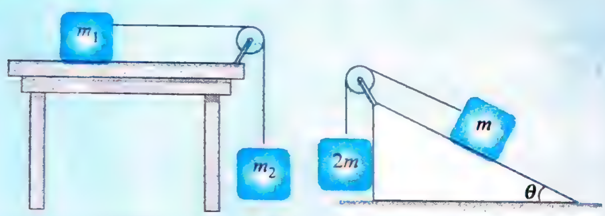
Here $m_1 g$ is taken negative as it is acting in the direction option in motion.

Acceleration of the system,

$$a = \frac{F_{net}}{M_{total}} = \frac{m_2 g - m_1 g}{(m_1 + m_2)} = \frac{(m_2 - m_1) g}{(m_1 + m_2)}$$

ILLUSTRATION 6.29

Calculate the acceleration of the system in case (1) and (2).

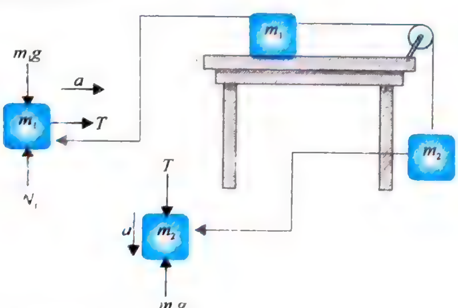


Case I Case II

Sol.

Case I

Free body diagrams



Equation of motion

For m_1 $T = m_1 a$ ...(i)

For m_2 $m_2 g - T = m_2 a$ ...(ii)

From (i) and (ii),

$$\Rightarrow \quad a = \frac{m_2 g}{(m_1 + m_2)}$$

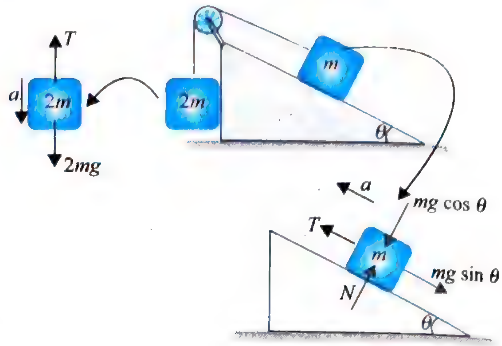
Approach 2

Driving force, $F_{net} = m_2 g$

Acceleration of the system, $a = \dfrac{F_{net}}{M_{total}} = \dfrac{m_2 g}{(m_1 + m_2)}$

Case II

Free body diagrams



Equation of motion

$T - mg \sin\theta = ma$ ...(i)

$2mg - T = 2ma$ ...(ii)

From equation (i) and (ii), $2mg - mg \sin\theta = (m + 2m)a$

$$a = \frac{(2 - \sin \theta)}{3} g$$

Approach 2

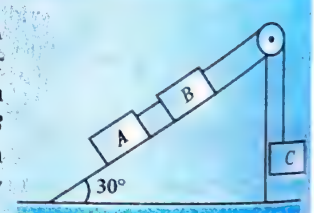
Driving force $F_{net} = 2mg - mg \sin\theta$

Acceleration of the system,

$$a = \frac{F_{net}}{M_{total}} = \frac{2mg - mg \sin \theta}{(2m + m)} = \frac{(2 - \sin \theta)}{3} g$$

ILLUSTRATION 6.30

In the given figure, blocks A and B are connected together by a string and placed on a smooth inclined plane. B is connected to C (which is suspended vertically) by another string which passes over a smooth pulley fixed to the plane. The mass of A is $m_A = 1$ kg and mass of B is $m_B = 2$ kg.



(a) If the system is at rest, find the mass of C.

(b) If the mass of C is twice the mass calculated in (a), then find the acceleration of the system.

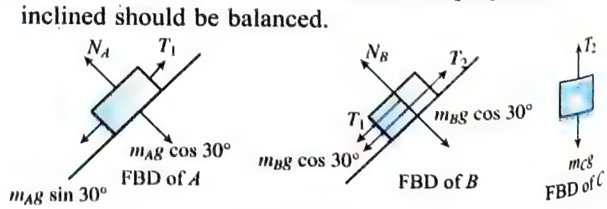
Sol.

(a) From the free-body diagram of A:

T_1: Tension in string between A and B

N_A: Normal reaction between A and incline

As A is at rest, net force parallel and perpendicular to inclined should be balanced.



$N_A = m_A g \cos \theta = 10 \cos 30° = 5\sqrt{3}$ N

$T_1 = m_A g \sin \theta = 10 \sin 30° = 5$ N

From the free-body diagram of B:

As B is connected to both strings, two tensions T_1 and T_2 will act on it.

T_2 is the tension (force) of string between B and C acting upwards

T_1 is the tension of string between A and B acting downwards

Balancing forces:

$$N_B = m_B g \cos 30^\circ = 20 \cos 30^\circ = 10\sqrt{3}\ \text{N}$$

$$T_2 = T_1 + m_B g \sin 30^\circ = 5 + 20 \sin 30^\circ$$

$$= 5 + 20 \times \frac{1}{2} = 15\ \text{N}$$

Force diagram of C:

T_2 is the pulling force of string on block C, therefore, $m_C g = T_2$, hence $m_C = 1.5$ kg

(b) In this case, mass of $C = 3$ kg

Let the acceleration of the system be a. We can assume an arbitrary direction of motion. Let the blocks move up the incline and block C move downward.

From FBD of A: $T_1 - 10 \sin 30^\circ = 1a$ ...(i)

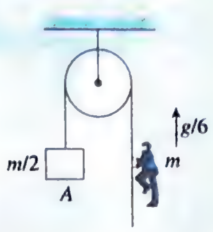
From FBD of B: $T_2 - T_1 - 20 \sin 30^\circ = 2a$ ...(ii)

From FBD of C: $30 - T_2 = 3a$ ...(iii)

Solving (i), (ii), and (iii), $a = 2.5\ \text{m s}^{-2}$

ILLUSTRATION 6.31

Block A of mass $m/2$ is connected to one end of light rope which passes over a pulley as shown in figure. A man of mass m climbs the other end of rope with a relative acceleration of $g/6$ with respect to rope. Find the acceleration of block A and tension in the rope.

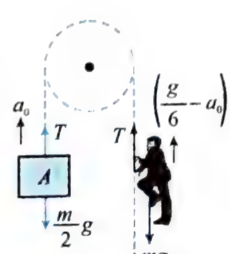


Sol. Let the acceleration of block be (a_0) in upward direction. As the block is tied with the rope, hence the acceleration of the rope connecting the block should also be a_0 (upward). It means the acceleration of the rope on right side of the man with respect to ground be a (upwards).

$$\vec{a}_{\text{man}} = \vec{a}_{\text{man,rope}} + \vec{a}_{\text{rope}} \Rightarrow a = \frac{g}{6} - a_0$$

From the force diagram of man,

$$T - mg = ma = m\left(\frac{g}{6} - a_0\right) \quad \text{...(i)}$$



From force diagram of block A,

$$T - \frac{m}{2} g = \frac{m}{2} a_0 \quad \text{...(ii)}$$

Subtracting (i) from (ii),

$$\frac{m}{2} g = \frac{m}{2} a_0 - \frac{mg}{6} + ma_0$$

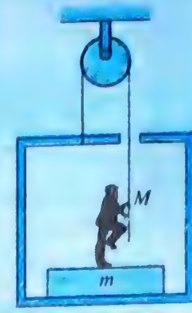
or $$\frac{3}{2} ma_0 = \frac{mg}{2} + \frac{mg}{6} \Rightarrow a_0 = \frac{4}{9} g$$

From (ii), $$T = \frac{m}{2} a_0 + \frac{m}{2} g$$

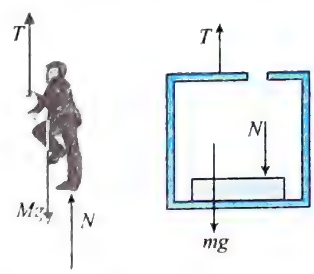
$$= \frac{m}{2} \times \frac{4}{9} g + \frac{m}{2} g \Rightarrow T = \frac{13mg}{18}$$

ILLUSTRATION 6.32

A man of mass M is standing on a plank kept in a box. The plank and box as a whole has mass m. A light string passing over a fixed smooth pulley connects the man and box. If the box remains stationary, find the tension in the string and force exerted by the man on the plank. (Given $M > m$)



Sol. The fixed pulley is taken as frame of reference. The forces on man and box with plank are shown in figure.



The forces are as follows:

(a) Weight of the man = Mg

(b) Tension in the string = T

(c) Normal contact force between the man and the plank = N

(d) The weight of the plank and box = mg

The box remains at rest, the man will have to be at rest

Referring to free-body diagram of man,

the equation of motion of the man is given as

$$T + N = Mg \quad \text{...(i)}$$

Referring to the free-body diagram of lift,

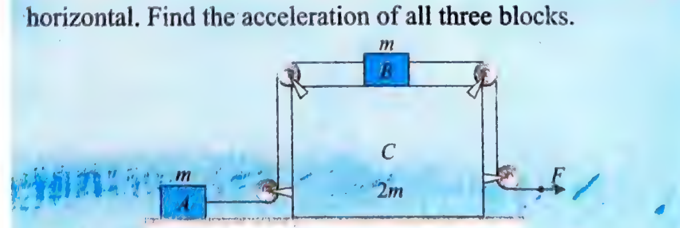
$$T = N + mg \quad \text{...(ii)}$$

Solving (i) and (ii), we obtain

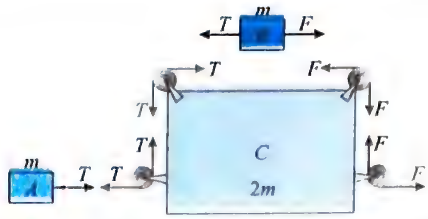
$$T = \frac{(M+m)g}{2} \text{ and } N = \frac{(M-m)g}{2}$$

ILLUSTRATION 6.33

In the system shown in the figure, all surfaces are smooth. Block A and B have mass m each and mass of block C is $2m$. All pulleys are massless and fixed to block C. Strings are light and the force F applied at the free end of the string is horizontal. Find the acceleration of all three blocks.



Sol. Let us draw the F.B.D of the blocks A, B and C. From F.B.D, we can observe the net force on the block 'C' in horizontal direction is zero. Hence, block 'C' will not move.



Blocks A and B are connected directly by a string. Hence, move with same acceleration say a

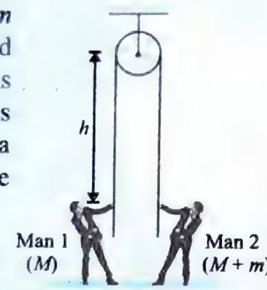
Equation of motion of the block A: $T = ma$ ...(i)

For block B: $F - T = ma$ ...(ii)

From (i) and (ii), we get $a = \dfrac{F}{2m}$

ILLUSTRATION 6.34

Two men of masses M and $M + m$ start simultaneously from the ground and climb with uniform accelerations up from the free ends of a massless inextensible rope which passes over a smooth pulley at a height h from the ground.



(a) Which man reaches the pulley first?

(b) If the man who reaches first takes time t to reach the pulley, then find the distance of the second man from the pulley at this instant.

Sol.

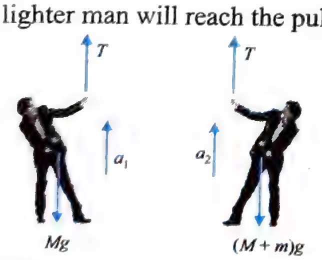
(a) Let a_1 and a_2 be the accelerations of the two men in upward direction, and T the tension in the rope. Then,

$$T - Mg = Ma_1 \Rightarrow a_1 = \frac{T}{M} - g \quad \text{... (i)}$$

and $$T - (M + m)g = (M + m)a_2 \Rightarrow a_2 = \frac{T}{M + m} - g \quad \text{...(ii)}$$

$\therefore \quad a_2 < a_1$

Hence the lighter man will reach the pulley first.



F.B.D of man '1' F.B.D of man '2'

(b) From (i) and (ii), we can find $a_2 = \dfrac{Ma_1 - mg}{M + m}$

The lighter man ascends a distance h in time t with acceleration a_1.

Hence, $h = \dfrac{1}{2}a_1 t^2 \Rightarrow a_1 = \dfrac{2h}{t^2}$ ...(iii)

Let s be the distance travelled by the heavier man in this time t. Then

$$s = \frac{1}{2}a_2 t^2 = \frac{t^2}{2}\left[\frac{M}{M + m}a_1 - \frac{mg}{M + m}\right]$$

$$= \frac{t^2}{2(M + m)}\left[M\left(\frac{2h}{t^2}\right) - mg\right] = \frac{1}{2(M + m)}[2Mh - mgt^2]$$

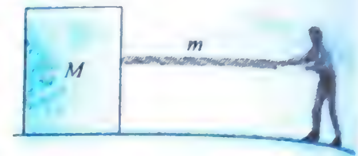
The distance of the second man from the pulley is

$$h - s = \frac{1}{(M + m)}\left[Mh + mh - Mh + \frac{mgt^2}{2}\right] = \frac{m}{(M + m)}\left[\frac{gt^2}{2} + h\right]$$

PROBLEMS OF STRING WITH MASS

ILLUSTRATION 6.35

A block of mass M is being pulled with the help of a string of mass m and length L. The horizontal force applied by the man on the string is F.



Determine

(a) Find the force exerted by the string on the block and acceleration of system.

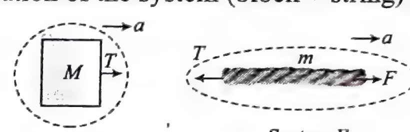
(b) Find the tension at the mid point of the string.

(c) Find the tension at a distance x from the end at which force is applied.

Assume that the block is kept on a frictionless horizontal surface and the mass is uniformly distributed in the string.

Sol.

(a) Let the force applied by string to the block be T. For part (a), consider one system is block and other string. Let the acceleration of the system (block + string) be a.



System I System II

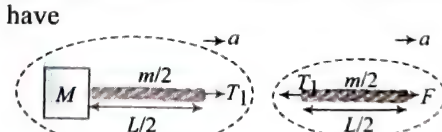
Now we apply Newton's law to each of them.

For system II: $F - T = ma$ ...(i)

For system I: $T = Ma$ ...(ii)

After solving (i) and (ii), $a = \dfrac{F}{M + m}$, $T = \dfrac{MF}{M + m}$

(b) Now we have to redefine our system. Choose system 1 as block and half string and system 2 as the other half string. On applying Newton's second law to system 1 and system 2, we have



System I System II

System II: $F - T_1 = \dfrac{m}{2} \times a$ ...(iii)

Mass per unit length of string $= m/L$

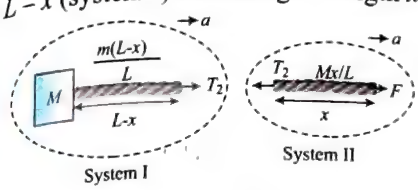
Hence, mass of $L/2$ length of string $= \dfrac{m}{L} \times \dfrac{L}{2} = \dfrac{m}{2}$

System I: $T_1 = \left(M + \dfrac{m}{2}\right)a$ ...(iv)

After solving, (iii) and (iv), we get

$$a = \frac{F}{M + m}, \quad T_1 = \frac{(M + m/2)F}{(M + m)}$$

(c) Now we can redefine our system in the block and string of length $L - x$ (system I) and string of length x (system II)



$$\text{System II: } F - T_2 = \left(\frac{m}{L} \times x\right) a \quad \text{...(v)}$$

$$\text{System I: } T_2 = \left\{\frac{m}{L}(L - x) + M\right\} a \quad \text{...(vi)}$$

Solving, (v) and (vi), we get

$$a = \frac{F}{M + m} \text{ and } T_2 = \frac{\{m(L - x)/L + M\}F}{M + m}$$

Here we see that acceleration in each part is same, but tension changes along the string.

ILLUSTRATION 6.36

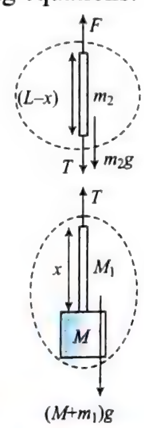
A body of mass M is hanging by an inextensible string of mass m. If the free end of the string accelerates up with constant acceleration a, find the variation of tension in the string as a function of the distance measured from the mass M (bottom of the string).

Sol. Let T be the tension in the string at a distance x from its bottom. Referring to the FBD, we have the following equations:

Force equation for m_1:

$$T - m_1 g - Mg = (M + m_1)a \quad \text{...(i)}$$

We are writing equation of motion for lower part as we are not given the force acting on top of the string



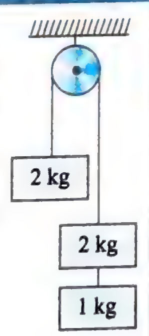
Substituting $m_1 = \frac{m}{l}x$ in equation (i), we have

$$T = \left(M + \frac{mx}{l}\right)(g + a).$$

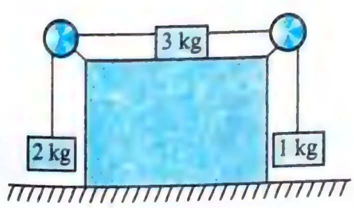
At $x = l$, $T = (M + m)(g + a) = F$(say)

CONCEPT APPLICATION EXERCISE 6.4

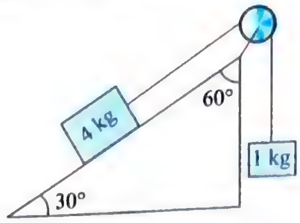
1. Consider the system shown in figure. The system is released from rest, find the tension in the cord connected between 1 kg and 2 kg blocks.



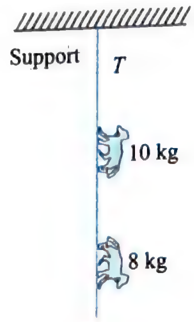
2. The system shown in figure is released from rest. Calculate the tension in the strings and the force exerted by the strings on the pulleys, assuming pulleys and strings are massless.



3. Find the acceleration of blocks and tension in the cord in the device shown in figure. Assume no friction anywhere.



4. Two monkeys of masses 10 kg and 8 kg are moving along a vertical rope as shown in figure. The former climbing up with an acceleration of 2 m s^{-2}, while the later coming down with a uniform velocity of 2 m s^{-1}. Find the tension in the rope at the fixed support.

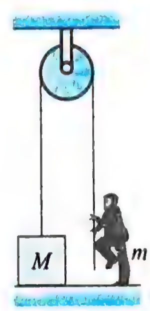


5. A homogeneous rod of length L is acted upon by two forces F_1 and F_2 applied to its ends and directed opposite to each other. With what force F will the rod be stretched at the cross section at a distance l from the end where F_1 is applied?

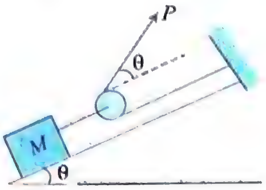
6. A 20-kg monkey has a firm hold on a light rope that passes over a frictionless pulley and is attached to a 20-kg bunch of bananas (as shown in figure). The monkey looks up, sees the bananas, and starts to climb the rope to get them.

   (a) As the monkey climbs, do the bananas move up, down, or remain at rest?
   (b) As the monkey climbs, does the distance between the monkey and the bananas decrease, increase, or remain constant?
   (c) The monkey releases its hold on the rope. What happens to the distance between the monkey and the bananas while it is falling?
   (d) Before reaching the ground, the monkey grabs the rope to stop its fall. What do the bananas do?

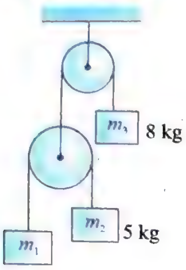
7. In the given figure, the block of mass M is at rest on the floor. At what acceleration with which should a boy of mass m climb along the rope of negligible mass so as to lift the block from the floor?



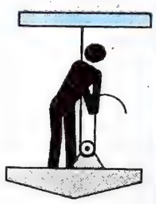
8. What should be the minimum force P to be applied to the string so that block of mass m just begins to move up the frictionless plane?



9. Three blocks m_1, m_2, and m_3 are arranged as shown in figure. If $m_2 = 5$ kg and $m_3 = 8$ kg, at what value of m_1 will 8 kg mass be at rest?

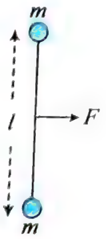


10. In the given figure, the man and the platform together weigh 950 N. The pulley can be treated as frictionless. Determine how hard the man has to pull on the rope to lift himself upward above the ground with constant velocity. If the weight of man is 550 N, what is the normal reaction between them?



11. A locomotive accelerates a train of identical railway carts. The carts are numbered consecutively with the cart next to locomotive having the number 1. The tension in the connection between the carts with numbers 4 and 5 is three times bigger than the tension in the connection between the carts with numbers 14 and 15. What is the number of the last cart? There is no resistance to the train's motion.

12. Two identical small masses each of mass m are connected by a light inextensible string on a smooth horizontal floor. A constant force F is applied at the mid point of the string as shown in figure. Find the acceleration of each mass towards each other.



13. A body hangs from a spring balance supported from the roof of an elevator.
   (a) If the elevator has an upward acceleration of 2.45 m s^{-2} and the balance reads 50 N, what is the true weight of the body?
   (b) Under what circumstances will the balance read 30 N?
   (c) What will be the balance reading if the elevator cable breaks?

ANSWERS

1. 8 N

2. $T_1 = \frac{7g}{6}$ N; $T_2 = \frac{5g}{3}$ N

   Force on pulley $P_1 = \sqrt{2}T_1$

   Force on pulley $P_2 = \sqrt{2}T_2$

3. 2 ms^{-2}, 12 N   4. 200 N   5. $\frac{(F_2 - F_1)l}{L} + F_1$

6. (a) The bananas move up.
   (b) Distance between them stays the same.
   (c) Distance between them does not change.
   (d) The bananas will slow down at the same rate as the monkey; if the monkey comes to a stop, so will the bananas.

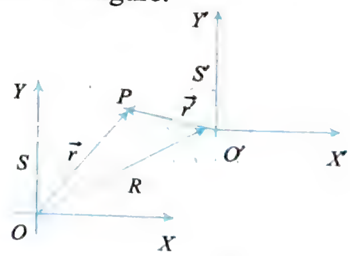
7. $a > \left(\frac{M}{m} - 1\right)g$ 8. $\frac{mg\sin\theta}{1+\cos\theta}$ 9. $\frac{10}{3}$ kg

10. 950 N, 1500 N   11. 19   12. $\frac{F}{2m}\tan\theta$

13. (a) 40 N   (b) $a = -\frac{g}{4}$ (c) 0 N

NON-INERTIAL FRAME OF REFERENCE AND PSEUDO (FICTITIOUS) FORCE

A coordinate system whose origin either accelerates or rotates is called a non-inertial frame of reference. The motion of a particle (P) is studied from two frames of references, S and S'. S is an inertial frame of reference, and S' is a non-inertial frame of reference. At any time, the position vectors of the particle with respect to those two frames are $\vec{r}$ and $\vec{r}'$, respectively. At the same moment, the position vector of the origin of S' is $\vec{R}$ with respect to S as shown in figure.



From the vector triangle $OO'P$, we get $\vec{r}' = \vec{r} - \vec{R}$

Differentiating this equation twice with respect to time, we get

$$\frac{d^2\vec{r}'}{dt^2} = \frac{d^2(\vec{r})}{dt^2} - \frac{d^2}{dt^2}(\vec{R}) \Rightarrow \vec{a}' = \vec{a} - \vec{A}$$

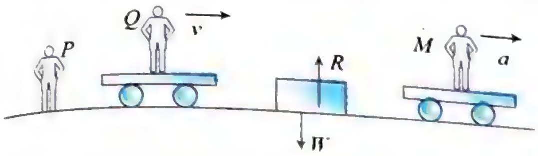
Here $\vec{a}'$ is the acceleration of the particle P relative to S', $\vec{a}$ is the acceleration of the particle relative to S, and $\vec{A}$ is the acceleration of S' relative to S.

Multiplying the above equation by m (mass of the particle), we get $m\vec{a}' = m\vec{a} - m\vec{A}$

$$\Rightarrow \quad \vec{F}' = \vec{F}_{(\text{real})} - m\vec{A} \Rightarrow \vec{F}' = \vec{F}_{(\text{real})} + (-m\vec{A})$$

We can observe an additional force $(-m\vec{A})$ apart from real face, that is acting on the particle as seen from an observer observing from reference frame S'. Let us learn about this additional force through an example

Suppose a box is placed on a railway platform. The only forces acting on it are its weight W acting downwards and the normal reaction R acting upwards. W and R are thus equal and opposite. Thus, the net external force acting on the box is zero. The box is at rest.



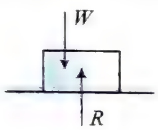
Let us consider the forces acting on this box and its state with respect to three observers, namely,

- A person P standing on the platform.
- A person Q sitting in a train moving with uniform velocity in a straight line parallel to the platform.
- A person M sitting in a train moving parallel to the platform but with some acceleration.

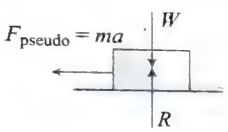
The box is at rest w.r.t. observer P. The box is in uniform motion in a straight line w.r.t. observer Q (moving with $-v$ velocity).

The box is having (negative) acceleration w.r.t. the observer M. For all the three observers (or frames of reference), the force acting on the box are the same: W and R. As W and R are equal and opposite, the net external force acting on the box is zero.

Let us reconsider the example of the box lying at rest on a railway platform. We saw that it has $-\vec{a}$ acceleration w.r.t. a train which is moving with an acceleration $\vec{a}$ even though no net external material force is acting on it. Neither first nor second law remains valid on the box with respect to the accelerating train.



Free-body diagram as seen from observer P and Q



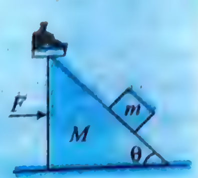
Free-body diagram as seen from observer M

But still, if we want to apply Newton's laws of motion in a non-inertial reference frame, we are allowed to do so, provided we include an additional force, the pseudo force, in the free-body diagram.

If we apply an imaginary force ma in the direction opposite to the direction of motion of observer M, the observer M can write the equation of motion of box. Thus, Newton's laws can also be applied with respect to a non-inertial reference frame, provided we include an extra force $-ma$ on the system. This force has no existence in reality but has been included only to suit the calculations involved, by Newton's second law, while working out a problem w.r.t. a non-inertial reference frame. This imaginary force is known as "pseudo force" or "fictitious force" or "inertial force." (The term "pseudo" means something which is not real.)

ILLUSTRATION 6.37

The block of mass m is in equilibrium relative to the smooth wedge of mass M which is pushed by a horizontal force F. Find pseudo force acting on (a) m, (b) M as viewed by the observer sitting on the wedge. Will these pseudo forces (c) equal and opposite, action-reaction pairs? Explain.



Sol. Taking block and wedge as system.

Net force on the system horizontally $= F$

Total mass $= M + m$

Therefore, acceleration of the system in +ve x-direction

$$a = \frac{F}{M+m} \Rightarrow \vec{a} = \frac{F}{M+m}\hat{i}$$

(a) Pseudo force acting on block m is in opposite direction of observer acceleration and magnitude equal to multiplication of mass of the object and the acceleration of the observer

$$\vec{F}_{PS} = -ma\hat{i} = -\frac{mF}{M+m}\hat{i}$$

(b) Pseudo force acting on block M $\vec{F}'_{PS} = -Ma\hat{i}$

$$\vec{F}'_{PS} = -\frac{MF}{M+m}\hat{i}$$

(c) Pseudo force is not a real force. It is applied to solve the problems of non-inertial frame. It does not have action reaction pairs.

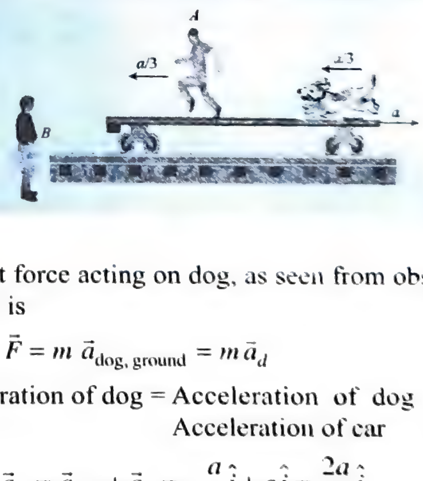
ILLUSTRATION 6.38

A rail-road car is moving towards right with acceleration a. A man A accelerating toward left with an acceleration of magnitude $a/3$ w.r.t. to car. A dog of mass m is following man A with an acceleration $a/3$ relative to the car. Observer B on ground is observing the dog and man A. Find the

(a) net force experienced by the dog as seen by observer B standing on ground.

(b) rate of change of linear momentum of the dog relative to the man A moving on trolley.

(c) pseudo-force on the dog as seen from man A.



Sol.

(a) The net force acting on dog, as seen from observer 'B' on ground is

$$\vec{F} = m\,\vec{a}_{\text{dog, ground}} = m\vec{a}_d \quad \text{(i)}$$

Acceleration of dog = Acceleration of dog w.r.t. car + Acceleration of car

$$\Rightarrow \quad \vec{a}_d = \vec{a}_{d,c} + \vec{a}_c = -\frac{a}{3}\hat{i} + a\hat{i} = \frac{2a}{3}\hat{i}$$

Hence, from (i), $\vec{F} = \frac{2ma}{3}\hat{i}$

(b) The rate of change of momentum of the dog relative to the man A

$$\frac{d\vec{p}}{dt} = m\vec{a}_{d,A} \quad \text{(ii)}$$

where $\vec{a}_{d,A} = \vec{a}_{d,C} - \vec{a}_{A,C} = -\frac{a}{3}\hat{i} - \left(\frac{-a}{3}\hat{i}\right) = 0$

Hence, from (ii) $\frac{d\vec{p}}{dt} = 0$

(c). Pseudo-force acting on dog viewed by the man A is

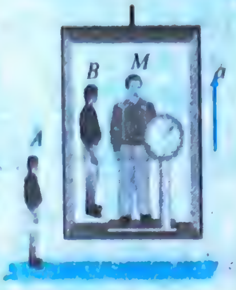
$$\vec{F}_{\text{pseudo}} = -m\vec{a}_{A,\text{ground}}$$

where $\vec{a}_{A,\text{ground}} = \vec{a}_{A,C} + \vec{a}_C = -\frac{a}{3}\hat{i} + a\hat{i} = \frac{2a}{3}\hat{i}$

This gives $\vec{F}_{\text{pseudo}} = -\frac{2ma}{3}\hat{i}$

ILLUSTRATION 6.39

A man of mass M stands on a weighing machine in an elevator accelerating upwards with an acceleration a. Draw the free-body diagram of the man as observed by the observer A (stationary on the ground) and observer B (stationary on the elevator). Also, calculate the reading of the weighing machine.

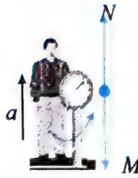


Sol.

Method 1: Observation from observer A on ground.

From FBD of man w.r.t. A, using Newton's second law,

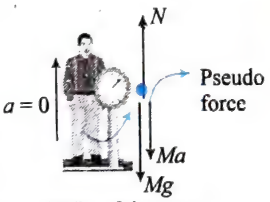
$$N - Mg = Ma \Rightarrow N = M(g + a)$$



FBD of the man w.r.t. observer A

N is the reading of weighing machine.

Method 2: Observation from observer B on elevator. The observer 'B' is moving in accelerated frame of reference. If he observes the man on weighing machine he will observe the man at rest w.r.t. elevator. If he is asked to draw a free-body diagram of M he will apply pseudo force on M apart from real forces acting on him.



FBD of the man w.r.t. observer 'B'

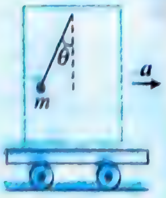
From free-body diagram:

$$N = Mg + ma \Rightarrow N = M(g + a)$$

The value of N calculated from both the methods is same.

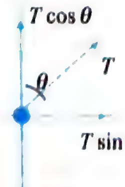
ILLUSTRATION 6.40

A bob of mass $m = 50$ g is suspended from the ceiling of a trolley by a light inextensible string. If the trolley accelerates horizontally, the string makes an angle $\theta = 37°$ with the vertical. Find the acceleration of the trolley.



Sol.

Method 1: Problem solving from ground frame. The observer on the ground will observe the bob moving with acceleration a same as trolley. In FBD, he will observe only two real forces, weight and tension force.



FBD of the bob w.r.t observer standing on ground

The bob is at equilibrium in vertical direction, hence writing equation of motion in vertical direction

$$T\cos\theta = mg \quad \text{...(i)}$$

Along horizontal direction, the bob is accelerating with acceleration a.

$$T\sin\theta = ma \quad \text{...(ii)}$$

Using (i) and (ii), $\tan\theta = a/g$

$$\therefore \quad a = g\tan\theta = 10\tan 37° = 7.5 \text{ m s}^{-2}$$

Therefore, acceleration of the trolley is 7.5 m s^{-2}

Method 2: Problem solving from non-inertial frame. If an observer sitting in the trolley and observes the bob, he will observe the bob at rest. If he asks to draw free-body diagram of bob he will apply pseudo force apart from real forces acting on the bob.

FBD of the bob w.r.t. observer in trolley

The bob is at equilibrium in vertical direction, hence writing equation of motion in vertical direction

$$T\cos\theta = mg \quad \text{...(iii)}$$

Along horizontal direction the bob is at rest.

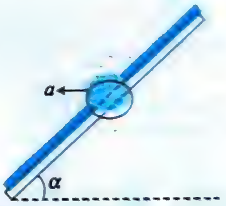
$$T\sin\theta = ma \quad \text{...(iv)}$$

Using (iii) and (iv) $\tan\theta = a/g$

$$\therefore \quad a = g\tan\theta = 10\tan 37° = 7.5 \text{ m s}^{-2}$$

ILLUSTRATION 6.41

A bead of mass m is fitted on to a rod and can move on it without friction. At the initial moment the bead is in the middle of the rod.



The rod moves translationally in a horizontal plane with an acceleration a in a direction forming an angle α with the rod. Find the acceleration of the bead relative to the rod.

Sol.

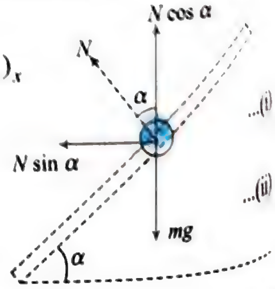
Method 1: Observation from ground frame. Let a_r be the acceleration of the bead relative to the rod. Then $a_r\cos\alpha$ is the leftward acceleration of the bead relative to the rod and $a_r\sin\alpha$ is downward relative acceleration of the rod. If a_x and a_y be the absolute leftward horizontal and downward vertical acceleration of the bead, then

$$(\vec{a}_{\text{bead}})_x = (\vec{a}_{\text{bead,rod}})_x + (\vec{a}_{\text{rod}})_x$$

or $\quad a_x = a_r\cos\alpha + a \quad$ ...(i)

and $\quad a_r\sin\alpha = a_y - 0$

or $\quad a_y = a_r\sin\alpha \quad$ ...(ii)



From FBD of the bead (projecting forces vertically and horizontally)

$$mg - N\cos\alpha = ma_r\sin\alpha \quad \text{...(i)}$$

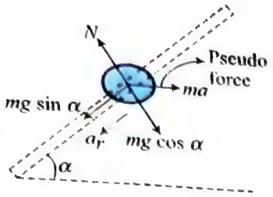
and $\quad N\sin\alpha = m(a_r\cos\alpha + a) \quad$ ...(ii)

Eliminating N between (i) and (ii)

$$mg \sin \alpha = ma_r + ma \cos \alpha$$

or $\quad a_r = g \sin \alpha - a \cos \alpha$

Method 2: Observation from an observer moving with rod. Considering bead w.r.t. rod, i.e., from non-inertial frame. A pseudo force of magnitude ma will act on the bead in the direction opposite to accelerator of rod, i.e., in right direction.



The bead is not moving perpendicular to rod. Hence,

$$N = mg \cos \alpha + ma \sin \alpha$$

Also in the direction along the rod, let acceleration of the bead w.r.t rod is a_r.

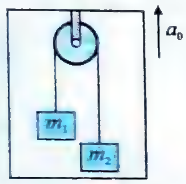
Equation of motion of bead with rod,

$$mg \sin \alpha - ma \cos \alpha = ma_r$$

$\Rightarrow \quad a_r = g \sin \alpha - a \cos \alpha$

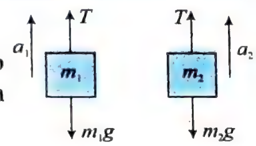
ILLUSTRATION 6.42

If the pulley is massless and moves with an upward acceleration a_0, find the acceleration of m_1 and m_2 w.r.t. to elevator.



Sol. **Method 1:** (Ground frame):

Let acceleration of block m_1 with respect to pulley be a (upward) and the acceleration of m_2 w.r.t. pulley is a (downward)



Equation of motion for 'm_1'

$$T - m_1g = m_1a_1 \quad \text{...(i)}$$

$$T - m_2g = m_2a_2 \quad \text{...(ii)}$$

$$\vec{a}_1 = \vec{a}_{1,P} + \vec{a}_P = a + a_0 \quad \text{...(iii)}$$

$$\vec{a}_2 = \vec{a}_{2,P} + \vec{a}_P = -a + a_0 \quad \text{...(iv)}$$

Substituting a_1 from (iii) in (i),

$$T - m_1g = m_1(a + a_0) \quad \text{...(v)}$$

Substituting a_2 from (iv) in (ii),

$$T - m_2g = m_2(-a + a_0) \quad \text{...(vi)}$$

Solving (v) and (iv),

$$T = \frac{2m_1m_2}{m_1 + m_2}(g + a_0) \text{ and } a = \frac{m_2 - m_1}{m_1 + m_2}(g + a_0)$$

Method 2: Solving problem from non-inertial frame of reference

Let us build the equations by using Newton's second law sitting on the accelerating pulley. Hence, we impose pseudo force $m_1a_0 \downarrow$ and $m_2a_0 \downarrow$ on both m_1 and m_2, respectively, in addition to the upward tension and their weights $m_1g \downarrow$ $m_2g \downarrow$, respectively. If m_1 accelerates up relative to the pulley, m_2 must accelerate down relative to the pulley with acceleration a.

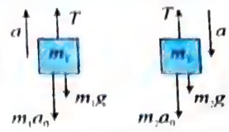
Force equation:

For m_1: $T - m_1g - m_1a_0 = m_1a \quad$ ...(i)

For m_2: $m_2g + m_2a_0 - T = m_2a \quad$ ...(ii)

Solving (i) and (ii), we have

$$a = \frac{m_2 - m_1}{m_1 + m_2}(g + a_0)$$



FBD of the m_1 and m_2 as seen from observer in elevator

ILLUSTRATION 6.43

Two smooth blocks of masses m and m' connected by a light inextensible strings are moving on a smooth wedge of mass M. If a force F acts on the wedge the blocks do not slide relative to the wedge. Find the (a) acceleration of the wedge and (b) value of F.

Sol.

(a) The force acting on the blocks m and m' parallel to the surface in contact are given as $mA \leftarrow$, $T \rightarrow$, $m'g\downarrow$, and $T\uparrow$.

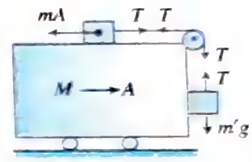
Force equation relative to M:

Since $a_{mM} = a_0 = 0$

For m: $T = Ma \quad$ ...(i)

For m': $T = m'g \quad$ ...(ii)

From Eq. (i) and (ii) $A = \dfrac{m'}{m}g$



(b) Force equation on system '$M + m + m'$'

$$\Sigma F_x = ma_x$$

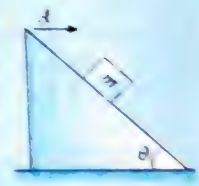
$$F = MA + mA + m'A$$

$$= (M + m + m')A$$

$$= (M + m + m')\frac{m'}{m}g$$

ILLUSTRATION 6.44

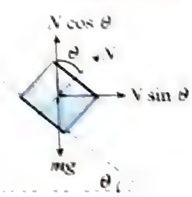
A block of mass m is placed on an inclined plane. With what acceleration A towards right should the system move on a horizontal surface so that m does not slide on the surface of inclined plane? Also calculate the force supplied by wedge on the block. Assume all surfaces are smooth.



Sol. **Method 1: Analysis of forces on m relative to ground**

If the motion of m is analyzed from ground, its acceleration is A and the forces acting on it are its weight mg and normal reaction N.

As m is at rest, moving with same acceleration as wedge in horizontal direction but in vertical direction, the block is at rest.



$$\Sigma \vec{F}_y = 0 \Rightarrow N \cos\theta = mg \quad \text{...(i)}$$

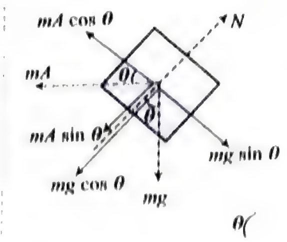
$$\Sigma \vec{F}_x = \Sigma m_i \vec{a}_i \Rightarrow N \sin\theta = mA \quad \text{...(ii)}$$

On solving (i) and (ii), we get $A = g \tan\theta$ and $N = \dfrac{mg}{\cos\theta}$

Method 2: Analysis of forces on m relative to the inclined plane

If the motion of m is analyzed from the view point of an observer standing on the inclined plane (i.e., relative to the plane), its

acceleration is 0 and the forces acting on it are: its weight, the normal reaction, and a pseudo force of magnitude mA towards left.



$$mA\sin\theta = mA\cos\theta$$

$$\Rightarrow \quad A = g\tan\theta$$

Also: $N = mg\cos\theta + mA\sin\theta$

$$= mg\cos\theta + mg\tan\theta\sin\theta = \frac{mg}{\cos\theta}$$

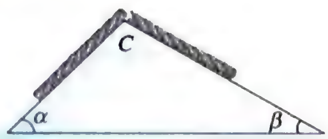
Important Points:

If we resolve the forces along the sloping side, we have net force along sloping direction $F = mA\cos\theta - mg\sin\theta$

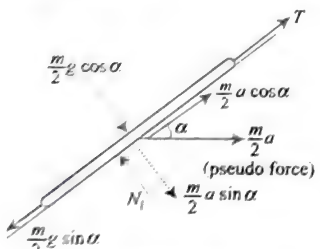
- If $ma\cos\theta > mg\sin\theta$, F is directed up along the slant. Then, the block will accelerate relative to the wedge with acceleration $a = A\cos\theta - g\sin\theta$
- If $mA\cos\theta = mg\sin\theta$, $F = 0$. Hence the block remains stationary relative to the wedge (or more with constant velocity relative to the wedge).
- If $mA\cos\theta < mg\sin\theta$, F is directed down along the slant. Hence, the block accelerates down the slant with acceleration $a = g\sin\theta - A\cos\theta$

ILLUSTRATION 6.45

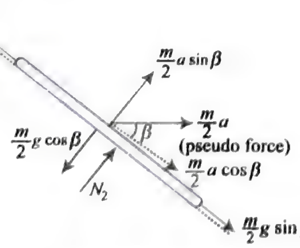
A homogeneous flexible rope rests on a wedge whose side edges make angles α and β with the horizontal (refer figure). The central part of the rope lies on the upper rib C of the wedge. With what acceleration should the wedge be pulled to the left along the horizontal plane in order to prevent the displacement of the rope with respect to the wedge? [Consider all surfaces to be smooth]



Sol. Let us assume the wedge is moving with acceleration a, considering the parts of the rope on either side of incline as two different elements and analyzing each part of the rope w.r.t wedge, the F.B.D of each part.



F.B.D of left part



F.B.D of right part

As rope is not sliding w.r.t wedge, hence

For left part: $T + \frac{m}{2}a\cos\alpha = \frac{m}{2}g\sin\alpha$ ...(i)

For right part: $T = \frac{m}{2}a\cos\beta + \frac{m}{2}g\sin\beta$ ...(ii)

From (i) and (ii), we get

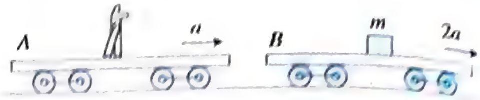
$$\frac{m}{2}a\cos\beta + \frac{m}{2}g\sin\beta + \frac{m}{2}a\cos\alpha = \frac{m}{2}g\sin\alpha$$

$$a[\cos\beta + \cos\alpha] = g[\sin\alpha - \sin\beta]$$

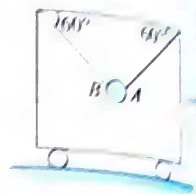
$$\Rightarrow \quad a = \frac{g[\sin\alpha - \sin\beta]}{(\cos\alpha + \cos\beta)}$$

CONCEPT APPLICATION EXERCISE 6.5

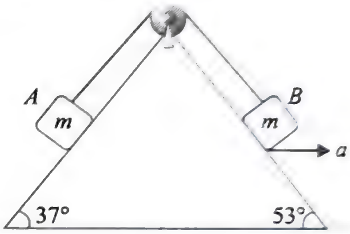
1. Two trolleys A and B are moving with acceleration a and $2a$, respectively, in the same direction. To an observer in trolley A, Find the magnitude of the pseudo force acting on a block of mass m on trolley B.



2. A steel ball is suspended from the ceiling of an accelerating carriage by means of two cords A and B. Determine the acceleration a of the carriage which will cause the tension in A to be twice that in B.



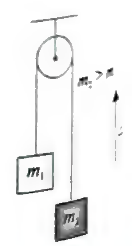
3. Two blocks A and B of equal masses m kg each are connected by a light thread, which passes over a massless pulley as shown. Both the blocks lie on wedge of mass m. Assume friction to be absent everywhere and both the blocks to be always in contact with the wedge. The wedge lying over smooth horizontal surface is pulled towards right with constant acceleration a (g is acceleration due to gravity).



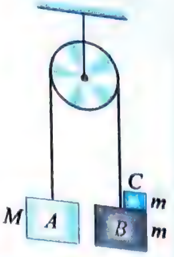
   (i) Find the normal reaction acting on block A
   (ii) Find the normal reaction acting on block B
   (iii) Find the accelerations of block A and block B w.r.t. wedge.
   (iv) Find the maximum value of acceleration a for which normal reactions acting on the block A and block B are non-zero.

4. A person is standing on a weighing machine placed on the floor of an elevator. The elevator starts going up with some acceleration, moves with uniform velocity for a while and finally decelerates to stop. The maximum and the minimum weights recorded are 72 kg and 60 kg. Assuming that the magnitudes of the acceleration and the deceleration are the same, find (a) the true weight of the person and (b) the magnitude of the acceleration. Take $g = 10\ \text{m/s}^2$.

5. In the given figure, the pulley and strings are light. The pulley is suspended in a elevator moving up with acceleration a_0. Find the accelerations of the blocks with respect to elevator.



6. In the given system, the pulley is light and frictionless; string is also massless. Initially the system is in equilibrium. Now a small block C of mass m is placed on the block B of mass M. If block C always remains on block B, find the normal reaction on C due to B.



ANSWERS

1. ma in backward direction.

2. $\frac{g}{3\sqrt{3}}$

3. (i) $\frac{m}{5}(4g-3a)$ (ii) $\frac{m}{5}(3g+4a)$

(iii) $\frac{1}{10}(7a-g)$ (iv) $\frac{4}{3}g$

4. 66 kg, $\frac{10}{11}$ m/s^2

5. $\frac{(m_2-m_1)}{(m_2+m_1)}(g-a_0)$

6. $\frac{2Mmg}{(2M+m)}$

CONSTRAINT RELATIONS

Constraints mean that two bodies (in this case the bodies which are attached to the pulley) are not free to move the way they want. The accelerations between them are dependent on each other. We need to find out the relationship to solve the problems of Newton's laws of motion.

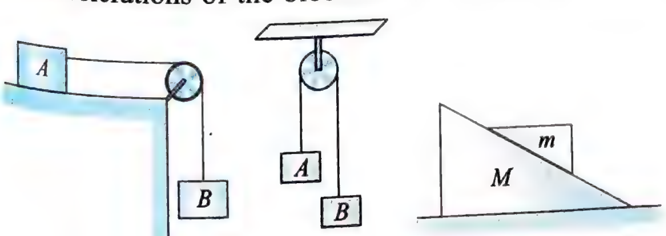
The equations showing the relation of the motions of a system of bodies, in which one motion is constrained by the other motions, are called *constrained relations*.

Constraints are the geometrical restrictions imposed on the motion of a body, which also govern the trajectory of the body. For example, a block placed on the table cannot move normal to the surface; it is bound to move parallel to the surface.

We have to use the method of constraint equations to relate the accelerations between the bodies.

First, we start our analysis with simple cases of pulleys. Consider the situation shown in figure. Two bodies are connected with a string which passes over a pulley at the corner of a table. Here if the string is inextensible, we can directly state that the displacement of A in the downward direction is equal to the displacement of B in the horizontal direction on table, and if displacements of A and B are equal in equal time, their speeds and acceleration magnitude must also be equal.

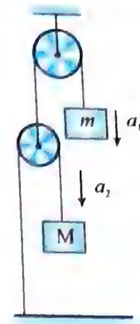
In figure, if the wedge and the block are free to move, it is obvious that the accelerations of the block and the wedge are related.



(a) (b) (c)

Look at figure. In this case, let us say that you have to find the acceleration of the masses. The number of unknowns will be

1. Tension, T
2. Acceleration, a_1, of the mass
3. Acceleration of the other body, a_2



(d)

There are three unknowns. However, we will get only two equations—one for one mass and another for the other mass.

Clearly, you can see that Newton's laws are not sufficient to solve the problem. In such cases, we need additional equations. Therefore, constraints provide additional equations. These are provided by what are called as *constraint equations*.

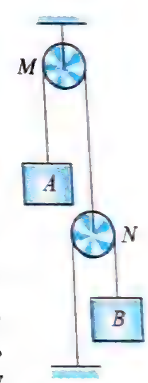
In many cases, you can write down the relation of acceleration by just looking at the situation. In other cases, for complex relationships, we can think of four types of constraints. (1) pulley constraints, (2) wedge constraints, (3) combination of wedge and pulleys, and (4) general constraints.

PULLEY CONSTRAINT

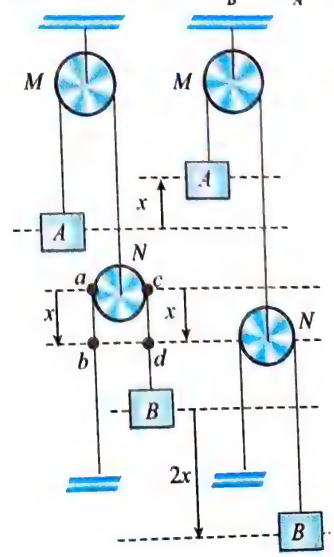
Pulley constraints are applicable when the bodies concerned are connected through pulleys and the rope connecting them is inextensible.

Let us learn the application of pulley constraint through the following cases through different methods:

Case I: Mass A is connected with a string which passes through a fixed pulley. The other end of the string is connected with a movable pulley N. Block B is connected with another string which passes through the pulley N as shown in figure. Find the reaction between the accelerations of A and B.



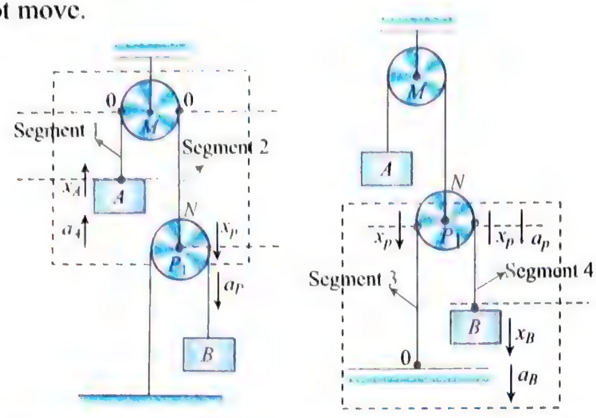
Method 1: Consider the situation shown in figure. If we consider that mass A is going up by distance x, pulley N, which is attached to the same string, will go down by the same distance x. Due to this, the string which is connected to mass B will now have free lengths ab and cd ($ab = cd = x$) which will go on the side of mass B due to its weight as the other end is fixed. Thus, mass B will go down by $2x$. Hence, its speed and acceleration will be twice that of block B. Hence, $v_B = 2v_A$ and $a_B = 2a_A$



Method 2: The above case can be understood by another approach in which we will consider that the total length of the string will always be constant.

To relate the acceleration of the bodies, assume that various bodies move by a distance x_1, x_2, ... Calculate the number of segments in the string.

The segments in the first string are marked 1 and 2. The distance moved by the various elements are also marked (as shown in Figure). Note that the pulley, which is connected to the ceiling, cannot move.



(a) Considering segments in string 1 (b) Considering segments in string 2

Relate the distance moved. (Total change in the length of the string must be zero.) To do these, calculate the change in the length of each segment of the string. Then add these changes to get the total change in the length of the string should be zero.

Change in the length of segment 1

$$\Delta l_1 = (-x_A) + 0 = -x_A$$

(Why negative? Because as block A moves up, the length of segment of string comes closer to the fixed pulley M and the length of the segment decreases.)

Change in the length of segment 2

$$\Delta l_2 = (+x_P) + 0 = +x_P.$$

Total change in length of string 1

$$-x_A + x_P = 0 \Rightarrow x_A = x_P$$

Once we have the relation between the distances, the relation between accelerations is simple. For the first string, it is $a_A = a_P$.

For second string, the distances moved by pulley N and block B are x_P and x_B, respectively, are shown in figure. The ground is at rest. Therefore, the end of the string that is connected to the ground will not move, its displacement can be taken zero.

Change in the length of segment 3

$$\Delta l_3 = (-x_P) + 0 = -x_P$$

(Why negative? Because as pulley moves down, the string comes closer to the ground and the length of the segment decreases.)

Change in the length of segment 4

$$\Delta l_4 = (+x_B) + (-x_P) = x_B - x_P.$$

Note: To see why this is so, let us consider a string with either points moving by a distance x_P and x_B. This is shown in figure.

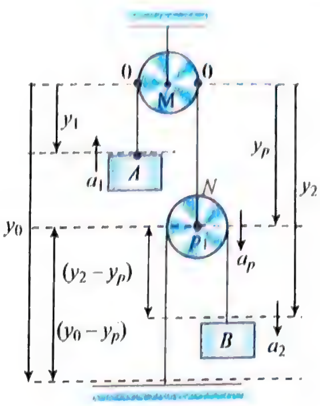
Because the right end moves by x_B, the length of the string increases by x_B. When the left end moves by x_P, the length reduces by x_P. The change in length is, therefore, $x_B - x_P$.

The total change in the length of the string is $-x_P + x_B - x_P = 0$

$$\Rightarrow \quad x_B - 2x_P = 0 \quad \text{or} \quad x_B = 2x_P \quad \text{or} \quad x_B = 2x_A$$

Once we get the relation between the distances moved, the acceleration relation will be the same. The acceleration relation is also $a_B = 2a_A$.

Method 3: The positions of blocks and movable pulley are taken from fixed pulley as shown in figure. The length of string is constant.



For string 1:

$$y_1 + y_2 + l_0 = \text{constant} \qquad ...(i)$$

l_0 is the length of part of string on pulley M.

Differentiating (i) w.r.t. time,

$$\frac{dy_1}{dt} + \frac{dy_P}{dt} + \frac{dl_0}{dt} = 0$$

or $\quad (-v_A) + v_P = 0$

or $\quad v_A = v_P$

Also $\quad a_A = a_P$

[dy_1/dt is negative as y_1 is decreasing with time and $dl_0/dt = 0$ as part of string on pulley is constant.

For string 2: $(y_0 - y_P) + (y_2 - y_P) + l'_0 = \text{constant}$

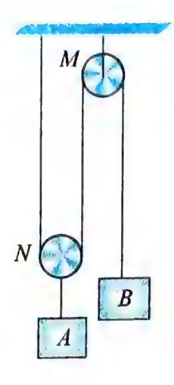
$$y_0 + y_2 - 2y_p + L'_s = \text{Constant} \qquad ...(ii)$$

Differentiating (ii) w.r.t. time,

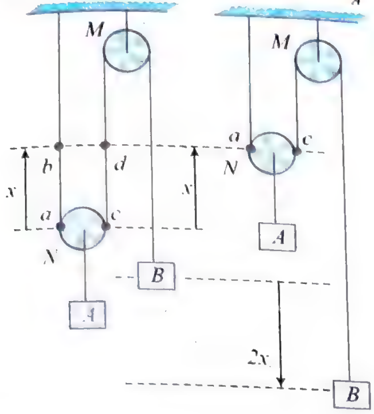
$$0 + (v_B) - 2(v_P) + 0 = 0 \Rightarrow v_B = 2v_P$$

or $\quad v_B = 2v_A \Rightarrow a_B = 2a_A$

Case II: In the given situation, two blocks A and B are arranged as shown in figure. M is a fixed pulley and N is a movable pulley. The system is released from rest. We need to find the relationship between displacements, velocities, and accelerations of the blocks. We will analyze this case through various methods:



Method 1: If mass A goes up by a distance x, we can observe that the string lengths ab and cd are slack. Due to the weight of block B, this length ($ab + cd = 2x$) will go on this side and block B will descend by a distance $2x$. As in equal time duration, B has travelled a distance twice that of A. So, $v_B = 2v_A$ and $a_B = 2a_A$.



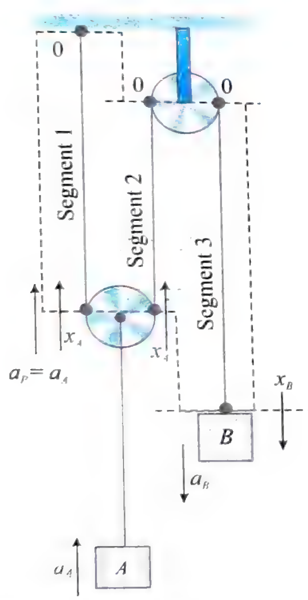
Method 2: Segment method

Change in the length of segment 1

$$\Delta l_1 = (0) + (-x_A) = -x_A$$

Change in the length of segment 2

$$\Delta l_2 = 0 + (-x_A) = -x_A$$



Change in the length of segment 3

$$\Delta l_3 = 0 + (+x_B) = +x_B$$

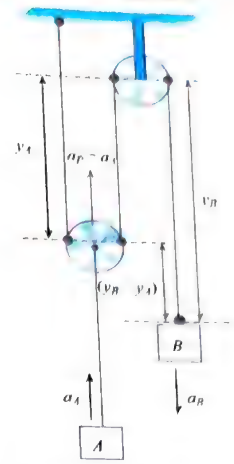
Total change in the length of string should be zero. Hence, the total sum of change of lengths of segments should be zero, i.e.,

$$\Delta l_1 + \Delta l_2 + \Delta l_3 = 0$$

$$(-x_A) + (-x_A) + (+x_B) = 0$$

$$\Rightarrow \quad x_B - 2x_A = 0 \text{ or } a_B = 2a_A$$

Method 3: Let us consider the system shown in figure. It is clear from the figure that the position of A is governed by the position of the center of movable pulley. Let at any instant the block B is at y_B and the center of movable pulley is y_A from the reference line (dotted line). The total length of the cord:



$$y_B + 2y_A + l_0 = \text{constant} = l \text{ (say)} \quad \text{...(i)}$$

l_0 is the part of the cord which is over the pulley (remain constant).
Differentiating (i) w.r.t. time, we get

$$\frac{dy_B}{dt} + \frac{2dy_A}{dt} + \frac{dl_0}{dt} = \frac{dl}{dt} \quad \text{...(ii)}$$

As l_0 and l are constant, therefore,

$$\frac{dl_0}{dt} = 0 \text{ and } \frac{dl}{dt} = 0$$

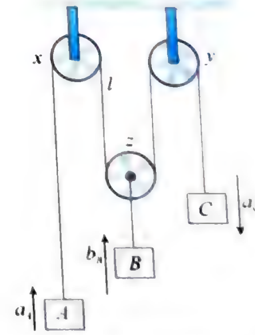
and $\dfrac{dy_B}{dt} = v_B$ and $\dfrac{dy_A}{dt} = -v_A$

Equation (ii) becomes $v_B - 2v_A = 0$ ...(iii)

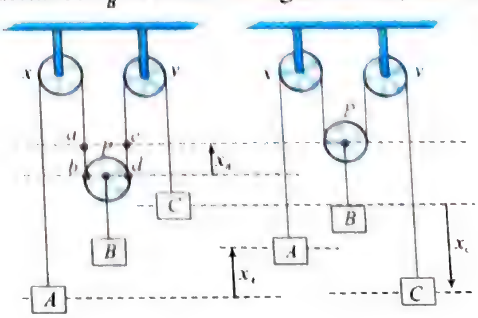
Differentiating once more, we get

$$a_B - 2a_A = 0 \quad \text{or} \quad a_B = 2a_A$$

Case III: Here three blocks A, B, and C are connected with strings and pulleys as shown in figure Here we develop constraint relation between the motion of masses A, B, and C.



Method 1: Let us assume that masses A and B would go up by distance x_A and x_B, respectively. These lengths of the string will slack as length ($ab + cd$) above the pulley P. Thus, block B will go up by a distance x_B as shown in figure. Thus, we have



$$(ab + cd) + x_A = x_C$$

or $\quad 2x_B + x_A = x_C$

Differentiating w.r.t. time, we get

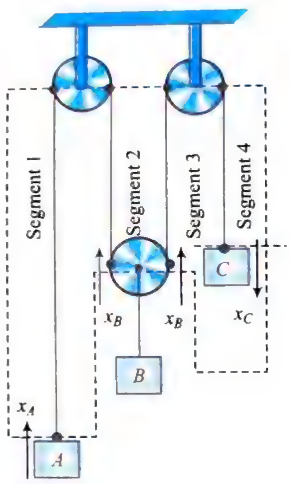
$$2v_B + v_A = v_C \quad \text{...(i)}$$

Differentiating again w.r.t. time,

$$2a_B = a_C - a_A \quad \text{...(ii)}$$

Equations (i) and (ii) are the constraint relations for motion of masses A, B, and C.

Method 2:



Change in the length of segment 1

$$\Delta l_1 = (-x_A) + (0) = -x_A$$

Change in the length of segment 2

$$\Delta l_2 = 0 + (-x_B) = -x_B$$

Change in the length of segment 3

$$\Delta l_3 = 0 + (-x_B) = -x_B$$

Change in the length of segment 4

$$\Delta l_4 = (0) + (x_C) = x_C$$

Total change in the segment length should be zero.

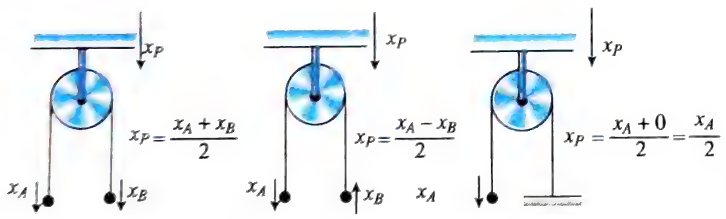
$$\Delta l_1 + \Delta l_2 + \Delta l_3 + \Delta l_4 = 0$$

$$(-x_A) + (-x_B) + (-x_B) + (x_C) = 0$$

$$2x_B = x_C - x_A \quad \Rightarrow \quad 2a_B = a_C - a_A$$

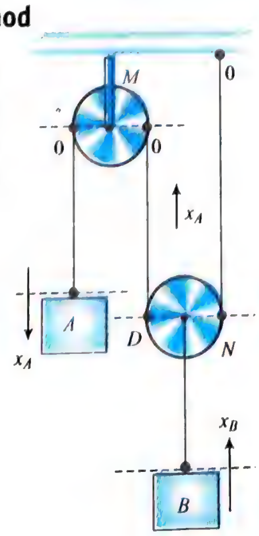
Shortcut Methods

In the cases, where pulley moves along with the blocks connected on both sides, we can say that the displacement of the pulley is the average of the displacement on both sides of the pulley.



If one end of the string is connected with the fixed end, the displacement of that end can be considered zero.

Analysis of case I using shortcut method



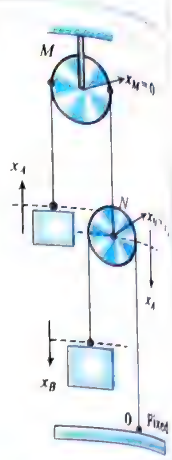
- As pulley M is fixed, the displacement should be zero. If the displacement of block A is x_A (down), then the displacement of other end should be x_A (up).

$$x_M = 0 = \frac{x_A + x_D}{2} \Rightarrow x_D = -x_A$$

- Displacement of block B = Displacement of pulley 2

$$x_B = \frac{x_A + 0}{2} \Rightarrow x_A = 2x_B$$

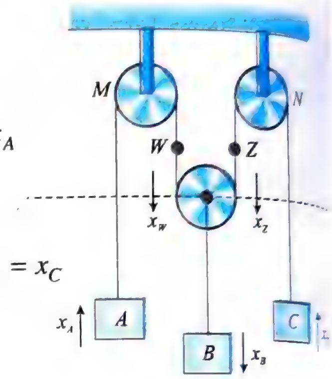
Analysis of case II using shortcut method



- As pulley M is fixed, $|x_{P,2}| = |x_A|$.
- If block A moves up by x_A, pulley N should move x_A in downward direction as shown in figure.
- For pulley 2,

$$x_{p,2} = x_A = \frac{x_B + 0}{2} \Rightarrow x_B = 2x_A$$

Analysis of case III using shortcut method



As pulley M is fixed, $x_M = 0$.

$$0 = \frac{x_A - x_W}{2} \Rightarrow x_W = x_A$$

Pulley N is also fixed.

$$x_N = 0 = \frac{x_C - x_Z}{2} \Rightarrow x_Z = x_C$$

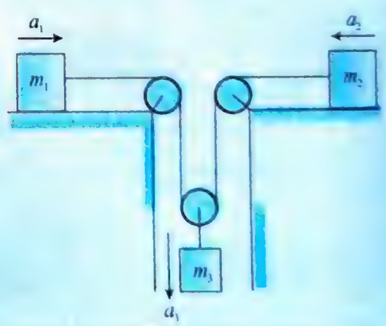
$$x_B = \frac{x_W + x_Z}{2}$$

$$2x_B = x_W + x_Z = x_A + x_C$$

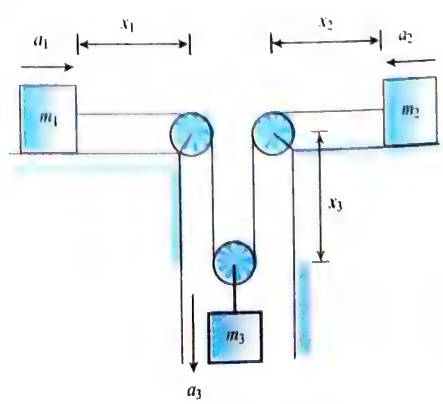
$$= 2a_B = a_C + a_A$$

ILLUSTRATION 6.46

In the arrangement of three blocks as shown in figure, the string is inextensible. If the directions of accelerations are as shown in the figure, then determine the constraint relation.



Sol. Let us assume the respective distance of each block as shown in figure. Since the length of the string is constant $x_1 + x_2 + 2x_3$ = constant.



On differentiating twice w.r.t. time, we get

$$\frac{d^2x_1}{dt}+\frac{d^2x_2}{dt^2}+2\frac{d^2x_3}{dt^2}=0$$

Since x_1 and x_2 are assumed to be decreasing with time,

$$\frac{d^2x_1}{dt^2}=-a_1 \text{ and } \frac{d^2x_2}{dt^2}=-a_2$$

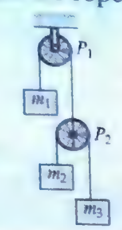
and x_3 is assumed to be increasing with time. Therefore,

$$\frac{d^2x_3}{dt^2}=+a_3$$

Thus, $-a_1-a_2+2a_3=0$ or $a_1+a_2=2a_3$

ILLUSTRATION 6.47

A pulley–rope–mass arrangement is shown in figure. Find the acceleration of block m_1 when the masses are set free to move. Assume that the pulley and the ropes are ideal.



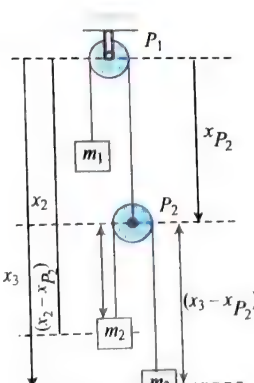
Sol. **Constraint relations. Method 1:**

For the upper string, the length of string l_1 not to change and for this string not to slacken, acceleration of m_1 w.r.t. the fixed pulley = acceleration of the movable pulley w.r.t. the fixed pulley

or $\quad |a_1|=|a_P|$. ...(i)

Constraint relations for string 2

Let us assume that block m_1 is moving up with acceleration a_1 and blocks m_2 and m_3 are moving down with acceleration a_2 and a_3, respectively w.r.t. ground.



Total length of string 2 is constant.

Therefore,

$$l_2=(x_2-x_{P_2})+(x_3-x_{P_2})+l_0'$$

where l_0 is the length of string on pulley 2 passing through the pulley P_2

$$l_2=x_2+x_3-2x_{P_2}+l_0' \quad \text{...(ii)}$$

Differentiating Eq. (ii) w.r.t. time,

$$\frac{dl_2}{dt}=0 \text{ (as total length of string is constant)}$$

$$\frac{dl_0'}{dt}=0 \text{ (as length of string over pulley is constant)}$$

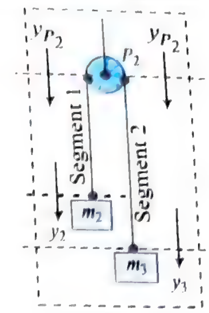
Differentiating Eq. (ii) twice w.r.t. time

$$2a_1=a_3+a_2 \quad \text{...(iii)}$$

Method 2: The acceleration of block m_1 and pulley 2 will be same in magnitude. Now considering pulley 2 and block 2 and 3.

Change in the length of segment (1)

$$\Delta l_1=(+y_2)+(-y_{P_2})=y_2-y_{P_2}$$



Considering string 2

Change in the length of segment (2),

$$\Delta l_2=(y_3)+(-y_{P_2})=y_3-y_{P_2}$$

Total sum of change in the segment length of string should be zero.

$$\Delta l=0=[y_2-y_{P_2}]+[y_2-y_{P_2}]$$

$$\Rightarrow \quad \Delta l=0=y_2+y_3-2y_{P_2}$$

$$\Rightarrow \quad 2y_{P_2}=y_3+y_2 \Rightarrow 2a_{P_2}=a_3+a_2$$

or $\quad 2a_1=a_3+a_2$

This equation is same as (iii) calculated in method 2.

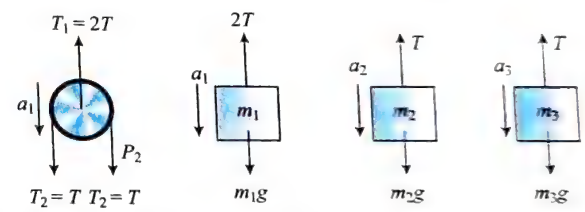
Method 3: For the rope length l_2 not to change and for this rope not to slacken acceleration of m_2 w.r.t. movable pulley = – acceleration of m_3 w.r.t. the movable pulley

or $\quad (a_2-a_P)=-(a_3-a_P)$

$$a_2+a_3-2a_P=0 \Rightarrow 2a_0=a_2+a_3$$

This equation is same as Eq. (iii) as calculated in method 1 and 2.

Applying Newton's laws of motion equations,



From free-body diagram (figure),

Equations of motion

For m_1: $2T-m_1g=m_1a_1$ ...(iv)

For m_2: $m_2g-T=m_2a_2$ ...(v)

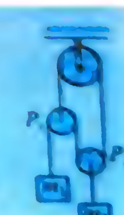
For m_3: $m_3g-T=m_3a_3$ ...(vi)

After solving (iii), (iv), (v), and (vi), we get

$$a_1=\frac{4m_2m_3g-m_1(m_2+m_3)g}{4m_2m_3+m_1(m_2+m_3)}$$

ILLUSTRATION 6.48

In the arrangement shown in figure, find the tensions in the rope and accelerations of the masses m_1 and m_2 and pulleys P_1 and P_2 when the system is set free to move. Assume the pulleys to be massless and strings are light and inextensible.



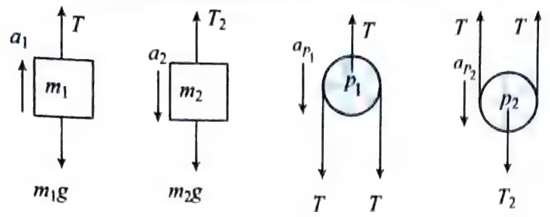
Sol. Applying the equations of motion for pulleys and blocks

For m_1 $\quad : T-m_1g=m_1a_1$ ...(i)

For m_2 $\quad : m_2g-T_2=m_2a_2$ ...(ii)

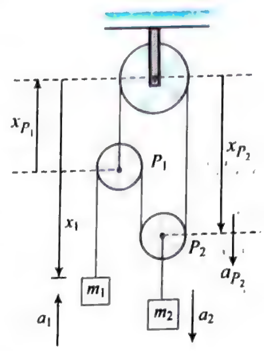
For pulley P_1 $\quad : 2T-T=0\times a_{P_1}$ ...(iii)

For pulley P_2 $\quad : T_2-2T=0\times a_{P_2}$ ...(iv)



From Eq (iii), $T = 0$; from (iv), $T_2 = 0$, and from (i) and (ii), $a_1 = -g$; also $a_2 = g$. Hence, blocks m_1 and m_2 both will move down with acceleration g downward direction.

Calculating a_{P_1} and a_{P_2}



Constraint relations: Consider the reference line and the position vectors of the pulleys and masses as shown in figure. Write the length of the rope in terms of position vectors and differentiate it to obtain the relations between accelerations of the masses and pulleys.

- For the length of the string connecting P_2 and m_2 not to change and for this rope not to slacken, $a_{P_2} = a_2 = g$.
- Length of the string connecting P_1 to m_1 not to change and for this rope not to slacken:

$$l = (x_1 - x_{P_1}) + x_{P_1} + (x_{P_2} - x_{P_1}) + x_{P_2}$$

$$= x_1 - x_{P_1} + x_{P_2}$$

Differentiating this equation w.r.t. t twice, we get

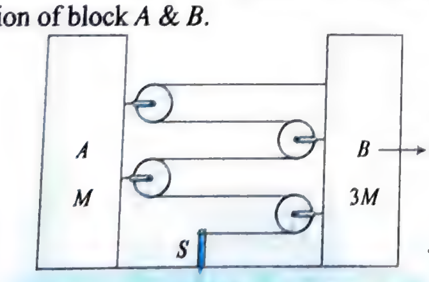
$$\frac{d^2 l}{dt^2} = \frac{d^2 x_1}{dt^2} - \frac{d^2 x_{P_1}}{dt^2} + 2\frac{d^2 x_{P_2}}{dt^2}$$

$$\Rightarrow \quad 0 = a_1 - a_{P_1} - 2a_{P_2}$$

$$\Rightarrow \quad 0 = g - a_{P_1} + 2g \Rightarrow a_{P_1} = 3g$$

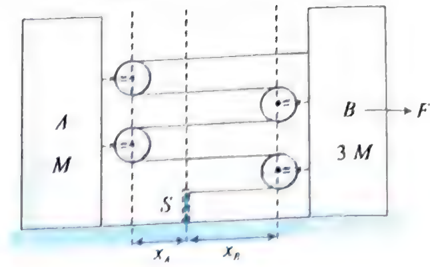
ILLUSTRATION 6.49

Two blocks A and B of mass M & $3M$ are connected through a light string. One end of the string is connected to the block B and its other end is connected to a fixed point S as shown in figure. Now a force F is applied to block B. Find the acceleration of block A & B.



Sol.

Method 1: Taking reference line through support S, let x_A and x_B are the distances of blocks A and B, respectively, from S. The total length of the string, $l = 4x_A + 5x_B + l_0$, where l_0 is some part of string which is over pulley and somewhere else which remains constant.



Differentiating l w.r.t. time, we get

$$\frac{dl}{dt} = \frac{d}{dt}(4x_A + 5x_B + l_0)$$

or $$0 = 4\frac{dx_A}{dt} + 5\frac{dx_B}{dt} \Rightarrow \frac{dx_A}{dt} = \frac{5}{4}\frac{dx_B}{dt}$$

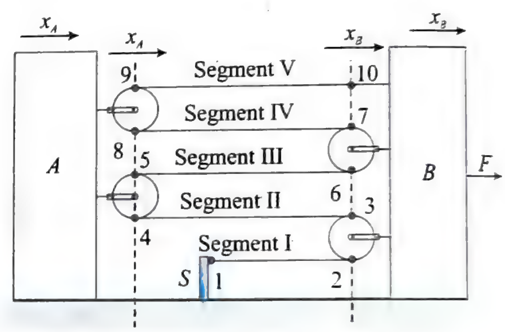
$$\frac{dx_A}{dt} = -v_A \text{ (-because } x_A \text{ is decreasing with time)}$$

$$\frac{dx_B}{dt} = v_B \text{ (+ because } x_B \text{ is increasing with time)}$$

or $$v_A = \frac{5}{4}v_B; \text{ also } a_A = \frac{5}{4}a_B$$

Method 2

Charge in the length of segment I, $\Delta l_{1,2} = 0 + (x_B)$



Segment II

$$\Delta l_{3,4} = (-x_A) + (x_B)$$

Segment III

$$\Delta l_{5,6} = (-x_A) + (x_B)$$

Segment IV

$$\Delta l_{7,8} = (-x_A) + (x_B)$$

Segment V

$$\Delta l_{9,10} = (-x_A) + (x_B)$$

Total charge in segment lengths should be zero. Therefore,

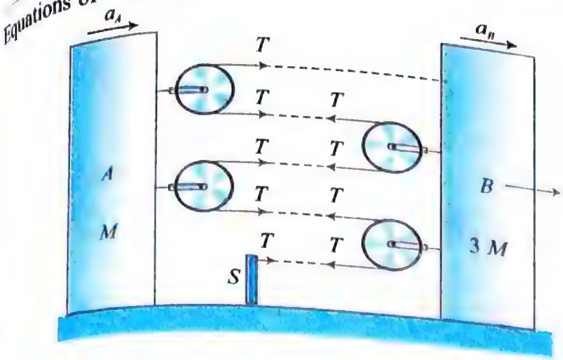
$$\Delta l = \Delta l_{1,2} + \Delta l_{3,4} + \Delta l_{5,6} + \Delta l_{7,8} + \Delta l_{9,10} = 0$$

$$= x_B + [-x_A + x_B] + [-x_A + x_B]$$

$$+ [-x_A + x_B] + [-x_A + x_B] = 0$$

$$= -4x_A + 5x_B = 0$$

or $$x_A = \frac{5}{4}x_B \Rightarrow v_A = \frac{5}{4}v_B \Rightarrow a_A = \frac{5}{4}a_B$$



For A:

$$4T = m_A a_A = Ma_A \quad ...(i)$$

For B:

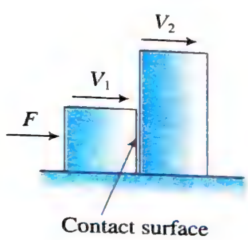
$$F - 5T = m_B a_B = 3Ma_B \quad ...(ii)$$

$$a_B = \frac{16F}{73M}$$

and $a_A = \frac{5}{4} \times \left(\frac{16F}{73M}\right) = \frac{20F}{73M}$

Wedge Constraint: Normal Constraint

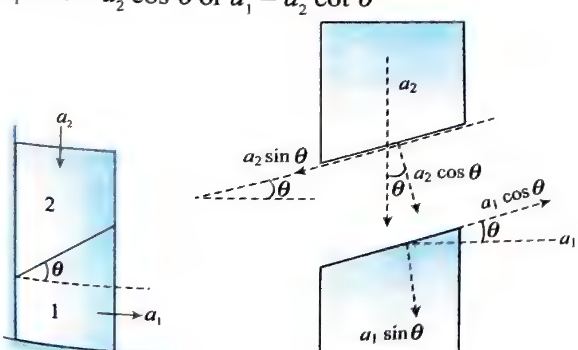
Consider two blocks moving on a surface and always remaining in contact. In order to maintain the contact component of velocity the vector perpendicular to the contact surface must be same, i.e., $\vec{v}_1 = \vec{v}_2$.



Similarly, $\vec{a}_1 = \vec{a}_2$

Now consider two wedges (1) and (2) are placed in contact. Wedge (1) is moving towards right with acceleration a_1 while wedge (2) is moving down with acceleration a_2. If wedges (1) and (2) are to remain in contact, the component of acceleration perpendicular to the contact surface must be zero.

$$a_1 \sin\theta = a_2 \cos\theta \text{ or } a_1 = a_2 \cot\theta$$



(a) (b)

Now finally consider a rod (1) and wedge (2) are arranged as shown in figure. The rod moves down with acceleration a_1 and wedge slides on the inclined with acceleration a_2. As the rod and the wedge always remain in contact with each other, then the acceleration of the rod should be equal to the component of acceleration of the wedge perpendicular to the surface of contact, i.e., $a_1 = a_2 \sin\theta$.

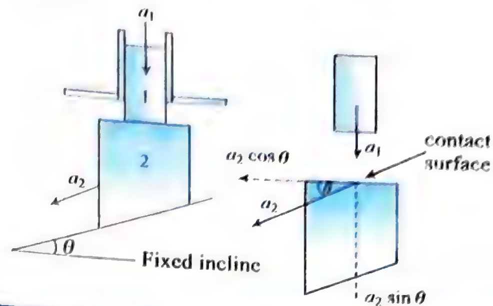
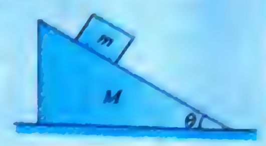


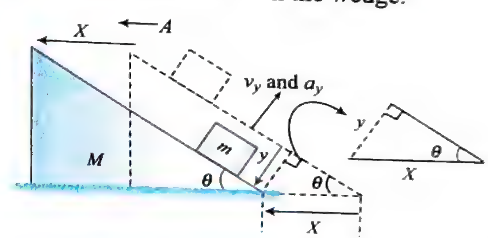
ILLUSTRATION 6.50

A block of mass m is placed on the inclined surface of a wedge as shown in figure. Calculate the acceleration of the wedge and the block when the block is released. Assume all surfaces are frictionless.



Sol. Constraint relation. Approach 1

We can observe that the wedge M can only move in horizontal direction towards left, and the block m can slide on inclined surface of M always in contact with the wedge.



- Let us defined our x and y axes parallel to the incline and perpendicular to incline, respectively.
- We can observe that the displacement of m and M in $-x'$ direction will be same as the block never lose contact with the wedge.
- If the wedge moves in the horizontal direction by a distance x', during this time, the block will move x in x' direction.
- We can relate these displacements x and X as

$$\frac{y}{X} = \sin\theta \Rightarrow y = X\sin\theta \quad ...(i)$$

Hence, velocity relation can be written as:

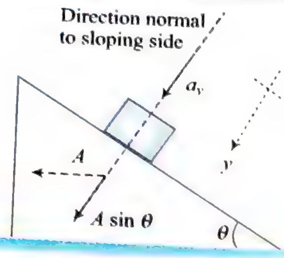
$$v_y = V\sin\theta \quad ...(ii)$$

and acceleration relation can be written as:

$$a_y = A\sin\theta \quad ...(iii)$$

Here v_y and a_y are the velocity and acceleration of the block, respectively, in the direction perpendicular to inclined surface.

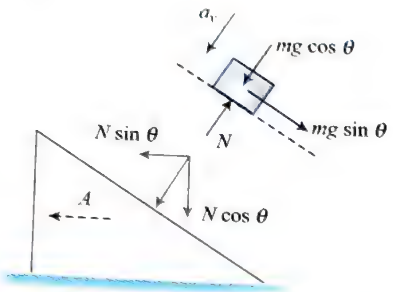
Approach 2: We consider the motion of the block parallel to incline and perpendicular to inclined surface. Let the components of acceleration of block with respect to ground along these direction are a_x and a_y, respectively.



Then we can write $a_y = A \sin \theta$.

Sol. For wedge:

$$N \sin \theta = MA \quad \text{...(i)}$$



For block: considering the block in the direction perpendicular to sloping surface.

$$mg \cos \theta - N = ma_y$$

But $a_y = A \sin \theta$

Hence,

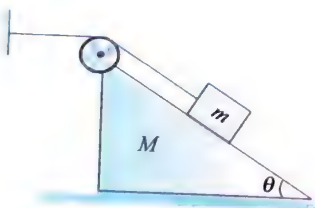
$$mg \cos \theta - N = mA \sin \theta \quad \text{...(ii)}$$

From (i) and (ii), we get

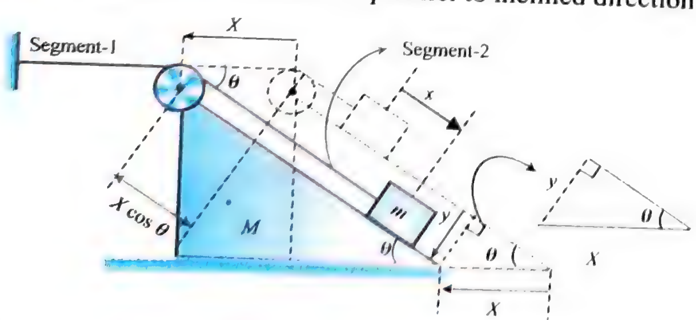
$$A = \frac{mg \sin \theta \cos \theta}{M + m \sin^2 \theta}$$

PULLEY AND WEDGE CONSTRAINT

Consider a block of mass m placed on the inclined surface of the wedge. The block is connected with a string and arranged as shown in figure. The system is released from rest.



We can divide the string in two segments, (1) and (2). let at any interval of time the wedge moves a distance X towards right and the block moves a distance x parallel to inclined direction.



From the diagrams it is clear that the length of segment 1 will decrease by X while the length of segment 2 will increase by $(x + X \cos \theta)$. As the overall length of the string is constant, then we can write

$$-X + (x + X \cos \theta) = 0$$

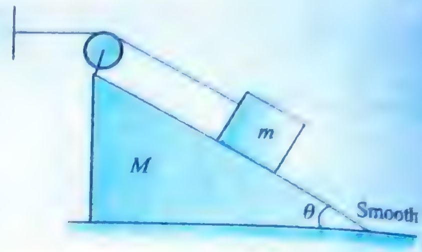
$$x = X(1 - \cos \theta) \text{ and } a_x = A(1 - \cos \theta)$$

From wedge constant, it is clear that

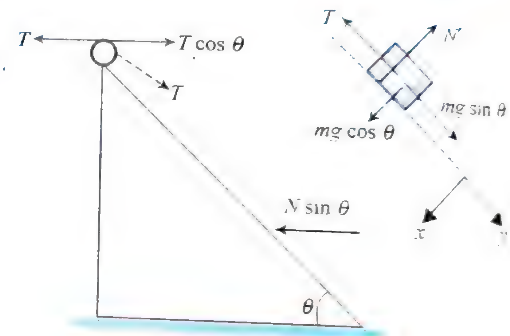
$$\frac{y}{x} = \sin \theta \Rightarrow y = X \sin \theta \text{ and } a_y = A \sin \theta$$

ILLUSTRATION 6.51

The mass of wedge, shown in figure, is M and that of block is m. Neglecting friction at all the places and mass of the pulley, calculate the acceleration of wedge. Thread is inextensible.



Sol. **Constraint relation:**



It is clear from above discussion.

$$a_x = A(1 - \cos \theta)$$

and $\quad a_y = A \sin \theta$

Equations of motion

Considering the motion of wedge in horizontal direction

For wedge:

$$T - T \cos \theta - N \sin \theta = MA$$

Considering the motion of block parallel and perpendicular the sloping side in horizontal direction.

For block: In the direction parallel to inclined surface.

$$mg \sin \theta - T = ma_x = MA(1 - \cos \theta)$$

or $\quad T = mg \sin \theta - MA(1 - \cos \theta)$

In the direction perpendicular to inclined plane.

$$mg \cos \theta - N = ma_y = mA \sin \theta$$

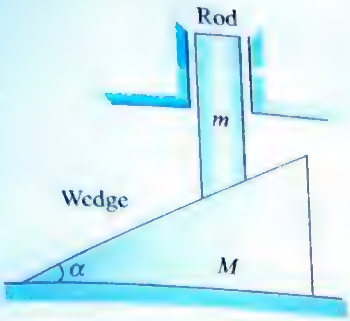
$$N = mg \cos \theta - mA \sin \theta \quad \text{...(iii)}$$

Substituting the value of N from (ii) and (iii) in (i), we get

$$A = \frac{mg \sin \theta}{M + 2m(1 - \cos \theta)}$$

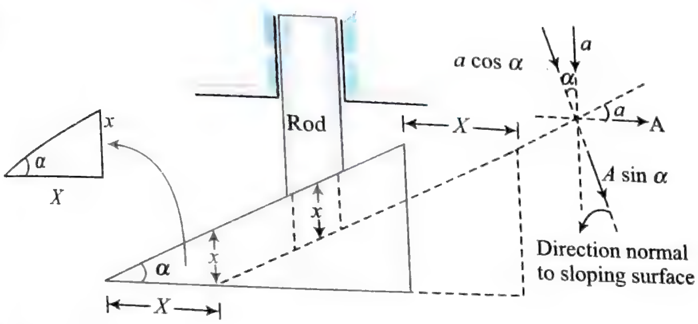
ILLUSTRATION 6.52

A rod of mass m is supported on a wedge of mass M shown in figure. Find the accelerations of rod a and wedge A in the arrangement. The friction between all contact surfaces is negligible.



Sol. The rod is constrained to move in the vertical direction (with the help of the guides) and the wedge will move along the surface in the horizontal direction. Initially, the system is held at rest.

Constraint relation: Approach 1 Let the acceleration of m w.r.t. ground be a vertically downwards and acceleration of M w.r.t. ground be A horizontally towards right.



The motion of the system is constrained by the fact that the "bottom face of the rod must always be in contact with the inclined plane." If in time t, X is the displacement of the wedge and x is the displacement of the rod, then the constraint demands that

From figure, we have $\dfrac{x}{X} = \tan \alpha$

$\therefore \quad x = X \tan \alpha$ (i)

Here α remains constant.

Differentiating (i) w.r.t. t twice, we get

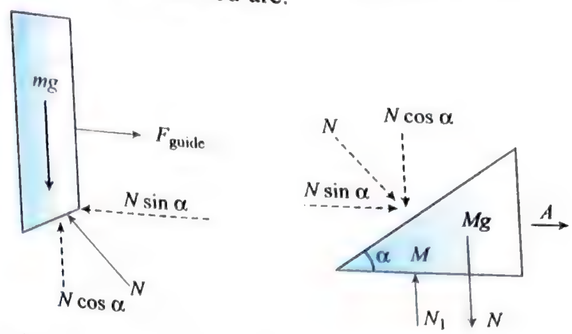
$$\left(\frac{d^2 x}{dt^2}\right) = \left(\frac{d^2 X}{dt^2}\right) \tan \alpha \qquad [\tan \alpha = \text{constant}]$$

Hence, $a = A \tan \alpha$

Approach 2: The fact that the rod (or a particle on the wedge) and the wedge must not lose contact is usually called *wedge constraints*. For this, the component of the acceleration of the rod perpendicular to the wedge plane = component of acceleration of the wedge perpendicular to wedge plane.

$$a \cos \alpha = A \sin \alpha \quad \Rightarrow \quad a = A \tan \alpha$$

The force acting on the rod are:



- The weight mg, vertically downwards
- The normal force N, normal to the bottom surface of the rod
- The force (F_{guide}) exerted by the guide to nullify the horizontal component of N as for the rod $a_{Horizontal} = 0$.

Motion in vertical direction

$$mg - N \cos \alpha = ma \qquad \text{...(ii)}$$

and the forces acting on the wedge are

- The weight, Mg
- N, reaction of N acting on the rod
- N_1, normal force by the surface

The force equations are

$$N_1 - Mg - N \cos \alpha = 0 \qquad \text{(iii)}$$

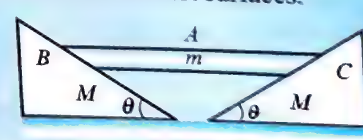
and $\quad N \sin \alpha = MA \qquad$ (iv)

Solving the above equations for a and A, we get

$$a = \frac{mg \tan \alpha}{m \tan \alpha + M \cot \alpha} \quad \text{and} \quad A = \frac{mg}{m \tan \alpha + M \cot \alpha}$$

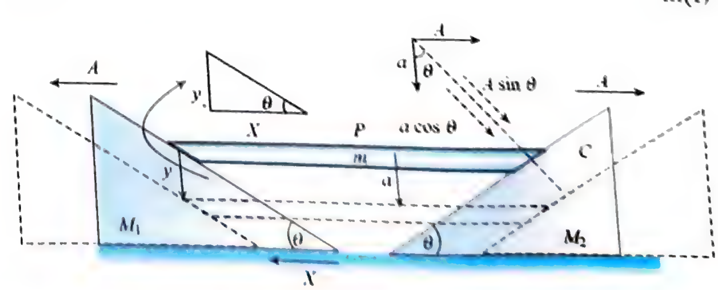
ILLUSTRATION 6.53

A plank of mass m rests symmetrically on two wedges B and C of mass M. What is the acceleration of the plank? Neglect friction between all the contact surfaces.



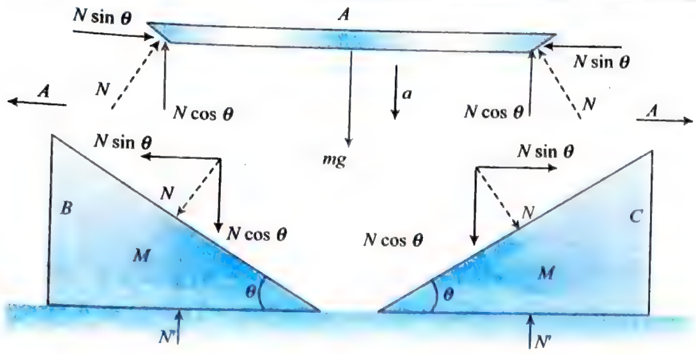
Sol. When the plank is released, let it fall through a distance y and let both the wedges move through a distance x.

$$y = x \tan \theta \qquad \text{...(i)}$$



On differentiating this expression twice, we obtain $a = A \tan \theta$

FBD of rod and wedge



Equations of wedge:

$$\Sigma F_x = N \sin\theta = MA \quad \text{...(ii)}$$

$$\Sigma F_y = N' - N\cos\theta - Mg = 0 \quad \text{...(iii)}$$

Equations of plank:

$$\Sigma F_x = N\sin\theta - N\sin\theta = 0 \quad \text{...(iv)}$$

$$\Sigma F_y = mg - 2N\cos\theta = ma \quad \text{...(v)}$$

From (ii), $N = \dfrac{MA}{\sin\theta}$

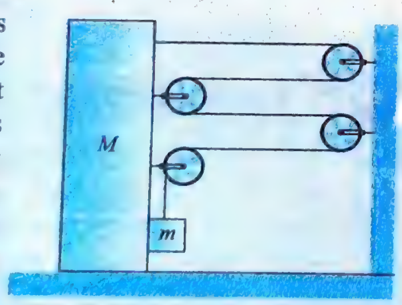
On substituting expression for N and a in (v), we obtain

$$mg - \frac{2MA\cos\theta}{\sin\theta} = \frac{mA\sin\theta}{\cos\theta}$$

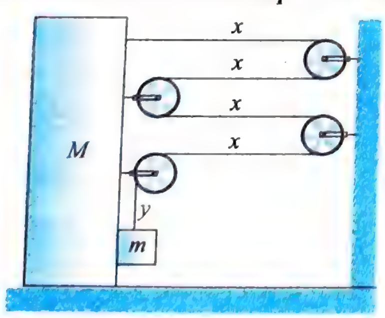
$$\Rightarrow \quad A = \frac{mg\sin\theta\cos\theta}{m\sin^2\theta + 2M\cos^2\theta}$$

ILLUSTRATION 6.54

A block of mass M is connected with a particle of mass m by a light inextensible string as shown in figure. Assuming all contacting surfaces as smooth, find the acceleration of the wedge after releasing the system.



Sol. From the constraint relationship shown in figure,



$$y + 4x = l$$

$$\Rightarrow \quad \frac{dy}{dt} + \frac{4dx}{dt} = 0$$

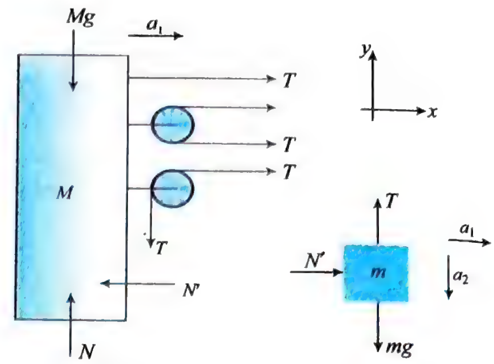
$$\Rightarrow \quad \frac{d^2y}{dt^2} + \frac{4d^2x}{dt^2} = 0$$

$$\Rightarrow \quad a_2 = 4a_1 \text{ (numerically)} \quad \text{...(i)}$$

where a_1 is the acceleration of M and a_2 is the acceleration of m with respect to M.

Equation for M

$$\Sigma F_x = Ma_1 \Rightarrow 4T - N' = Ma_1 \quad \text{...(ii)}$$



F.B.D. of M F.B.D. of m

For m, $\Sigma F_y = ma_2 \Rightarrow mg - T = ma_2$ ...(iii)

$$\Sigma F_x = ma_1 \Rightarrow N' = ma_1 \quad \text{...(iv)}$$

as m is always in contact with M. Hence, the acceleration of m in horizontal direction should be same as the acceleration of M

$$\text{(ii)} + \text{(iv)} \Rightarrow a_1 = \frac{4T}{M+m} \quad \text{...(v)}$$

Putting $a_2 = 4a_1$ from (i) and $T = \dfrac{(M+m)a_1}{4}$

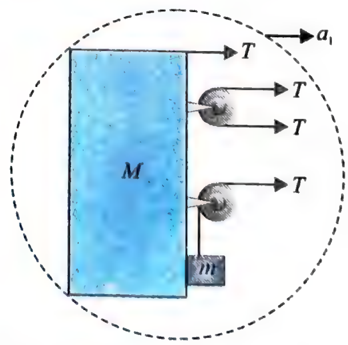
From (v) in (iii), we obtain

$$mg - \frac{(M+m)a_1}{4} = m(4a_1)$$

$$\Rightarrow \quad a_1 = \frac{4\,mg}{M+17\,m}$$

Approach 2:

If we take 'block (M) and particle (m)' as system, then the system has acceleration a_1 in horizontal direction. The external force on the system in horizontal direction is $4T$. Now consider the system in horizontal direction.



We can write equation of motion for system

$$4T = (M+m)a_1 \quad \text{...(vi)}$$

Now consider the motion of the particle (m) in vertical direction.

Equation of motion of m : $mg - T = ma_2$

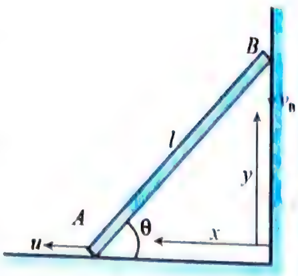
or $\quad mg - T = m(4a_1)$ ...(vi) [As $a_2 = 4a_1$]

Now on solving equations (vi) and (vii), we get $a_1 = \dfrac{4mg}{(M+17m)}$

GENERAL CONSTRAINTS

ILLUSTRATION 6.55

Figure shows a rod of length l resting on a wall and the floor. Its lower end A is pulled towards left with a constant velocity u. As a result of this, end A starts moving down along the wall. Find the velocity of the other end B downward when the rod makes an angle θ with the horizontal.



Sol. Let us first find the relation between the two displacements. Then differentiate with respect to time. Here if the distance from the corner to the point A is x and that up to B is y, the left velocity of point A can be given as $v_A = dx/dt$ and that of B can be given as $v_B = -dy/dt$ (negative sign indicates y decreasing).

If we relate x and y: $x^2 + y^2 = l^2$

Differentiating with respect to t, $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$$\Rightarrow \quad xv_A = yv_B \Rightarrow xu = yv_B \Rightarrow v_B = u\frac{x}{y} = u\cot\theta$$

Alternatively: In cases where distance between two points is always fixed, we can say the relative velocity of one point of an object with respect to any other point of the same object in the direction of the line joining them will always remain zero, as their separation always remains constant.

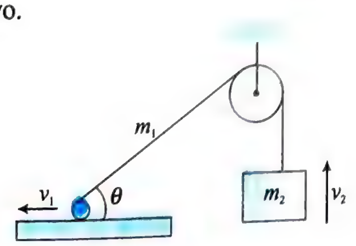
Here, in the above example, the distance between the points A and B of the rod always remains constant; thus, the two points must have the same velocity components in the direction of their line joining, i.e., along the length of the rod.

If point B is moving down with velocity v_B, its component along the length of the rod is $v_B \sin\theta$. Similarly, the velocity component of point A along the length of rod is $u\cos\theta$. Thus, we have

$$v_B \sin\theta = u\cos\theta \text{ or } v_B = u\cot\theta.$$

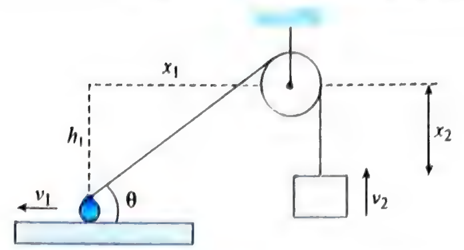
ILLUSTRATION 6.56

In figure, a ball of mass m_1 and a block of mass m_2 are joined together with an inextensible string. The ball can slide on a smooth horizontal surface. If v_1 and v_2 are the respective speeds of the ball and the block, then determine the constraint relation between the two.



Sol.

Method 1: Distances are assumed from the center of the pulley as shown in figure.



Constraint: Length of the string remains constant.

$$\sqrt{x_1^2 + h_1^2} + x_2 = \text{constant}$$

Differentiating both the sides w.r.t. time, we get

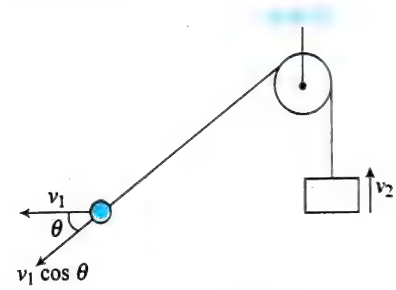
$$\frac{2x_1}{2\sqrt{x_1^2 + h_1^2}}\frac{dx_1}{dt} + \frac{dx_2}{dt} = 0$$

Since the ball moves so as to increase x_1 with time and block moves so as to decrease x_2 with time,

$$\frac{dx_1}{dt} = +v_1 \quad \text{and} \quad \frac{dx_2}{dt} = -v_2$$

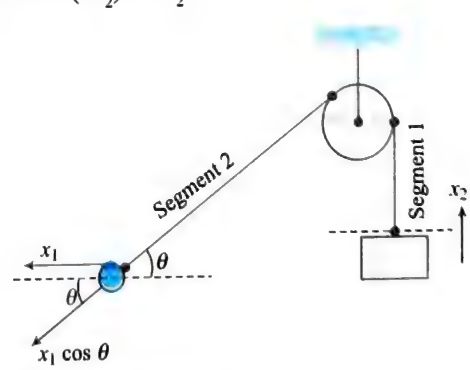
Also $\dfrac{x_1}{\sqrt{x_1^2 + h_1^2}} = \cos\theta$ or $v_2 = v_1 \cos\theta$

Method 2: The problem can be solved very easily if we look at the problem from a different viewpoint and identify a different constraint, i.e., the velocity of any two points along the string is same. Obviously, from figure, we have $v_1 \cos\theta = v_2$.



Method 3: Change in length of segment I,

$$\Delta l_1 = 0 + (-x_2) = -x_2$$



Change in length of segment II,

$$\Delta l_2 = (x_1 \cos\theta) + 0 = x_1 \cos\theta$$

Total change in the length of all segments should be zero, as the length of string is constant.

$$\Delta l = \Delta l_1 + \Delta l_2$$

$$= -x_2 + x_1 \cos\theta = 0$$

$$\Rightarrow \quad x_2 = x_1 \cos\theta \text{ or } v_2 = v_1 \cos\theta$$

ILLUSTRATION 6.57

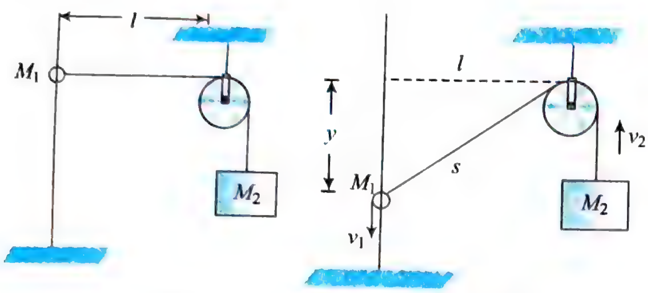
A ring M_1 and a block M_2 are held in the position shown in figure (a). Now the system is released. If $M_1 > M_2$, find V_1/V_2 when the ring m_1 slides down along the smooth fixed vertical rod by the distance h.

Sol.

Method 1: Let ring has moved down a distance y. From figure (b), we have

$$y^2 + l^2 = s^2 \quad \text{...(i)}$$

Here l is the length of string which is constant. Differentiating (i) w.r.t. time, we get



(a) (b)

$$2y\frac{dy}{dt}+0=2s\frac{ds}{dt} \quad \text{or} \quad \frac{dy}{dt}=\frac{s}{y}\cdot\frac{ds}{dt} \quad \text{...(ii)}$$

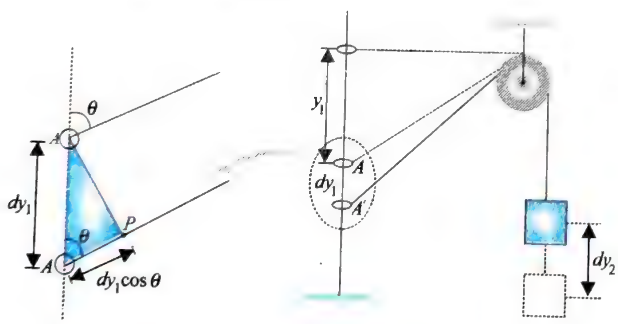
Here $\frac{dy}{dt}=v_1$ and $\frac{ds}{dt}=v_2$.

For $y = h$, $s = \sqrt{h^2+l^2}$

Therefore, Eq. (ii) takes the form

$$v_1=\frac{\sqrt{h^2+l^2}}{h}\cdot v_2 \quad \text{or} \quad \frac{v_1}{v_2}=\frac{\sqrt{h^2+l^2}}{h}$$

Method 2: Let in time t the ring falls to point A which is a distance y_1 below the ring's initial position.



At this position, the string makes an angle θ with the rod in small time interval dt, the ring moves down to dy_1 and the block rises by dy_2. Let us draw a perpendicular AP form A to AB.

For very small displacement, $AB \approx PB$.

As the length of the string is constant

$\therefore \quad A'P = dy_2$

But $A'P = dy_1 \cos\theta$

$\therefore \quad dy_1 \cos\theta = dy_2$

Dividing both sides by dt, we get $\left(\frac{dy_1}{dt}\right)\cos\theta=\left(\frac{dy_2}{dt}\right)$.

Here $\frac{dy_1}{dt}=V_1$, the rate of change of position of ring with time

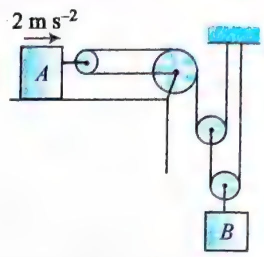
and $\frac{dy_2}{dt}=V_2$, the rate of change position of block with time.

Thus, we have $V_1 \cos\theta = V_2$

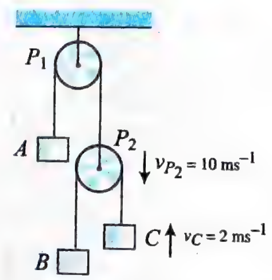
or $$\frac{V_1}{V_2}=\frac{1}{\cos\theta}=\frac{1}{\frac{h}{\sqrt{h^2+l^2}}}=\frac{\sqrt{h^2+l^2}}{h}$$

CONCEPT APPLICATION EXERCISE 6.6

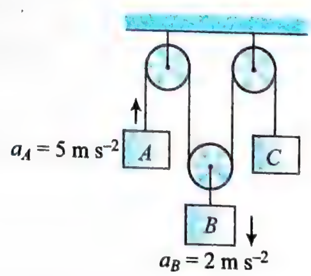
1. In the given figure, find the acceleration of B, if the acceleration of A is 2 m s^{-2}.



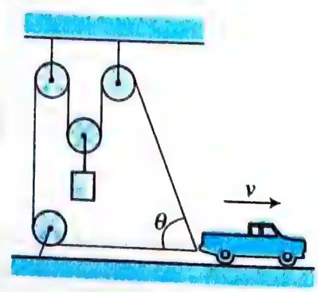
2. The three blocks shown in figure move with constant velocities. Find the velocity of blocks A and B. Given $V_{P2} = 10\text{ ms}^{-1}\downarrow$, $V_C = 2\text{ ms}^{-1}\uparrow$.



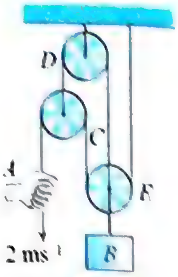
3. For the system as shown in figure, find the acceleration of C. The accelerations of A and B with respect to ground are marked.



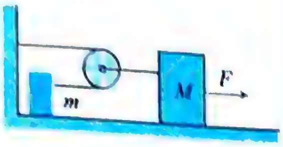
4. In the figure shown, the speed of the truck is v to the right. Find the speed with which the block is moving up at $\theta = 60°$.



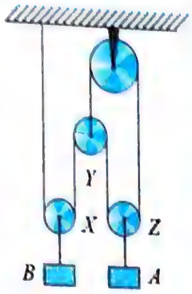
5. Determine the speed with which block B rises in figure if the end of the chord at A is pulled down with a speed of $2\ \text{ms}^{-1}$.



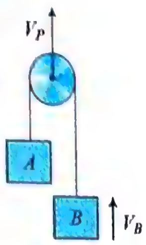
6. Find the acceleration of blocks in the given figure. The pulley and the strings are massless.



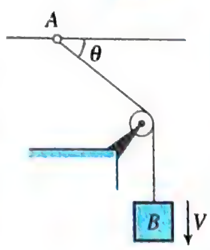
7. The given figure shows a system of four pulleys with two masses A and B. Find, at an instant
   (a) Speed of block A when block B is going up at $1\ \text{ms}^{-1}$ and pulley Y is going up at $2\ \text{ms}^{-1}$.
   (b) Acceleration of block A if block B is going up at $3\ \text{ms}^{-2}$ and pulley Y is going down at $1\ \text{ms}^{-2}$.



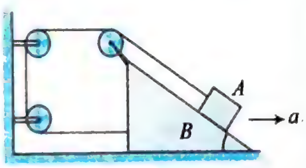
8. Figure shows a pulley over which is string passes and connected to two masses A and B. Pulley moves up with a velocity V_P and mass B is also going up at a velocity V_B. Find the velocity of mass A if
   (a) $V_P = 5\ \text{ms}^{-1}$ and $V_B = 10\ \text{ms}^{-1}$
   (b) $V_P = 5\ \text{ms}^{-1}$ and $V_B = -20\ \text{ms}^{-1}$



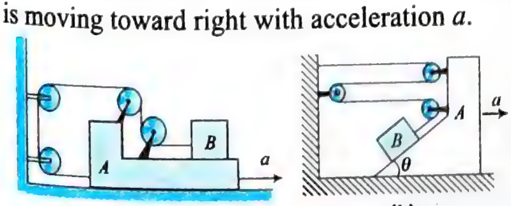
9. A ring A which can slide on a smooth wire is connected to one end of a string as shown in figure. Other end of the string is connected to a hanging mass B. Find the speed of the ring when the string makes an angle θ with the wire and mass B is going down with a velocity v.



10. The given figure shows block A constrained to slide along the inclined plane of the wedge B shown. Block A is attached with a string which passes through three ideal pulleys and connected to the wedge B. If wedge is pulled toward right with an acceleration a, find
   (a) the acceleration of the block with respect to wedge.
   (b) the acceleration of the block with respect to ground.

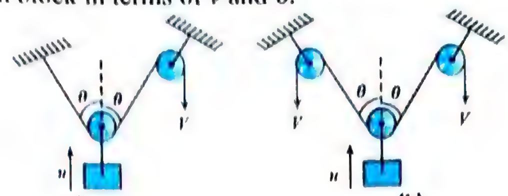


11. Find the acceleration of block B as shown in figures (a) and (b) relative to block A and relative to ground if block A is moving toward right with acceleration a.



(a) (b)

12. If the string is inextensible, determine the velocity u of each block in terms of v and θ.

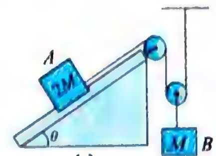


(a) (b)

(a) Fig. (a): $u = \ldots$ (b) Fig. (b): $u = \ldots$

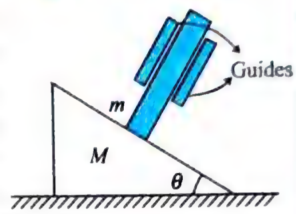
13. Calculate the accelerations of blocks A and B in cases (a), (b), and (c).

(a) (b)

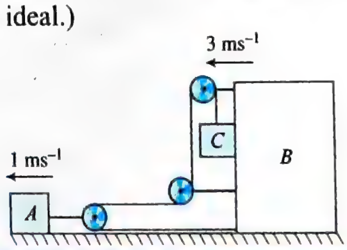


(c)

14. A rod of mass m is constrained to move along the guide and always in contact with the wedge of mass M as shown in figure. Assume no friction anywhere calculate the acceleration of the wedge and rod.



15. The velocities of A and B are shown in figure. Find the speed (in ms^{-1}) of block C. (Assume that the pulleys and string are ideal.)



ANSWERS

1. $1\ \text{ms}^{-2}$ **2.** $v_A = 10\ \text{ms}^{-1}, v_B = 22\ \text{ms}^{-1}$ **3.** $1\ \text{ms}^{-2}$

4. $v_{\text{block}} = \dfrac{3}{4} v_{\text{truck}}$ **5.** $0.5\ \text{ms}^{-1}$

6. $A = \dfrac{F}{M+4m}, a = \dfrac{2F}{(M+4m)}$

7. (a) 0 **(b)** $a_1 = 7/2\ \text{ms}^{-2}$

8. (a) $V_A = 0$ **(b)** $V_A = 30\ \text{ms}^{-1}$ **9.** $v \sec\theta$

10. (a) $a_A = 2a$ **(b)** $a_1 = a\sqrt{5-4\cos\theta}$

11. (a) (i) $a_{BA} = 2a$, **(ii)** $a_B = -a$
(b) (i) $a_{BA} = 3a$, **(ii)** $a_{BG} = a\sqrt{10+6\cos\theta}$

12. (a) $v/(2\cos\theta)$, $v/\cos\theta$

13. (a) $g/3$ **(b)** $\dfrac{g[\sin\theta - 1]}{3}$ **(c)** $\dfrac{g[4\sin\theta - 1]}{9}$

14. $a_w = \dfrac{mg\sin\theta\cos\theta}{M+m\sin^2\theta}, a_r = \dfrac{mg\sin^2\theta\cos\theta}{M+m\sin^2\theta}$

15. $5\ \text{ms}^{-1}$

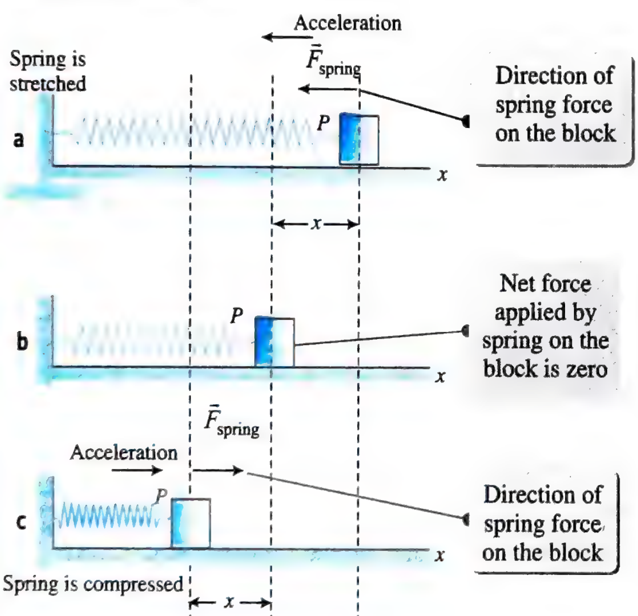
SPRING FORCE AND COMBINATIONS OF SPRINGS

In mechanics, we come across the system of bodies connected with springs. For this purpose, we need to develop the ideas how exactly a spring responds to an external force applied on it.

First of all, for the sake of simplicity, let us consider a light (massless) spring. Then fix one of its end and pull the other end slowly through a distance x (say). When we go on pulling the end P of the spring towards right, the spring pulls us towards left with a gradually increasing force F_s. Likewise, when we go on pushing the end P towards left, the spring will push us towards right with a gradually increasing force F_s.

From the above simple experiment, we can easily understood that

- the force offered by the spring, that is, "spring force" F_s points (acts) opposite to the displacement of the force end of the spring.
- the amount of springs force increase linearly with the deformation (compression or elongation) of the spring; when we plot the variation of F_s versus x, we obtain a straight line up to certain (limited) value of x, which is known as elastic limit.

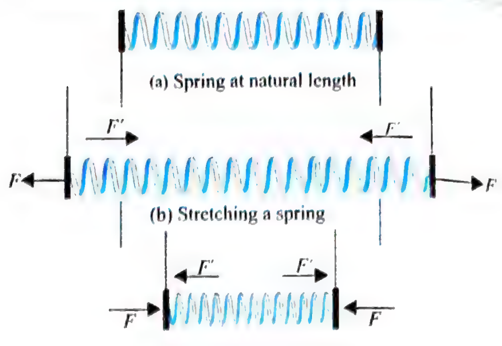


Robert Hooke experimented in seventh century who put forth all the above facts in a single vector formula in scalar form.

Force $\propto$ Stretch (or compression), i.e., $F = kx$ ...(i)

i.e., restoring force is linear. This force in a spring is not constant and depends on stretch (or compression) x. Greater the stretch (or compression), greater will be the force and vice-versa.

k is called the force constant of the spring and is equal to the slope of force versus–stretch curve. It has dimensions $[F/L] = [MT^{-2}]$ and units N m^{-1}. Greater the force constant of a spring, lesser will be the stretch (or compression) for a given force and more stiffer is said to be the spring. The force constant k of a spring depends on wire (its length, radius r, and material) used to make the spring, radius of spring R, and length l of spring.



(a) Spring at natural length

(b) Stretching a spring

(c) Compressing a spring

To produce extension or compression in a spring, two equal and opposite forces (F) are to be applied and in equilibrium restoring force (F') developed due to the elasticity of spring is equal to either force, i.e.,

$F = F'$ and always opposite to applied force.

For small stretch or compression, springs obey Hooke's law.

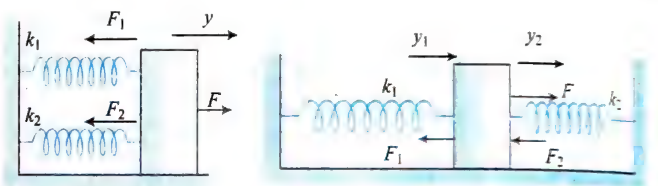
FORCE CONSTANT OF COMPOSITE SPRINGS

If a number of springs are connected to a body and we want to produce it to a single spring, following three cases of common interest are possible.

SPRINGS IN PARALLEL

This situation is shown in figure. If the force F pulls the mass m by y, the stretch in each spring will be y,

i.e., $y_1 = y_2 = y$ ...(i)



Now as for a spring $F = ky$ and as force constants are not equal, so $F_1 \neq F_2$, but for equilibrium,

$F = F_1 + F_2$, i.e., $ky = k_1y_1 + k_2y_2$ [as $F = ky$]

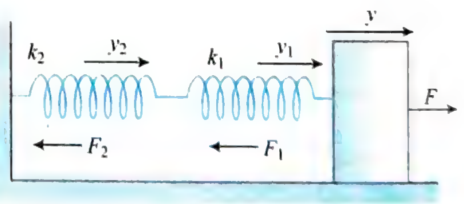
which in the light of (i) reduces to

$k = k_1 + k_2 + \cdots$

This is like capacitors in parallel or resistance in series.

SPRINGS IN SERIES

This situation is shown in figure.



As springs are massless, force in these must be same, i.e.,

$F_1 = F_2 = F$ ...(i)

Now as $F = ky$ and force constants are not equal, stretches will not be equal, i.e., $y \neq y_2$.

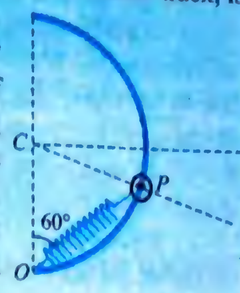
But, $y = y_1 + y_2$ or $\frac{F}{k} = \frac{F_1}{k_1} + \frac{F_2}{k_2}$

[as for $F = ky$, $y = (F/k)$]

which in the light of Eq. (i) reduces to $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$

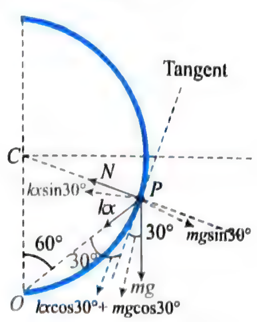
ILLUSTRATION 6.58

A smooth semicircular wire track of radius R is fixed in a vertical plane (see figure). One end of a massless spring of natural length $(3/4)\,R$ is attached to the lowest point O of the wire track. A small ring of mass m, which can slide on the track, is attached to the other end of the spring. The ring is held stationary of point P such that the spring makes an angle of $60°$ with the vertical. The spring constant $k = (mg/R)$. Consider the instant when the ring is released. Determine the tangential acceleration of the ring and the normal reaction between ring and track.



Sol.

1. The free-body diagram of the ring is shown in figure. The forces acting on the ring are:



(a) The weight mg acting vertically downwards

(b) Normal force N by the wire track.

Normal force on the ring could be either radially outwards or radially inwards depending on whether the ring presses against the inner surface or outer surface of the track. To ascertain whether normal force is inwards or outwards, assume that, to begin with, it is inwards. Then from $\sum \vec{F} = m\vec{a}$, find the value of normal force. If it is positive. it is inwards and if it is negative, it is outwards.

(c) Force of the spring kx. In the given physical situation, the spring is extended, it will pull the ring. So the spring force kx is along the spring towards O.

2. Length of the spring in the position shown $= R$. ($CP = CO = R$; $\angle COP = \angle OPC = 60°$; $\angle COP$ is equilateral)

Change in length of the spring $= R - \left(\dfrac{3}{4}\right)R = \left(\dfrac{R}{4}\right)$

$$kx = \left(\frac{mg}{R}\right)\left(\frac{R}{4}\right) = \frac{mg}{4}$$

Now from $F_t = ma_t$,

$$\left(\frac{mg}{4}\right)\cos 30° + mg\cos 30° = ma_t \Rightarrow a_t = \frac{5\sqrt{3}}{8}g$$

Now consider in radial direction

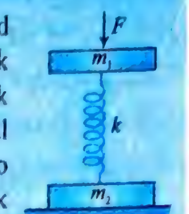
$$N + kx\sin 30° = mg\sin 30°$$

$$N = mg\sin 30° - kx\sin 30°$$

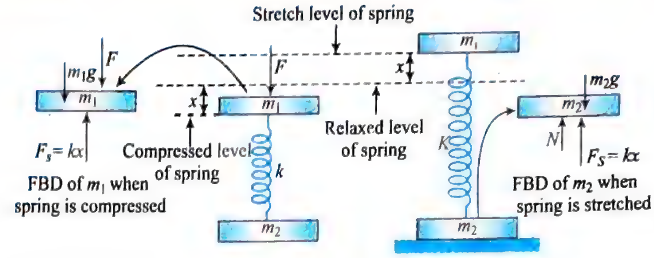
$$= mg\left(\frac{1}{2}\right) - \left(\frac{mg}{4}\right)\cdot\left(\frac{1}{2}\right) = \frac{3}{8}mg$$

ILLUSTRATION 6.59

Two disks of masses m_1 and m_2 are connected by a spring of force constant k. The lower disk of mass m_2 lies on a table and the upper disk is vertically above it (Figure). What vertical force F should be applied to the upper disk so that when the force is withdraw, the lower disk is lifted off the table?



Sol.



The situation is shown in figure. Let the applied force F compresses the spring by x, so $F_S = kx$. After removal of force, the spring recovers its compressed length and further extends by x. At this instant the force exerted by spring is along upward direction. Thus, to lift off the lower disk, normal reaction N becomes zero.

When force is applied and spring is compressed,

$$F = F_s + m_1 g \quad \text{...(i)}$$

When m_2 leaves contact (we have, $N = 0$) with ground and spring is stretched,

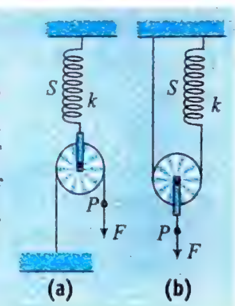
$$F_S = m_2 g \quad \text{...(ii)}$$

From Eqs (i) and (ii), we have $F_{min} = (m_1 + m_2)g$

So to lift off the lower disk, $F \geq (m_1 + m_2)g$

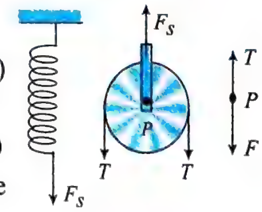
ILLUSTRATION 6.60

On applying a force F, point P is displaced vertically down by y from equilibrium position. Find force F in terms of the force constant k of the spring and displacement y, for the cases (a) and (b), as shown in figure.



Sol.

Case (a)



At point P: $F = T$ ...(i)

And for the equilibrium of pulley,

$$2T = F_s \quad \text{...(ii)}$$

But as due to the shift of point P by y, the spring stretches by $(y/2)$. So

$$F_s = k(y/2) \quad \text{...(iii)}$$

So substituting F_s from (iii) in (ii) and then T from (ii) in (i), we get

$$F = (k/4)y \quad \text{(A)}$$

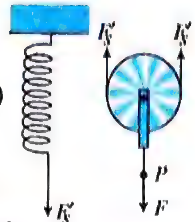
Case (b) As the tension in massless string and spring will be same,

$$T = F'_s \quad \text{...(i)}$$

For pulley: $F = 2F'_s$ ...(ii)

Now if the mass M shifts by y, the spring will stretch by $2y$ (as string is inextensible)

$$F'_s = k(2y) \qquad \text{...(iii)}$$



So substituting F' from Eq. (ii) in (iii),

$$F = (4k)y \qquad \text{(B)}$$

Force of spring does not change instantaneously.

ILLUSTRATION 6.61

Two blocks are connected by a spring. The combination is suspended, at rest, from a string attached to the ceiling, as shown in figure. The string breaks suddenly.

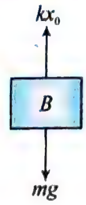
Immediately after the string breaks, what is the initial downward acceleration of the upper block of mass $2m$?

Sol.

Step I: Discuss the problem before cutting the string:

From the force diagram of lower block,

$$kx_0 = mg$$



From the force diagram of upper block,

$$T = 2mg + kx_0$$

Step II: Discuss the problem after cutting the string,

$$2mg + kx_0 = 2ma$$

or $\quad 2mg + mg = 2ma$

or $\quad 3mg = 2ma$

$\therefore \quad a = \dfrac{3}{2}g$

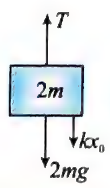
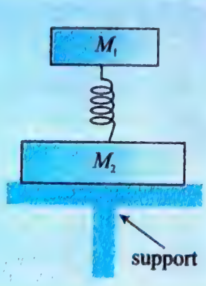
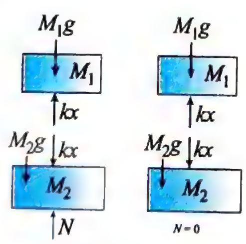


ILLUSTRATION 6.62

The system of two weights with masses M_1 and M_2 are connected with weightless spring as shown in figure. The system is resting on the support S. Find the acceleration of each of the weights just after the support S is quickly removed.



Sol. The force of spring does not change instantaneously, so find spring force at initial instant.



FBD initially FBD when support is removed

Initially, $M_1g = kx$

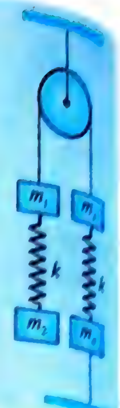
When support is removed, spring force does not change.

For M_1: $M_1g - kx = M_1a_1$ or $a_1 = 0$

For M_2: $M_2g + kx = M_2a_2$ or $a_2 = \dfrac{(M_1 + M_2)g}{M_2}$

ILLUSTRATION 6.63

Four blocks and two springs are arranged as shown in figure. The system at rest, determine the acceleration of all the loads immediately after the lower thread keeping the system in equilibrium has been cut. Assume that the threads are weightless, the mass of the pulley is negligible small, and there is no friction at the point of suspension.



Sol. For the equilibrium of system of loads,

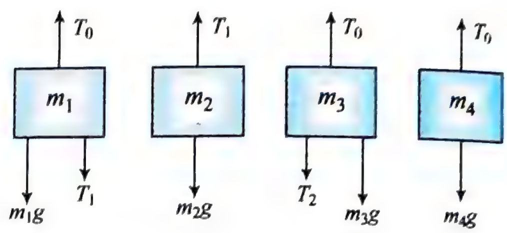
$$(m_1 + m_2) > (m_3 + m_4)$$

The force in the left spring, $T_1 = m_2g$

Let T_2 is the force in the right sprig. For the equilibrium of load m_3, we have

$$m_3g + T_2 = T_0 \qquad \text{...(i)}$$

For m_1, $m_1g + T_1 = T_0$



As $T_1 = m_2g$, therefore

$$m_1g + m_2g = T_0$$

Substituting this value in (i), we get

$$m_3g + T_2 = (m_1g + m_2g)$$

or $\quad T_2 = (m_1 + m_2 - m_3)g \qquad \text{...(ii)}$

After cutting, the lower thread, the equations of motion for the loads are

$$m_1g + T_1 - T_0 = m_1a_1 \qquad \text{...(iii)}$$

$$m_2g - T_1 = m_2a_2 \qquad \text{...(iv)}$$

$$T_2 + m_3g - T_0 = m_3a_3 \qquad \text{...(v)}$$

and $\quad T_2 - m_4g = m_4a_4 \qquad \text{...(vi)}$

Solving above equations, we get

$$a_1 = a_2 = a_3 = 0 \text{ and } a_4 = \frac{(m_1 + m_2 - m_3 - m_4)g}{m_4}$$

ILLUSTRATION 6.64

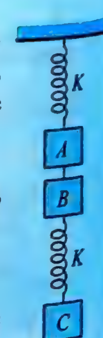
The system shown in figure is in equilibrium. Find the acceleration of the blocks A, B, and C, all of equal masses m at the instant when (assume springs to be ideal).

(a) The spring between ceiling and A is cut.

(b) The string (inextensible) between A and B is cut.

(c) The spring between B and C is cut.

Also find the tension in the string in the above three cases.



Sol. The system is in equilibrium:

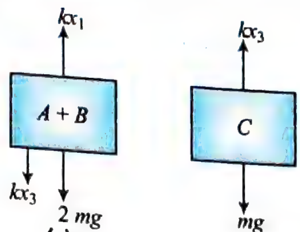
From Fig. (a),

$2mg + kx_3 = kx_1$ ...(i)

From Fig. (b),

$kx_3 = mg$ ...(ii)

$\therefore$ $3mg = kx_1$



(a) (b)

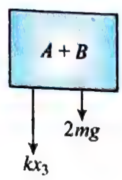
(a) When the spring between ceiling and block is cut, then the elongation of spring between B and C remains same just after cutting.

$\therefore$ $a_c = 0$ $(kx_3 = mg)$

For $(A + B)$,

$kx_3 + 2mg = 3mg$

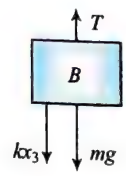
$\therefore$ $3mg = 2ma$



$\therefore$ $a = \frac{3}{2}g = 15 \text{ m s}^{-2}$

$\therefore$ $a_A = a_B = 15 \text{ m s}^{-2}$

For tension (from FBD of B),

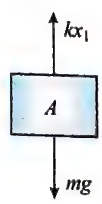


$mg + kx_3 - T = ma_B$

$mg + mg - T = \frac{3mg}{2}$

$\therefore$ $T = \frac{mg}{2}$

(b) When string between A and B is cut, the elongation in springs do not change just after cutting the string.



$mg - kx_1 = ma_A$

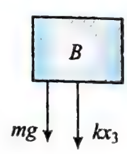
$mg - 3mg = ma_A$ $\{kx_1 = 3mg\}$

$-2mg = ma_A$

$a_A = -2g$

$\therefore$ $a_A = 2g$ (upward)

For B,

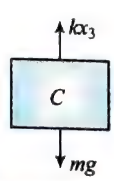


$mg + kx_3 = m\,a_B$

$mg + mg = m\,a_B$ $\{\therefore kx_3 = mg\}$

$\therefore$ $a_B = 2g$ (downward)

For C,



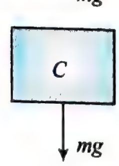
$mg - kx_3 = ma_C$

or $mg - mg = ma_C$ $\{kx_3 = mg\}$

$a_C = 0$

$T = 0$

(c) When spring between B and C is cut.



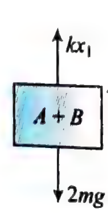
For C, $mg = ma_C$

$\therefore$ $a_c = g$ (downward)

The acceleration of A and B will be equal; taking "$A + B$" together as system

$2mg - kx_1 = 2ma_B$ $\{\therefore a_A = a_B\}$

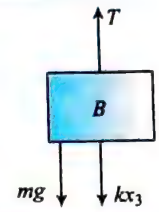
$2mg - 3mg = 2ma_B$ $\{\therefore kx_1 = 3mg\}$



$\therefore$ $a_B = -\frac{g}{2}$

$\therefore$ $a_A = a_B = \frac{g}{2}$ (upward)

For tension in string between A and B, considering the FBD of B only

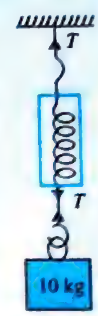


$T - (mg) = ma_B$

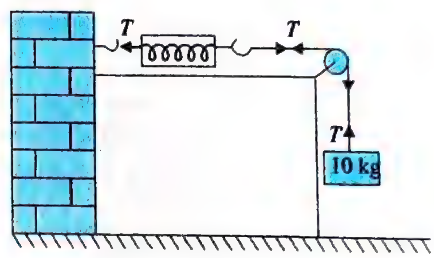
$T = mg + \frac{mg}{2} = \frac{3mg}{2}$

CONCEPT APPLICATION EXERCISE 6.7

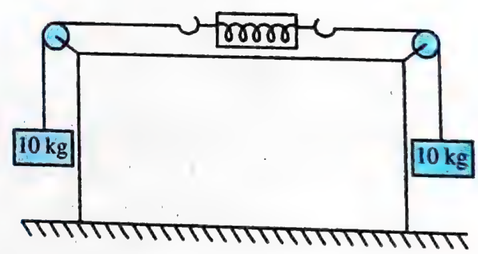
1. (a) A 10-kg block is supported by a cord that runs to a spring scale, which is supported by another cord from the ceiling as shown in figure. What is the reading on the scale?



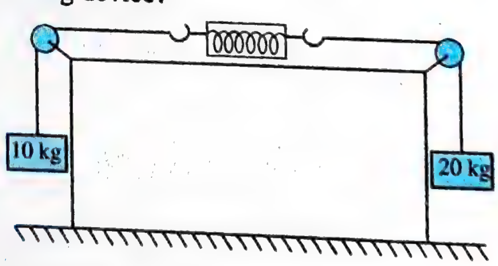
(b) In figure, the block is supported by a cord that runs around a pulley and to a scale. The opposite end of the scale is attached by a cord to a wall. What is the reading of the scale?



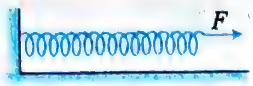
(c) In figure, the wall has been replaced with a second 10-kg block. What is the reading on the scale now?



2. What is the reading of the spring balance in the following device?



3. In the given figure, three identical massless springs are kept horizontal. The left end of the first is tied to a wall. The left end of the second spring is tied to a block of mass m placed on rough ground and the left end of the third spring is tied to a block of mass m placed on frictionless ground. The right end of each spring is pulled by a force that is increased gradually from zero to F. Extensions in these springs are x_1, x_2, and x_3, respectively. Find the relationship between x_1, x_2, and x_3.



(a)

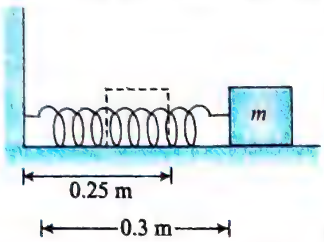
Rough surface

(b)

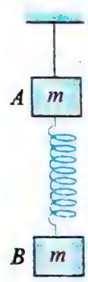
Frictionless surface

(c)

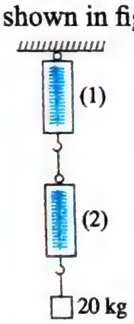
4. A smooth block of mass m is connected with a spring of stiffness k (= 20 N m^{-1}) and natural length l_0 = 0.25 m. If the block is pulled such that the new length of the spring becomes l = 0.3 m, find the acceleration of the block at the moment when it is released from given position.



5. Two blocks A and B of same mass m attached with a light spring are suspended by a string as shown in figure. Find the acceleration of block A and B just after the string is cut.

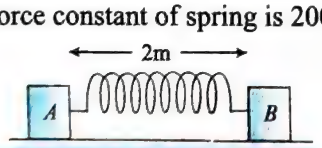


6. A block of mass 20 kg is suspended through two light spring balances as shown in figure. Calculate the:



(a) reading of spring balance (1).
(b) reading of spring balance (2).

7. Two blocks are connected by a spring of natural length 2 m. The force constant of spring is 200 N m^{-1}.



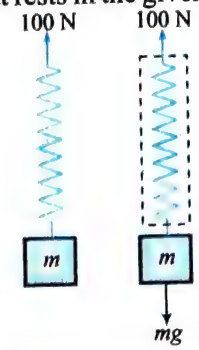
Find the spring force in the following situations:

(a) A is kept at rest and B is displaced by 1 m in right direction.

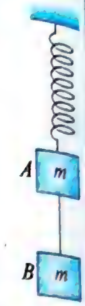
(b) B is kept at rest and A is displaced by 1 m in left direction.

(c) A is displaced by 0.75 m in right direction and B is 0.25 m in left direction.

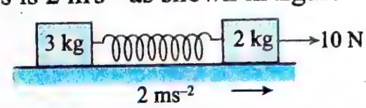
8. If the force constant of spring is 50 N m^{-1}, find mass of the block, if it at rests in the given situation (g = 10 m s^{-2}).



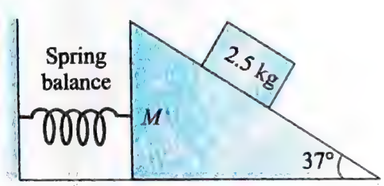
9. Two blocks A and B of same mass m attached with a light string are suspended by a spring as shown in figure. Find the acceleration of block A and B just after the string is cut.



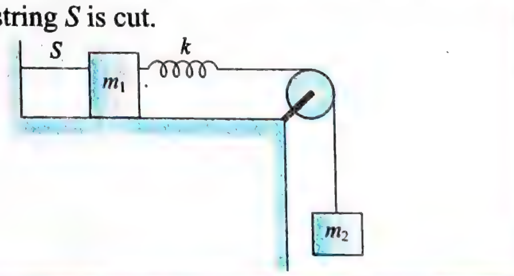
10. Find the acceleration of 3 kg mass when acceleration of 2 kg mass is 2 m s^{-2} as shown in figure.



11. Find the reading of spring balance as shown in figure. Assume that mass M is in equilibrium. (All surfaces are smooth.)



12. Two blocks of masses m_1 and m_2 are in equilibrium. The block m_2 hangs from a fixed smooth pulley by an inextensible string that is fitted with a light spring of stiffness k as shown in figure. Neglecting friction and mass of the string, find the acceleration of the bodies just after the string S is cut.



ANSWERS

1. In all the three cases the spring balance reads 10 kg.

2. $\frac{40}{3}$ kg 3. Stretch of all three springs will be equal.

4. 1/m (←) 5. $a_A = 2g$ (downwards), $a_B = 0$

6. 20 kg 7. (a) 200 N (b) 200 N (c) 200 N

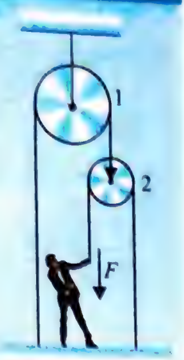
8. 10 kg 9. $a_A = g$ (upwards), $a_B = g$ (downwards)

10. 2ms^{-2} 11. 12 N 12. $a_1 = \left(\frac{m_2}{m_1}\right)g, a_2 = 0$

Solved Examples

EXAMPLE 5.1

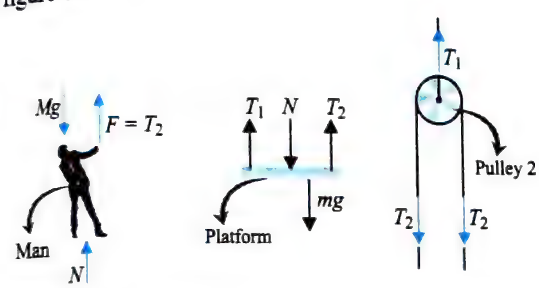
A man of mass 60 kg is standing on a platform of mass 20 kg. With what force F a man pulls on a rope in order to support the platform on which he stands? With what forces N does the man press the platform? What is the maximum weight of the platform that the man can support?



Sol. Let N be the reaction between man and platform. T_1 is the tension in string passing over pulley 1 and T_2 is the tension in string passing over pulley 2.

M = mass of man, m = mass of platform

The free body diagrams of man, platform and pulley 2 are shown in the figure below.



The force exerted by man on platform = Normal reaction exerted by platform on man = N

For equilibrium of man: $Mg = T_2 + N$ ...(i)

For equilibrium of platform: $T_1 + T_2 = N + mg$ ...(ii)

For equilibrium of pulley 2, $T_1 = 2T_2$ ...(iii)

Adding eqs.(i) and (ii), we get

$$T_1 + T_2 + (T_2 + N) = N + mg + Mg$$

$$T_1 + 2T_2 = (M + m)g$$

Using (iii), we get $2T_2 + 2T_2 = (M + m)g \Rightarrow T_2 = \dfrac{(M+m)}{4}g$

$$\therefore \quad F = T_2 = \frac{(M+m)}{4}g = \left(\frac{60+20}{4}\right) \times 10\,\text{N} = 200\,\text{N}$$

Now from (i) $N = Mg - T_2 = Mg - F$

$$= Mg - \frac{(M+m)}{4}g = \frac{(3M-m)}{4}g \qquad \text{...(iv)}$$

$$\Rightarrow N = \left(\frac{3\times 60 - 20}{4}\right) \times 10 = 400\,\text{N}$$

From (iv), $mg = 3Mg - 4N$

Clearly mg is maximum when N is minimum.

Minimum $\quad N = 0$

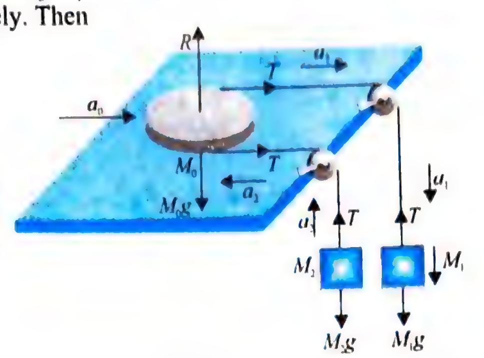
So $\quad (mg)_{max} = 3Mg = 3 \times 60g = 180$ kg wt.

$\quad (mg)_{max} = 3Mg = 3 \times 60 \times 10 = 1800$ N

EXAMPLE 5.2

A smooth pulley A of mass M_0 is lying on a frictionless table. A massless rope passes round the pulley and has masses M_1 and M_2 attached to its ends, the two portions of the string being perpendicular to the edge of the table so that the masses hang vertically. Find the acceleration of the pulley.

Sol. Let a_0, a_1 and a_2 are accelerations of M_0, M_1 and M_2, respectively. Then



Constraint relation: $a_0 = \dfrac{a_1 - a_2}{2} \Rightarrow 2a_0 = a_1 - a_2$ ...(i)

Equation of motion:

For M_0: $2T = M_0 a_0$ ...(ii)

For M_1: $M_1 g - T = M_1 a_1$ ...(iii)

For M_2: $T - M_2 g = M_2 a_2$ ...(iv)

Substituting values of a_0, a_1 and a_2 from Eqs. (ii), (iii) and (iv) in (i), we get

$$2\left(\frac{2T}{M_0}\right) = \left(g - \frac{T}{M_1}\right) - \left(\frac{T}{M_2} - g\right)$$

$$\frac{4T}{M_0} = 2g - T\left(\frac{1}{M_1} + \frac{1}{M_2}\right)$$

i.e., $$\left(\frac{4}{M_0} + \frac{1}{M_1} + \frac{1}{M_2}\right)T = 2g$$

This gives $$T = \frac{2M_0 M_1 M_2 g}{4M_1 M_2 + M_0(M_1 + M_2)} \qquad \text{...(v)}$$

Acceleration of pulley of A from eq. (ii),

$$a_0 = \frac{2T}{M_0} = \frac{4M_1 M_2 g}{4M_1 M_2 + M_0(M_1 + M_2)}$$

EXAMPLE 5.3

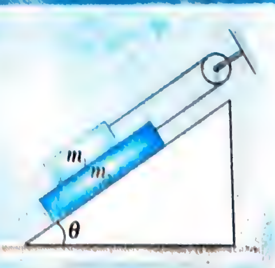
Figure shows a block of mass m_1 sliding on a block of mass m_2, with $m_1 > m_2$. Find

(a) acceleration of each block

(b) tension in the string

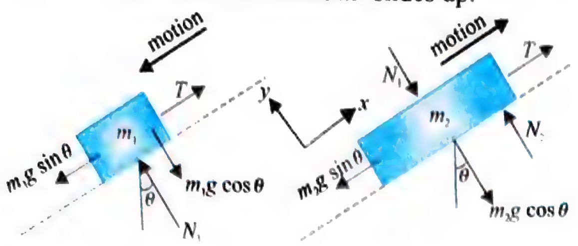
(c) force exerted by m_1 on m_2

(d) force exerted by m_2 on the incline



Sol. Let block m_1 moves down with acceleration a and block m_2 moves up with acceleration a. The figure below shows free body diagram of each block. We will apply Newton's second

law along x and y-axis shown in free body diagram. Block m_1 is heavy, hence it slides down whereas m^2 slides up.



$$\sum F_x = m_1 g \sin\theta - T = m_1 a$$
$$\sum F_y = N_1 - m_1 g\cos\theta = 0$$
$$\sum F_x = T - m_2 g\sin\theta = m_2 a$$
$$\sum F_y = N_2 - N_1 - m_2 g\cos\theta = 0$$

From eqns. (i) and (iii), $a = \dfrac{m_1 g\sin\theta - m_2 g\sin\theta}{m_1+m_2}$

$$T = m_2 a + m_2 g\sin\theta$$
$$= \frac{m_2(m_1 g\sin\theta - m_2 g\sin\theta)}{m_1+m_2} + m_2 g\sin\theta = \frac{2m_1m_2 g\sin\theta}{m_1+m_2}$$

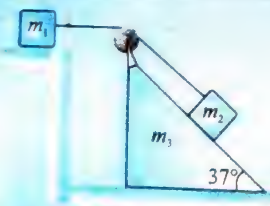
From eqns. (ii) and (iv), $N_1 = m_1 g\cos\theta$

$$N_2 = N_1 + m_2 g\cos\theta = (m_1 g\ \cos\theta + m_2 g\ \cos\theta)$$

N_1 is force exerted by block m_1 on m_2 and N_2 is force exerted by block m_2 on inclined surface.

EXAMPLE 5.4

In the arrangement shown in the figure, a wedge of mass $m_3 = 3.45$ kg is placed on a smooth horizontal surface. Small and light pulley is connected on its top edge, as shown. A light, flexible thread passes over the pulley. Two blocks having mass $m_1 = 1.3$ kg and $m_2 = 1.5$ kg are connected at the ends of the thread.



m_1 is on smooth horizontal surface and m_2 rests on inclined smooth surface of the wedge. The base length of wedge is 2 m and inclination is 37°, m_2 is initially near the top edge of the wedge. If the whole system is released from rest, calculate:

(a) Velocity of wedge when m_2 reaches its bottom

(b) Velocity of m_2 at that instant ($g = 10\ \text{ms}^{-2}$)

Let acceleration of m_1 be a (rightwards) and that of wedge be b (leftwards). Acceleration of m_2 (relative to wedge) becomes $(a+b)$, down the plane w.r.t wedge.

Therefore, resultant acceleration of m_2 is vector sum of the two acceleration.

(a) $(a+b)$ down the plane and

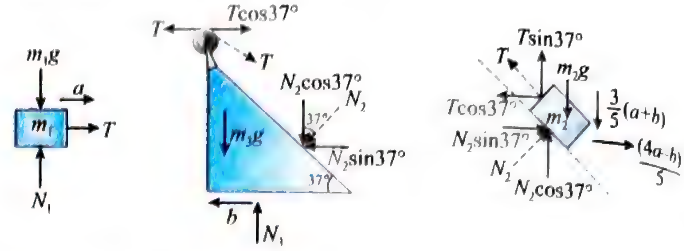
(b) b leftwards.

Hence, components of this resultant acceleration are

(i) $[(a+b)\cos 37° - b] = \dfrac{(4a-b)}{5}$ horizontal rightward, and

(ii) $(a+b)\sin 37° = \dfrac{3}{5}(a+b)$ vertically downward.

Considering free body diagrams,



For horizontal forces on m_1, $T = m_1 a$ ...(i)

For horizontal forces on wedge,

$$T - T\cos 37° + N_2\sin 37° = m_3 b \qquad \text{...(ii)}$$

For horizontal forces on m_2,

$$N_2.\sin 37° - T\cos 37° = m_2\left(\frac{4}{5}a - \frac{1}{5}b\right) \qquad \text{...(iii)}$$

For vertical forces on m_2,

$$m_2 g - N_2\cos 37° - T\sin 37° = m_2\frac{3}{5}(a+b) \qquad \text{...(iv)}$$

From above equations, $a = 3\ \text{ms}^{-2}$, $b = 2\ \text{ms}^{-2}$

Since, base angle and base length of wedge are 37° and 2m respectively, therefore, height of its vertical forces is $2\tan 37° = 1.5$ m.

Now considering vertical motion of m_2 from top to bottom of the wedge, $u = 0$, acceleration $= \dfrac{3}{5}(a+b) = 3\ \text{ms}^{-2}$ and displacement $= 1.50$ m.

Using $s = ut + \dfrac{1}{2}at^2$

$t = 1$s

At this instant, horizontal component of velocity of m_2 is

$$v_{2x} = \left(\frac{4}{5}a - \frac{1}{5}b\right)t = 2\ \text{ms}^{-1}$$

and vertical component, $v_{2y} = \dfrac{3}{5}(a+b)t = 3\ \text{ms}^{-1}$

Vertical of m_2 is $v_2 = \sqrt{v_{2x}^2 + v_{2y}^2} = \sqrt{13}\ \text{ms}^{-1}$

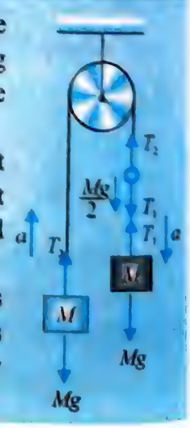
Velocity of wedge at this instant $= bt = 2\ \text{ms}^{-1}$.

EXAMPLE 5.5

Two blocks of equal mass, M each, are connected to two ends of a massless string passing over a massless pulley. On one side of the string, there is a bead of mass $M/2$.

(a) When the system is released from rest the bead continues to remain at rest while the two blocks accelerate. Find the acceleration of the blocks.

(b) Find the acceleration of the two blocks if it was observed that the bead was sliding down with a constant velocity relative to the string.



Sol.

(a) The frictional force between the bead and the string causes the tension to change in the string on the two sides of the bead.

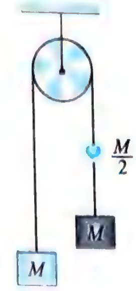
For the bead, $T_2 = T_1 + \frac{Mg}{2}$ ...(i)

For the block, $T_2 - Mg = Ma$ ...(ii)

$Mg - T_1 = Ma$ ...(iii)

(ii) + (iii) $\Rightarrow T_2 - T_1 = 2Ma$

Using (i), $\frac{Mg}{2} = 2Ma \Rightarrow a = \frac{g}{4}$



(b) In this case acceleration of the bead is same as acceleration of the two blocks.

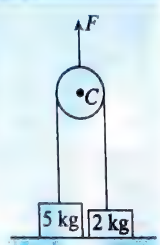
$$\frac{Mg}{2} + T_1 - T_2 = \frac{M}{2}a$$

The other two equations remain same.

Solving the three equations, we get $a = \frac{g}{5}$

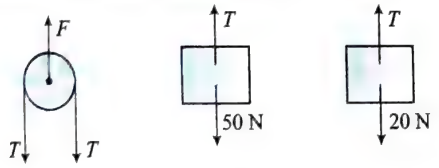
EXAMPLE 5.6

Two blocks of masses 5 kg and 2 kg are initially at rest on the floor. They are connected by a light string, passing over a light frictionless pulley. An upward force F is applied on the pulley and maintained at a constant value. Calculate the acceleration a_1 and a_2 of the 5-kg and 2-kg masses, respectively, when F is (take $g = 10\ \text{m s}^{-2}$)



(a) 30 N (b) 60 N (c) 140 N

Sol. Apart from the constraint that the string is unstretchable, the additional constraint is that neither of the masses can go downward. So the block will be lifted only when the tension of the string exceeds the gravitational pull on them.



(a) **When $F = 30$ N:** Considering the free-body diagram of the pulley

$30 - 2T = 0$ (pulley is massless)

or $T = 15$ N

So, tension is less than gravitational pull on both the blocks. So, no acceleration is produced in them. Therefore, $a_1 = a_2 = 0$

(b) **When $F = 60$ N:** Now $60 - 2T = 0$ or $T = 30$ N

So, the 5-kg weight will not be lifted but the 2-kg weight will. For acceleration of 2-kg block, writing equation of motion

$30 - 20 = 2 \times a_2$ or $a_2 = 5\ \text{m s}^{-2}$

Thus, $a_1 = 0$ and $a_2 = 5\ \text{m s}^{-2}$

(c) **When $F = 140$ N:** Now $140 - 2T = 0$ or $T = 70$ N

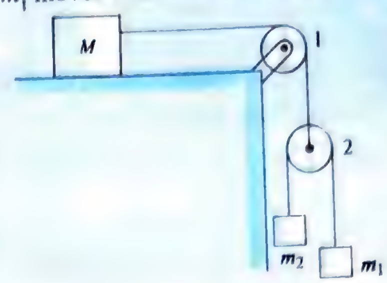
So both the weights are lifted. Writing equation of motion

For 5-kg block: $70 - 50 = 5a_1$ or $a_1 = 4\ \text{m s}^{-2}$

For 2-kg block: $70 - 20 = 2a_2$ or $a_2 = 25\ \text{m s}^{-2}$

EXAMPLE 5.7

In the arrangement shown in figure, $m_1 = 1$ kg, $m_2 = 2$ kg. Pulleys are massless and strings are light. For what value of M, the mass m_1 moves with constant velocity.



Sol. Mass m_1 moves with constant velocity if tension in the lower string,

$T_1 = m_1 g = (1)(10) = 10$ N ...(i)

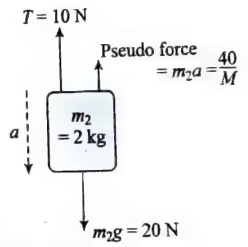
Tension in the upper string,

$T_2 = 2T_1 = 20$ N ...(ii)

Acceleration of block M is, therefore,

$a = \frac{T_2}{M} = \frac{20}{M}$ ...(iii)

This is also the acceleration of pulley 2.



The absolute acceleration of mass m_1 is zero. Thus, the acceleration of m_1 relative to pulley 2 is a upward or acceleration of m_2 with respect to pulley 2 is a downward. Drawing free-body diagram of m_2 with respect to pulley 2 (figure above).

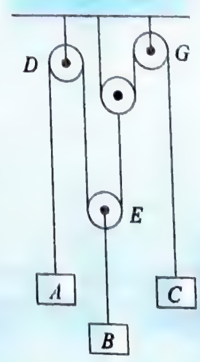
Equation of motion gives

$$20 - \frac{40}{M} - 10 = 2a = \frac{40}{M}$$

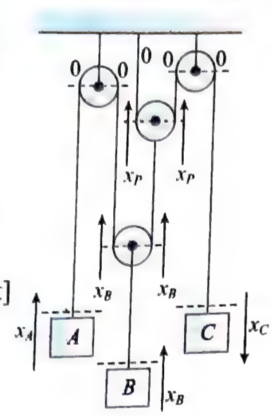
Solving this, we get $M = 8$ kg.

EXAMPLE 5.8

Pulleys shown in the system are massless and frictionless. Threads are inextensible. The mass of blocks A, B, and C are $m_1 = 2$ kg, $m_2 = 4$ kg, and $m_3 = 2.75$ kg, respectively. Calculate the acceleration of each block.



Sol. Let the acceleration of blocks A and B be a and b vertically upwards, and acceleration of block C is c downward, respectively.



As length of strings are constant, the sum of change in segment lengths should be zero.

For string 1: [writing from left to right]

$$(-x_A)+(-x_B)+(x_P-x_B)=0$$

$$x_P = x_A + 2x_B \quad \text{...(i)}$$

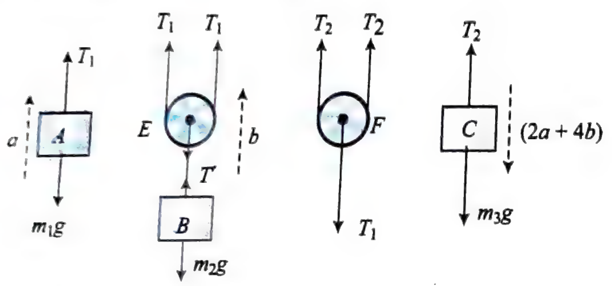
For string 2: $(-x_P)+(-x_P)+(x_C)=0$

$$x_C = 2x_P \quad \text{...(ii)}$$

From (i) and (ii): $x_C = 2x_A + 4x_B$

$$\Rightarrow \quad c = 2a + 4b \quad \text{...(iii)}$$

Now considering FBDs, we get the following:



For block A, $\quad T_1 - m_1 g = m_1 a \quad$ ...(i)

For block B, $\quad 2T_1 - m_2 g = m_2 b \quad$ ...(ii)

For pulley F, $\quad T_1 = 2T_2 \quad$ ...(iii)

For block C, $\quad m_3 g - T_2 = m_3(2a+4b) \quad$ ...(iv)

Solving above equations,

$$T_1 = 22\text{ N},\ T_2 = 11\text{ N}$$

$$a = 1\text{ ms}^{-2},\ b = 1\text{ ms}^{-2}$$

Hence, acceleration of block A,

$$a = 1\text{ ms}^{-2}\ (\uparrow)$$

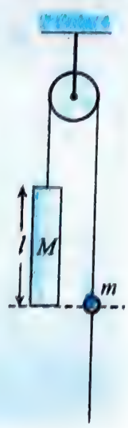
Acceleration of block B,

$$b = 1\text{ ms}^{-2}\ (\uparrow)$$

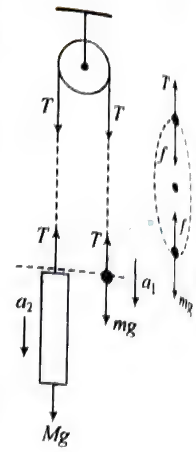
Acceleration of block $C = (2a+4b) = 6\text{ ms}^{-2}\ (\downarrow)$

EXAMPLE 5.9

In the arrangement shown in figure, mass of the rod M exceeds the mass m of the ball. The ball has an opening permitting it to slide along the thread with some friction. The mass of the pulley and the friction in its axle are negligible. At the initial moment, the ball was located opposite the lower end of the rod. When set free, both bodies began moving with constant accelerations. Find the friction force between the ball and the thread t seconds after the beginning of motion, the ball got opposite to the upper end of the rod. The rod length equals l.



Sol. The friction force between ball and thread is f_r. Acceleration of both the bodies is downward.



$$mg - T = ma_1 \quad \text{...(i)}$$

$$Mg - T = Ma_2 \quad \text{...(ii)}$$

But $\quad f_r = T$

$$mg - f_r = ma_1 \quad \text{...(iii)}$$

$$a_2 - a_1 = \left(\frac{Mg - f_r}{M}\right) - \left(\frac{mg - f_r}{m}\right)$$

According to problem,

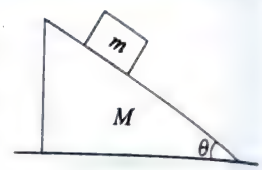
$$l = \frac{1}{2}(a_2 - a_1)\,t^2 \text{ or } (a_2 - a_1) = \frac{2l}{t^2}$$

$$\frac{m(Mg - f_r) - M(mg - f_r)}{Mm} = \frac{2l}{t^2}$$

$$(M-m)f_r = \frac{2lmM}{t^2} \Rightarrow f_r = \frac{2lmM}{(M-m)t^2}$$

EXAMPLE 5.10

A block of mass m is placed on the inclined surface of a wedge as shown in figure. Calculate the acceleration of the wedge and the block when the block is released. Assume all surfaces are frictionless.



Sol. **Method 1: Analysis from ground frame:** Let the acceleration of wedge be A and that of block be a (w.r.t. wedge). Then, acceleration of m w.r.t. ground is

$$\vec{a}_m = \vec{a}_{m,M} + \vec{a}_M = (a\cos\theta\hat{i} - a\sin\theta\hat{j}) - A\hat{i}$$

$$= (a\cos\theta - A)\hat{i} - a\sin\theta\hat{j} \quad \text{...(i)}$$

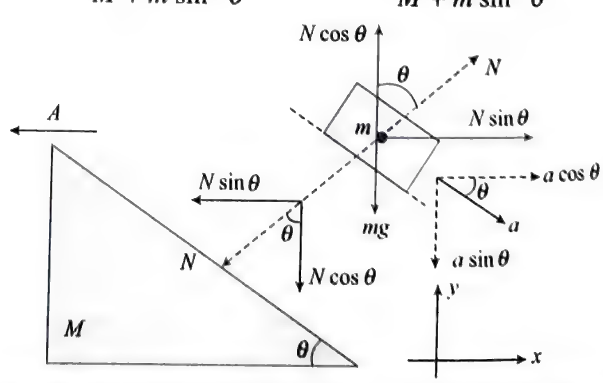
For M: $N\sin\theta = MA \quad$ ...(ii)

For m: $mg - N\cos\theta = ma\sin\theta \quad$ ...(iii)

$$N\sin\theta = m(a\cos\theta - A) \quad \text{...(iv)}$$

Solve (ii), (iii), and (iv) to get

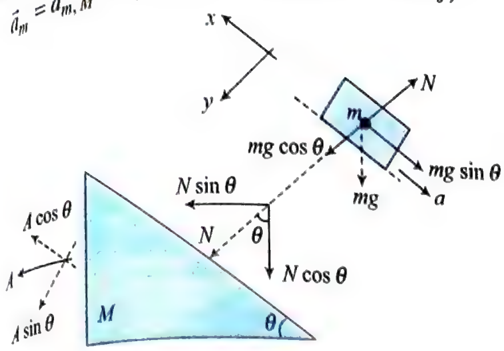
$$A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \quad \text{and} \quad a = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$



Now acceleration of block can be found by putting the values of A and a in (i).

Method 2: If we consider the motion of block parallel and perpendicular to sloping surface w.r.t. ground, acceleration of m can also be written as:

$\vec{a}_m = \vec{a}_{m,M} + \vec{a}_M = -a\hat{i} + (A\cos\theta\hat{i} + A\sin\theta\hat{j})$



$\vec{a}_m = (A\cos\theta - a)\hat{i} + A\sin\theta\,\hat{j}$ ...(v)

For M: $N\sin\theta = MA$ ...(vi)

For m: $-mg\sin\theta = m(A\cos\theta - a)$ ...(vii)

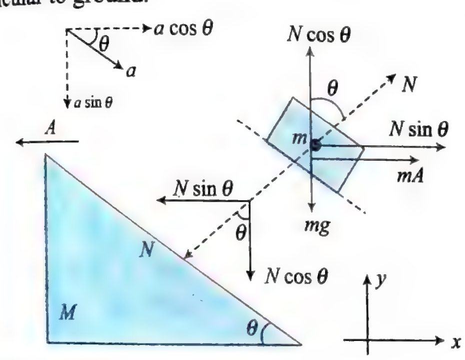
$mg\cos\theta - N = mA\sin\theta$ ...(viii)

From above equations, we can get the values of

$$A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \quad \text{and} \quad a = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

Note: In method 2, we see that in a direction perpendicular to the inclined plane of wedge, accelerations of both wedge and block are same, which is equal to A sin θ. This is necessary if both have to remain in contact.

Method 3: Analysis from non-inertia frame: With the help of pseudo force. If we consider the motion of block parallel and perpendicular to ground.



Let us make the FBD of m w.r.t. M.

For M: $N\sin\theta = MA$ ...(ix)

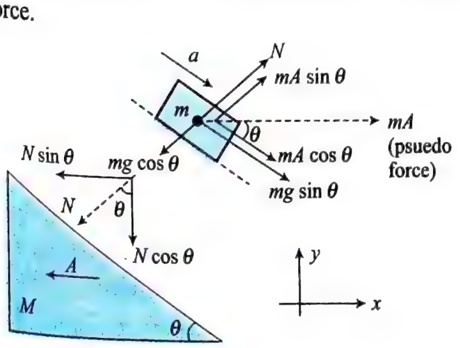
For m: $N\sin\theta + mA = ma\cos\theta$ ...(x)

$mg - N\cos\theta = ma\sin\theta$ ...(xi)

From above equations, we can get the values of

$$A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \quad \text{and} \quad a = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

Method 4: Analyses from non-inertia frame: With the help of pseudo force.



When the forces are resolved in directions along the plane and perpendicular to the plane:

For M: $N\sin\theta = MA$

For m: $mg\sin\theta + mA\cos\theta = ma$

$mg\cos\theta = N + mA\sin\theta$

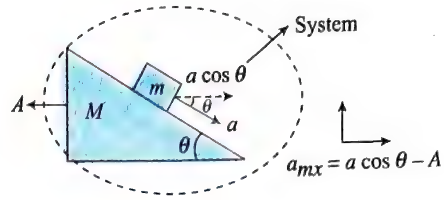
From above equations, we can get the values of

$$A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \quad \text{and} \quad a = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

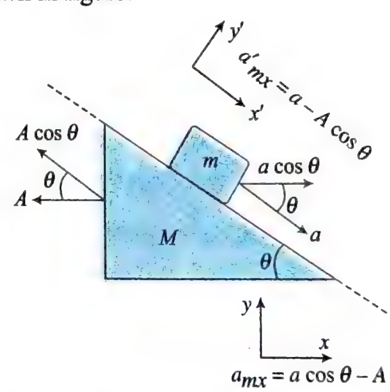
Method-5:

Taking system together:

Considering wedge and block together as a system as shown in figure.



When the system is released, the wedge will start moving towards left with acceleration A and the block will start sliding on the sloping surface of the wedge with acceleration a with respect to wedge as shown in figure.



Now analyzing the system in the horizontal direction, the system is not having any external force in horizontal direction. So we can apply

$$\Sigma\vec{F}_x = \Sigma m_i\vec{a}_i = m\vec{a}_m + M\vec{a}_M$$

$$\Sigma\vec{F}_x = 0 \Rightarrow -MA + m(a\cos\theta - A) = 0$$

$$\Rightarrow \quad a = \frac{(m+M)A}{m\cos\theta} \qquad \text{...(i)}$$

Now considering the motion of block parallel to the sloping side. Writing the equation of motion with respect to ground.

$$\Sigma\vec{F}_{x'} = \Sigma m\vec{a}_{x'}$$

$$\Rightarrow \quad mg\sin\theta = ma'_x$$

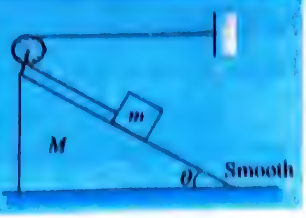
$$= m(a - A\cos\theta) \qquad \text{...(ii)}$$

Solve (i) and (ii) to get

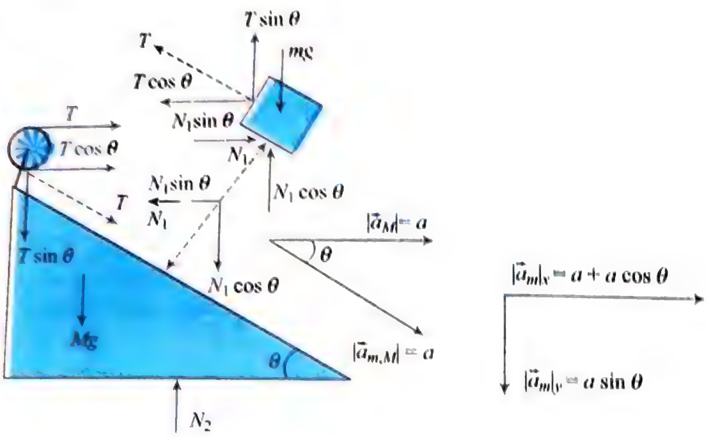
$$A = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} \quad \text{and} \quad a = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

EXAMPLE 5.11

The mass of wedge, shown in figure, is M and that of the block is m. Neglecting friction at all the places and mass of the pulley, calculate acceleration of the wedge. Thread is inextensible.



Sol. **Method 1:** Let the acceleration of wedge be a rightward. Then acceleration of block relative to wedge will also be a, but down the incline. Hence, the net acceleration of the block will be equal to the vector sum of these two as shown in figure.



$$(\bar{a}_m)_x = (\bar{a}_{mM})_x + (\bar{a}_M)_x$$

$$(\bar{a}_m)_x = a\cos\theta + a = a(1+\cos\theta)$$

$$(\bar{a}_m)_y = (\bar{a}_{mM})_y + (\bar{a}_M)_y = a\sin\theta + 0 = a\sin\theta$$

Now considering FBDs.

For the wedge: $N_2 = N_1 \cos\theta + T\sin\theta + Mg$ ...(i)

$T + T\cos\theta - N_1 \sin\theta = Ma$ ...(ii)

For the block:

$N_1 \sin\theta - T\cos\theta = m(a + a\cos\theta)$ ...(iii)

$mg - N_1 \cos\theta - T\sin\theta = ma\sin\theta$ ...(iv)

From (iii), $N_1 = \dfrac{T\cos\theta + m(a + a\cos\theta)}{\sin\theta}$

Putting the value of N_1 in (iv),

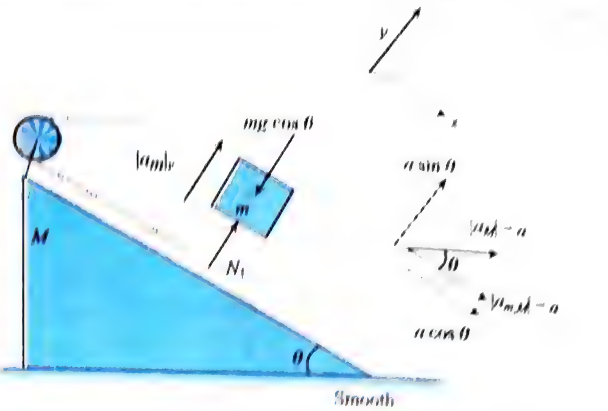
$$mg - \frac{\cos\theta}{\sin\theta}(T\cos\theta + m(a + a\cos\theta)) - T\sin\theta = ma\sin\theta$$

Simplify to get $T = mg\sin\theta - ma(1 + \cos\theta)$ ...(v)

From (ii) and (iii), $T = Ma + m(a + a\cos\theta)$ ...(vi)

Solving (v) and (vi), we get $a = \dfrac{mg\sin\theta}{M + 2m(1+\cos\theta)}$

Method 2: The accelerations of wedge and block will be same in a direction perpendicular to the incline of wedge which is $a\sin\theta$.



Writing the equation of block for this direction, we get

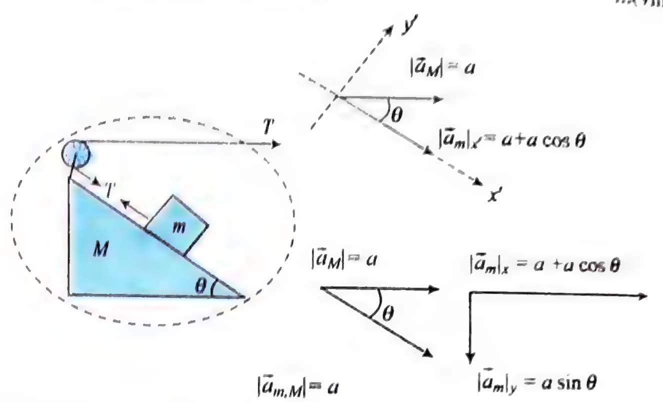
$$N_1 - mg\cos\theta = ma\sin\theta \quad ...(vii)$$

Now solving (ii), (iii), and (vii), we can get the same answers as earlier.

Method 3: If we consider block and wedge as a system, then the only horizontal force acting on the system is tension in the string as shown in figure.

Writing the equations for the combined system in horizontal direction, we get

$$T = Ma + m(a + a\cos\theta) \quad ...(viii)$$



Consider the motion of block parallel to sloping surface and writing the equation of motion w.r.t. ground,

$$(\bar{a}_m)_{x'} = (\bar{a}_{m,M})_{x'} + (\bar{a}_M)_{x'} = a + a\cos\theta$$

$$mg\sin\theta - T = m(a + a\cos\theta)$$

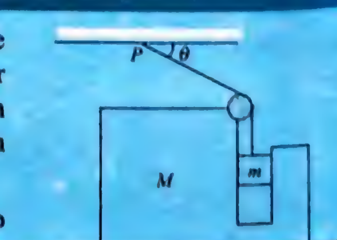
$$T = mg\sin\theta - m(a + a\cos\theta) \quad ...(ix)$$

Solving (viii) and (ix), we get

$$a = \frac{mg\sin\theta}{M + 2m(1+\cos\theta)}$$

EXAMPLE 5.12

A block of mass m can slide freely in a slot made in a bigger block of mass M as shown in figure. There is no friction anywhere in the system.



The block m is connected to one end of a string whose other end is fixed at point P. System is released from rest when the string at P makes an angle θ with horizontal. Find the accelerations of m and M just after release.

Method 1: Just after the release of system, block M will move towards left (say with acceleration A) and block m will move down with respect ot M (say with acceleration a). As there is no motion of M in vertical direction, hence the acceleration of m in vertical direction will be same as a. The acceleration of m in horizontal direction will be same as that of M, i.e., A, because horizontally both move together.

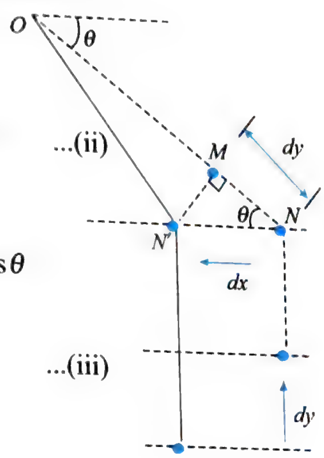
Finding constraint: Method 1: Let just after release, during a small time dt, the displacement of m (w.r.t. M) is dy downwards and that of M is dx horizontally as shown in figure.

Decrease in the length of segment I = $dx \cos\theta$

Increase in the length of segment II = dy

As the total length of the string is constant, so

$$dy = dx\cos\theta \quad \text{...(i)}$$



Dividing both sides by dt, we get

$$\frac{dy}{dt} = \frac{dx}{dt}\cos\theta \Rightarrow v = V\cos\theta \quad \text{...(ii)}$$

Differentiating w.r.t. t,

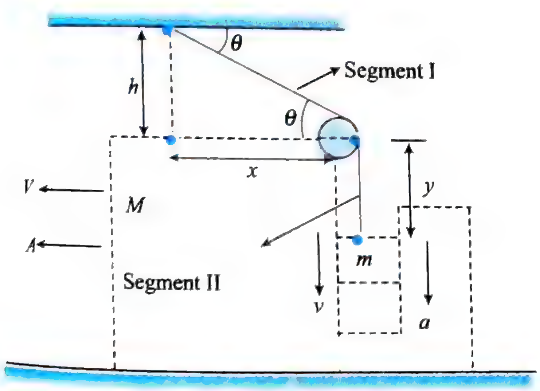
$$\frac{dv}{dt} = V(-\sin\theta)\frac{d\theta}{dt} + \frac{dV}{dt}\cos\theta$$

But initially

$$V = 0, \text{ so } a = A\cos\theta \quad \text{...(iii)}$$

$$\left(\because \frac{dv}{dt} = a, \frac{dV}{dt} = A\right)$$

Method 2: We can obtain (i) in the following way.



The displacement of M towards left is dx, then its component along AP is $dx\cos\theta$. So $dx\cos\theta$ should be decreased in the length of segment I and this should be equal to increase in the length of segment II which is dy. Hence, $dy = dx\cos\theta$.

$\Rightarrow \quad a = A\cos\theta$.

Method 3:

Total length of string: $l = \sqrt{h^2 + y^2} + x$

Differentiating w.r.t. t, $\dfrac{dl}{dt} = \dfrac{d(h^2+x^2)^{1/2}}{dt} + \dfrac{dx}{dt}$

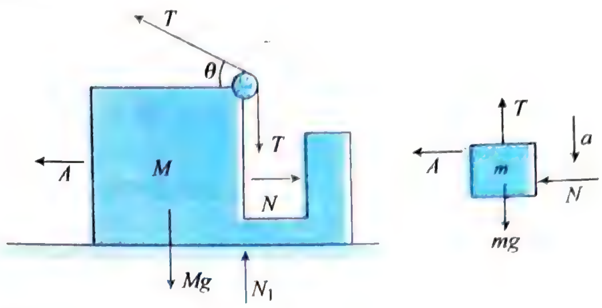
$$\Rightarrow \quad 0 = \frac{2y\dfrac{dy}{dt}}{2\sqrt{h^2+y^2}} + \frac{dx}{dt} \Rightarrow 0 = \cos\theta(-V) + v$$

$$\left(\because \frac{dx}{dt} = v, \frac{dy}{dt} = -V(\text{as } y \text{ decreases with time})\right.$$
$$\left.\text{and } \cos\theta = \frac{y}{\sqrt{h^2+y^2}}\right)$$

$\Rightarrow \quad v = V\cos\theta$ which is same as (ii).

Now finding accelerations of blocks

We can use any of the following methods to find the values of A and a.



Method 1:

From FBD of M:

$$T\cos\theta - N = MA \quad \text{...(iv)}$$

From FBD of m: $N = mA \quad$ ...(v)

$$mg - T = ma \quad \text{...(vi)}$$

Combining (iv) and (v), we get

$$T\cos\theta = (m+M)A \quad \text{...(vii)}$$

Putting $a = A\cos\theta$ in (vi), we get

$$mg - T = mA\cos\theta \quad \text{...(viii)}$$

Put the value of T from (viii) into (vii), we get

$$(mg - mA\cos\theta)\cos\theta = (m+M)A$$

$$\Rightarrow \quad A = \frac{mg\cos\theta}{M + m(1+\cos^2\theta)}$$

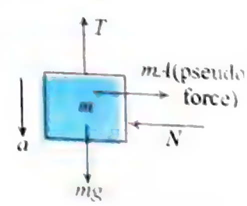
$$\text{and} \quad a = A\cos\theta = \frac{mg\cos^2\theta}{M + m(1+\cos^2\theta)}$$

The net acceleration of m is

$$\sqrt{A^2 + a^2} = A\sqrt{1+\cos^2\theta} = \frac{mg\cos\theta\sqrt{1+\cos^2\theta}}{M + m(1+\cos^2\theta)}$$

Method 2: Analyzing the block with w.r.t. wedge

W.r.t. M, m has only vertical acceleration which is a downward. Here we have to apply pseudo force because M is a non-inertial frame.



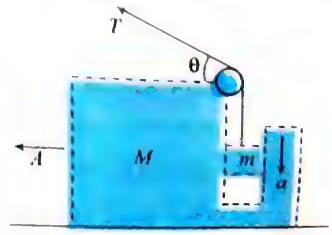
FBD of m w.r.t. M:

$$N = mA, \; mg - T = ma$$

These equations are same as (v) and (vi). Now solve to get answers.

Method 3: Analyzing the block and wedge together

Analyzing the system of $(M + m)$, i.e., block and wedge together in horizontal direction, in horizontal direction both masses have same acceleration which is A.

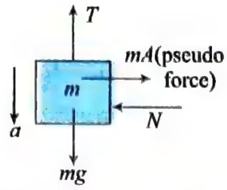


$$T\cos\theta = (M+m)A \qquad \text{...(ix)}$$

Analyzing the motion of block in vertical direction.

$$mg - T = ma \qquad \text{...(x)}$$

Solving (ix) and (x), we get

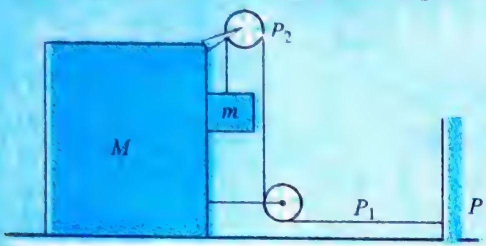


$$A = \frac{mg\cos\theta}{M+m(1+\cos^2\theta)}$$

and $$a = \frac{mg\cos^2\theta}{M+m(1+\cos^2\theta)}$$

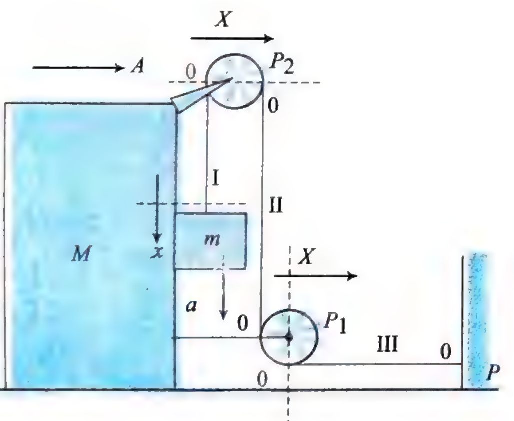
EXAMPLE 5.13

A block of mass m can slide freely along the vertical surface of a bigger block of mass M as shown in figure. There is no friction anywhere in the system. The block m is connected to one end of a string whose other end is fixed at point P.



The string between P_1 and P is horizontal and other parts of the string are vertical. System is released from rest. Find the accelerations of m and M during their subsequent motion.

Sol. Let us divide the string in three segments as shown in figure. As we release the system, the block m will try to move downward due to which the length of segment I will increase. Then the length of segment III should decrease because the length of segment II is fixed. This will make M to move toward right. Let the acceleration of m downward be a and the acceleration of M toward right be A. Then in horizontal direction, the acceleration of m will also be A, because m is constrained to move horizontally with M.



Constraint relation: As the length of string is constant, there should not be any change in the length of string although there can be change in the individual segments of string. Let during some time, m moves downwards a distance x and M moves towards right a distance X.

Change in the length of segment I = $+x$

Change in the length of segment II = 0

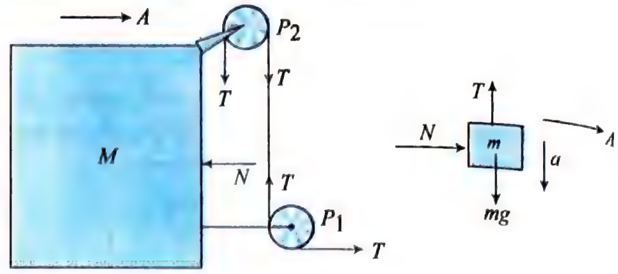
Change in the length of segment III = $-X$

But net change is zero, so $(+x) + 0 + (-X) = 0 \Rightarrow x = X$

Differentiating twice w.r.t. time, we get $a = A$ ...(i)

Now finding the accelerations.

Method 1: Analyses from ground frame:



If we make FBDs of M and m separately, then

From FBD of M:

$$T - N = MA \qquad \text{...(ii)}$$

From FBD of m:

$$N = mA \qquad \text{...(iii)}$$

and $mg - T = ma$ ...(iv)

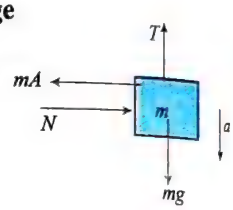
From (ii) and (iii), we get

$$T = (M+m)A \qquad \text{...(v)}$$

Equation (v) is same as (ii). Now solve to get the answers.

Method 2: Analyzing block w.r.t. wedge

W.r.t. M, m has only vertical acceleration which is a downward. Here we have to apply pseudo force (mA in figure) because M is non-inertial frame.

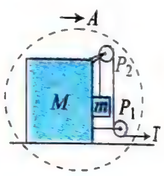


From FBD of m w.r.t. M, writing the equation of motion of block:

$$N = mA, \quad mg - T = ma$$

These equations are same as (iii) and (iv). Now solve to get answers.

Method 3: Considering the motion of m and M together in horizontal direction.



$$T = (M+m)A \qquad \text{...(vi)}$$

Now consider the motion of m in vertical direction only

$$mg - T = ma = mA \qquad \text{...(vii)}$$

From (vii),

$$T = m(g-A) \qquad \text{...(viii)}$$

From (vi) and (viii),

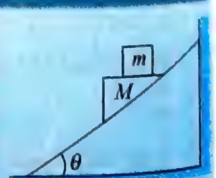
$$m(g-A) = (M+m)A \Rightarrow A = \frac{mg}{M+2m} = a$$

Hence, the net acceleration of m

$$a_{\text{net}} = \sqrt{a^2+A^2} = \sqrt{2}A = \sqrt{2}\frac{mg}{(M+2m)}$$

EXAMPLE 5.14

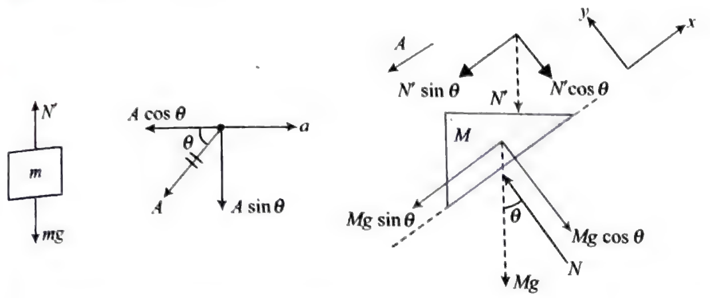
Consider a system of a small body of mass m kept on a large body of mass M placed over an inclined plane of the angle of inclination θ to the horizontal. Find the acceleration of m when the system is set in motion. Assume an inclined plane to be fixed. All the contact surfaces are smooth.



Sol. **Method 1:** Analysis in an inertial reference frame attached to the ground:

Let a be the acceleration of m w.r.t. M directed horizontally towards right.

Let A be the acceleration of M w.r.t. the ground directed along the incline downwards.



Acceleration of m w.r.t. ground will be the vector sum of the acceleration of m w.r.t. M and the acceleration of M w.r.t. ground.

Force equations for m are

$$0 = m(a - A\cos\theta) \quad \text{(i)}$$

$$mg - N' = m(A\sin\theta) \quad \text{(ii)}$$

Force equations for M are

$$N'\sin\theta + Mg\sin\theta = MA \quad \text{(iii)}$$

$$N'\cos\theta + Mg\cos\theta - N = 0 \quad \text{(iv)}$$

From (ii),

$$N' = mg - mA\sin\theta$$

Substituting N' in (iii),

$$(mg - mA\sin\theta)\sin\theta + Mg\sin\theta = MA$$

$$\Rightarrow \quad A = \frac{(M+m)g\sin\theta}{M+m\sin^2\theta}$$

Acceleration of m w.r.t. ground is $A\sin\theta$ $\quad(\because A\cos\theta - a = 0)$.

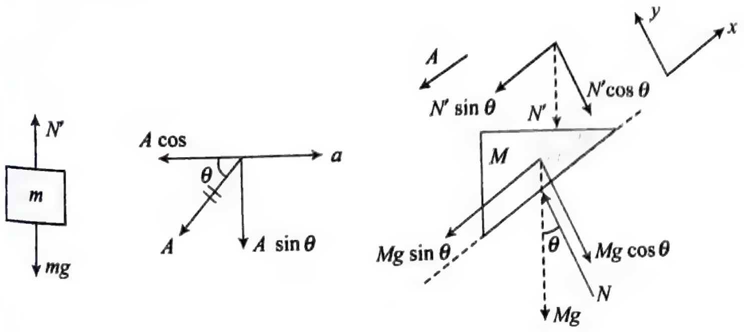
Thus, the acceleration of m w.r.t. ground is

$$a_{mG} = A\sin\theta = \frac{(M+m)g\sin\theta}{M+m\sin^2\theta}$$

Method 2: Analysis of motion of m in the non-inertial reference frame attached to M:

Let the acceleration of M w.r.t. ground be A along the inclined plane downwards. Consider that ground is an inertial reference frame, the reference frame attached to M will be non-inertial. For applying Newton's second law of motion to any object w.r.t. M, you have to apply an inertial force (also called pseudo force) on the object which equals mass of the object times acceleration of M, directed opposite to A.

Force equations for m:



Let the acceleration of m w.r.t. M be a, in the horizontal direction toward right.

The forces acting on m are:

- Weight of m, mg, acting vertically downwards
- Normal force on m by M, N', vertically upwards, and
- (mA), the inertial force acting along the incline upwards as the acceleration of M is A along the incline downwards.

$$N' + (mA)\sin\theta = mg \quad \text{...(i)}$$

$$(mA)\cos\theta = ma \quad \text{...(ii)}$$

Force equations for M:

The forces acting on M are

- The weight Mg
- N', normal force exerted by m
- N, normal force exerted by the incline

$$Mg\sin\theta + N'\sin\theta = MA \quad \text{...(iii)}$$

(Along the incline)

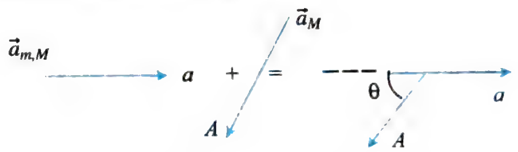
$$Mg\cos\theta + N'\cos\theta = N \quad \text{...(iv)}$$

(Perpendicular to the incline)

From (i), (iii) and (iv),

$$A = \frac{(M+m)g\sin\theta}{M+m\sin^2\theta}$$

Acceleration of m relative to ground,



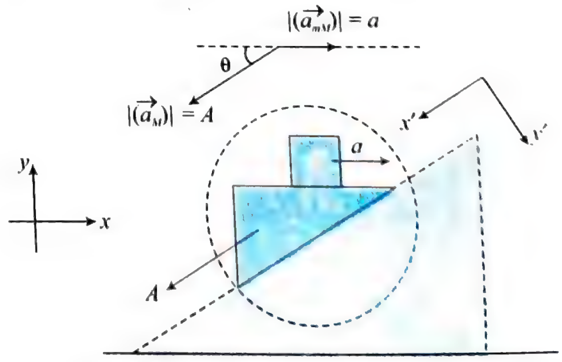
$$\vec{a}_m = \vec{a}_{mM} + \vec{a}_M = a\hat{i} + A\cos\theta(-\hat{i}) + A\sin\theta(-\hat{j})$$

But $\quad a = A\cos\theta$ [from (ii)], therefore,

$a_m = A\sin\theta$, vertically downward

$$= \frac{(M+m)g\sin^2\theta}{M+m\sin^2\theta}$$

Method 3: Considering M and m as system.



The system has $(M+m)g\sin\theta$ external force parallel to incline plane.

$$\vec{F}_{x'} = M(\vec{a}_M)_{x'} + m(\vec{a}_m)_{x'}$$

$$(\vec{a}_m)_{x'} = (\vec{a}_{m,M})_{x'} + (\vec{a}_M)_{x'}$$

$$= -a\cos\theta + A = (A - a\cos\theta)$$

$$MA + m(A - a\cos\theta) = (M+m)g\sin\theta$$

Considering the motion of block in horizontal direction only and writing equation of motion w.r.t. ground.

$$(\vec{a}_m)_x = a - A\cos\theta$$

The block is not having any external force in horizontal direction.

$$\Sigma \vec{F}_x = 0 = m(\vec{a}_m)_x = m(a - A\cos\theta)$$

or $\quad a = A\cos\theta$ ...(ii)

From (i) and (ii),

$$MA + m[A - (A\cos\theta)\cos\theta] = (M+m)g\sin\theta$$

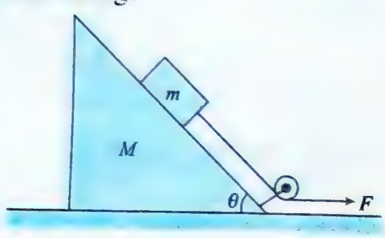
$$MA + mA\sin^2\theta = (M+m)g\sin\theta$$

After solving, we get

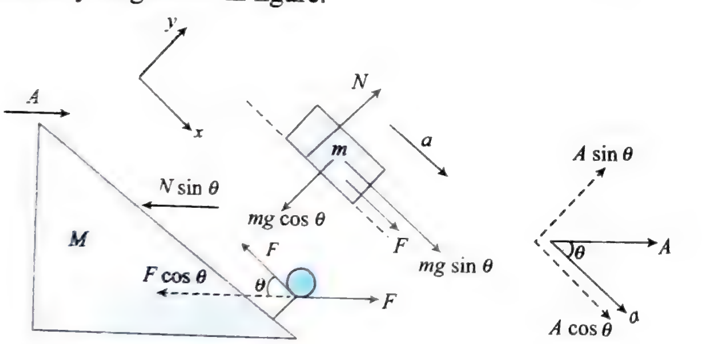
$$A = \frac{(M+m)g\sin\theta}{M + m\sin^2\theta}$$

EXAMPLE 5.15

In the given figure, mass m is being pulled on the incline of a wedge of mass M. All the surfaces are smooth. Find the acceleration of the wedge.



Sol. Method 1: Let the acceleration of block relative to wedge be $\vec{a}_{mM} = \vec{a}$ and acceleration of wedge on ground be $\vec{a}_M = \vec{A}$. Free-body diagram as in figure.



Equation of motion of M,

$$F - N\sin\theta - F\cos\theta = MA \quad \text{(i)}$$

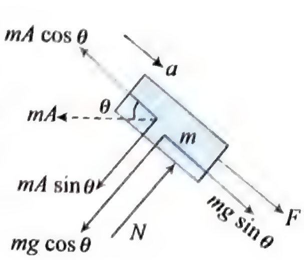
Equation of motion of m in y-direction,

$$N - mg\cos\theta = ma_y = mA\sin\theta \quad \text{(ii)}$$

Substituting the value of N from (ii) in (i), we can get the value of A.

$$A = \frac{F(1-\cos\theta) - mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

Method 2: If we make the FBD of m w.r.t. M, pseudo force mA is to be applied because M is non-inertial frame. W.r.t. M, there is no acceleration of m in y-direction, so balancing the forces in y-direction,

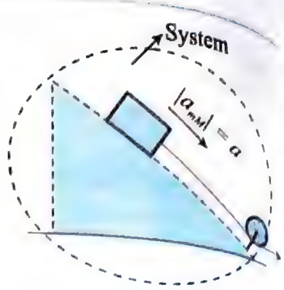


$$N = mg\cos\theta + mA\sin\theta$$

which is same as (ii).

Method 3: Taking block and wedge as system.

External force acting in horizontal direction is F.



$$\Sigma F_x = m(a_m)_x + M(a_M)_x$$

Let the acceleration of block w.r.t wedge is a. or $|\vec{a}_{m,M}| = a$

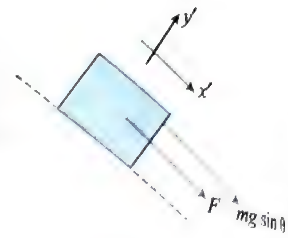
$$(\vec{a}_m)_x = (\vec{a}_{m,M})_x + (\vec{a}_M)_x$$

$$|(\vec{a}_m)_x| = a\cos\theta + A$$

Hence,

$$F = m(a\cos\theta + A) + MA \quad \text{...(i)}$$

Now analyzing the block parallel to the sloping surface and writing equation w.r.t. ground,



$$\Sigma \vec{F}_{x'} = m(\vec{a}_m)_{x'}$$

$$(\vec{a}_m)_{x'} = (\vec{a}_{m,M})_{x'} + (\vec{a}_M)_{x'} = a + A\cos\theta$$

$$F + mg\sin\theta = m(a + A\cos\theta) \quad \text{...(ii)}$$

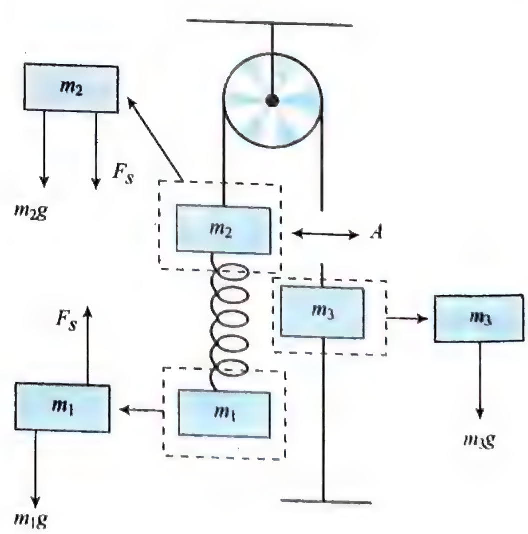
From (i) and (ii),

$$A = \frac{F(1-\cos\theta) - mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

EXAMPLE 5.16

Three blocks are arranged with pulley and spring as shown in figure. If the string connecting blocks m_2 and m_3 is cut at point A, find the accelerations of masses m_1, m_2, and m_3, just after the string is cut at point A.

Sol. Let us analyze the system at equilibrium. The forces acting on the blocks are shown in figure.



Just after cutting the string at A, the tensions T and T' will be zero. But the spring force will remain unchanged just after cutting. The figure shows the forces acting on the blocks just after cutting the string.

The forces acting on m_1 just before and just after cutting the string are same and just before cutting string m_1 were in equilibrium. Hence, the acceleration of m_1 will be zero just after cutting the string.

At equilibrium, $F_S = m_1 g$

As F_S does not change just before and just after cutting the string, acceleration of m_2

$$a_2 = \frac{m_2 g + F_S}{m_2} = \frac{m_2 g + m_1 g}{m_2} = \left(1 + \frac{m_1}{m_2}\right) g$$

From the free-body diagram of block m_3 just after cutting the string, only $m_3 g$ will act and tensions in the string will disappear. Hence, acceleration of m_3,

$$a_3 = \frac{m_3 g}{m_3} = g$$

Finally, acceleration of $m_1 = 0$

Acceleration of $m_2 = \left(1 + \frac{m_1}{m_2}\right) g$ and acceleration of $m_3 = g$.

Exercises

Single Correct Answer Type

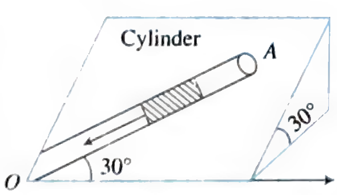
1. A wooden box is placed on a table. The normal force on the box from the table is N_1. Now another identical box is kept on first box and the normal force on lower block due to upper block is N_2 and normal force on lower block by the table is N_3. For this situation, mark out the correct statement(s).
   (1) N_1 [illegible] N_2 [illegible] N_3 (2) $N_1 < N_2 = N_3$
   (3) N_1 [illegible] N_2 [illegible] N_3 (4) $N_1 = N_2 > N_3$

2. Three forces are acting on a particle of mass m initially in equilibrium. If the first two forces (R_1 and R_2) are perpendicular to each other and suddenly the third force (R_3) is removed, then the acceleration of the particle is
   (1) $\frac{R_3}{m}$ (2) $\frac{R_1+R_2}{m}$
   (3) $\frac{R_1-R_2}{m}$ (4) $\frac{R_1}{m}$

3. Two skaters have weight in the ratio 4 : 5 and are 9 m apart, on a smooth frictionless surface. They pull on a rope stretched between them. The ratio of the distance covered by them when they meet each other will be
   (1) 5 : 4 (2) 4 : 5
   (3) 25 : 16 (4) 16 : 25

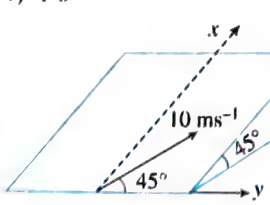
4. A plumb bob is hung from the ceiling of a train compartment. The train moves on an inclined track of inclination 30° with horizontal. The acceleration of train up the plane is $a = g/2$. The angle which the string supporting the bob makes with normal to the ceiling in equilibrium is
   (1) 30° (2) $\tan^{-1}(2/\sqrt{3})$
   (3) $\tan^{-1}(\sqrt{3}/2)$ (4) $\tan^{-1}(2)$

5. An inclined plane makes an angle 30° with the horizontal. A groove (OA) of length 5 m cut in the plane makes an angle 30° with OX. A short smooth cylinder is free to slide down under the influence of gravity. The time taken by the cylinder to reach from A to O is ($g = 10\ \text{m s}^{-2}$)



   (1) 4 s (2) 2 s
   (3) 3 s (4) 1 s

6. The small marble is projected with a velocity of 10 m s^{-1} in a direction 45° from the horizontal y-direction on the smooth inclined plane. Calculate the magnitude v of its velocity after 2 s.

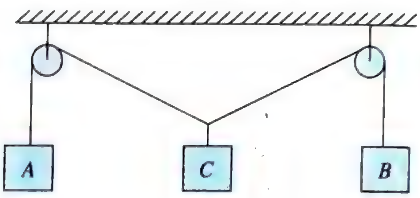


   (1) $10\sqrt{2}$ m s^{-1} (2) 5 m s^{-1}
   (3) 10 m s^{-1} (4) $5\sqrt{2}$ m s^{-1}

7. A lift is moving down with an acceleration a. A man in the lift drops a ball inside the lift. The acceleration of the ball as observed by the man in the lift, and a man standing stationary on the ground are, respectively,
   (1) a, g (2) $(g-a), g$
   (3) a, a (4) g, g

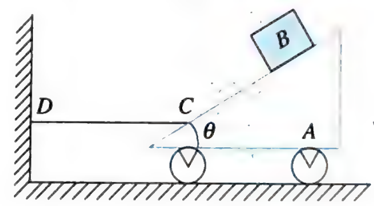
8. A particle of mass 2 kg moves with an initial velocity of $\vec{v} = 4\hat{i} + 4\hat{j}$ m s^{-1}. A constant force of $\vec{F} = 20\hat{j}$ N is applied on the particle. Initially, the particle was at (0, 0). The x-coordinate of the particle when its y-coordinate again becomes zero is given by
   (1) 1.2 m (2) 4.8 m
   (3) 6.0 m (4) 3.2 m

9. Three blocks A, B, and C are suspended as shown in figure. Mass of each of blocks A and B is m. If the system is in equilibrium, and mass of C is M, then



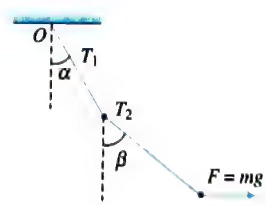
   (1) $M > 2m$ (2) $M = 2m$
   (3) $M < 2m$ (4) None of these

10. Block B has mass m and is released from rest when it is on top of wedge A, which has a mass $3m$. Determine the tension in cord CD needed to hold the wedge from moving while B is sliding down A. Neglect friction.



   (1) $2mg\cos\theta$ (2) $\frac{mg}{2}\cos\theta$
   (3) $\frac{mg}{2}\sin 2\theta$ (4) $mg\sin 2\theta$

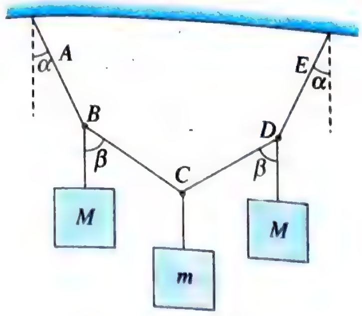
11. Two particles A and B, each of mass m, are kept stationary by applying a horizontal force $F = mg$ on particle B as shown in figure. Then



   (1) $2\tan\beta = \tan\alpha$ (2) $2T_1 = 5T_2$
   (3) $T_1\sqrt{2} = T_2\sqrt{5}$ (4) None of these

12. The given figure represents a light inextensible string $ABCDE$ in which $AB = BC = CD = DE$ and to which are attached masses M, m, and M at the points B, C, and D,

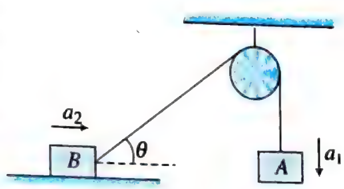
respectively. The system hangs freely in equilibrium with ends A and E of the string fixed in the same horizontal line. It is given that $\tan\alpha = 3/4$ and $\tan\beta = 12/5$. Then the tension in the string BC is



(1) $2mg$ (2) $(13/10)mg$
(3) $(3/10)mg$ (4) $(20/11)mg$

13. A particle is moving in the x-y plane. At certain instant of time, the components of its velocity and acceleration are as follows: $v_x = 3\ \text{m s}^{-1}$, $v_y = 4\ \text{m s}^{-1}$, $a_x = 2\ \text{m s}^{-2}$ and $a_y = 1\ \text{m s}^{-2}$. The rate of change of speed at this moment is
(1) $\sqrt{10}\ \text{m s}^{-2}$ (2) $4\ \text{m s}^{-2}$
(3) $\sqrt{5}\ \text{m s}^{-2}$ (4) $2\ \text{m s}^{-2}$

14. The given figure shows two blocks, each of mass m. The system is released from rest. If accelerations of blocks A and B at any instant (not initially) are a_1 and a_2, respectively, then



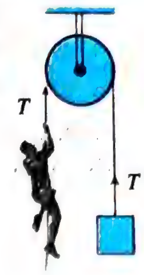
(1) $a_1 = a_2 \cos\theta$ (2) $a_2 = a_1 \cos\theta$
(3) $a_1 = a_2$ (4) None of these

15. A block is lying on the horizontal frictionless surface. One end of a uniform rope is fixed to the block which is pulled in the horizontal direction by applying a force F at the other end. If the mass of the rope is half the mass of the block, the tension in the middle of the rope will be
(1) F (2) $2F/3$
(3) $3F/5$ (4) $5F/6$

16. A 60-kg man stands on a spring scale in a lift. At some instant, he finds that the scale reading has changed from 60 kg to 50 kg for a while and then comes back to original mark. What should be concluded?
(1) The lift was in constant motion upward.
(2) The lift was in constant motion downward.
(3) The lift while in motion downward suddenly stopped.
(4) The lift while in motion upward suddenly stopped.

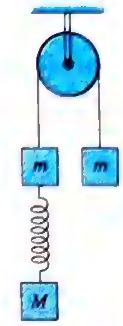
17. Inside a horizontally moving box, an experimenter finds that when an object is placed on a smooth horizontal table and is released, it moves with an acceleration of $10\ \text{m s}^{-2}$. In this box, if 1-kg body is suspended with a light string, the tension in the string in equilibrium position. (w.r.t. experimenter) will be (take $g = 10\ \text{m s}^{-2}$)
(1) 10 N (2) $10\sqrt{2}$ N
(3) 20 N (4) Zero

18. In order to raise a mass of 100 kg, a man of mass 60 kg fastens a rope to it and passes the rope over a smooth pulley. He climbs the rope with acceleration $5g/4$ relative to the rope. The tension in the rope is (take $g = 10\ \text{m s}^{-2}$)



(1) $\frac{4875}{8}$ N (2) $\frac{4875}{2}$ N
(3) $\frac{4875}{4}$ N (4) $\frac{4875}{6}$ N

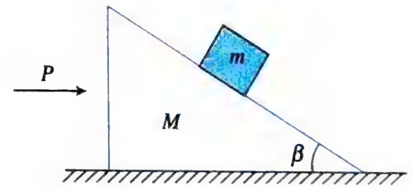
19. The system shown in figure is released from rest. The spring gets elongated



(1) If $M > m$
(2) If $M > 2m$
(3) If $M > m/2$
(4) For any value of M
(Neglect the friction and masses of pulley, string, and spring.)

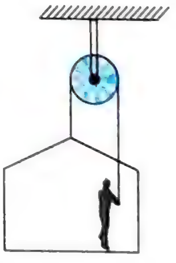
20. A balloon of mass M is descending at a constant acceleration α. When a mass m is released from the balloon, it starts rising with the same acceleration α. Assuming that its volume does not change, what is the value of m?
(1) $\frac{\alpha}{\alpha+g}M$ (2) $\frac{2\alpha}{\alpha+g}M$
(3) $\frac{\alpha+g}{\alpha}M$ (4) $\frac{\alpha+g}{2\alpha}M$

21. Two wooden blocks are moving on a smooth horizontal surface such that the mass m remains stationary with respect to the block of mass M as shown in figure. The magnitude of force P is



(1) $(M + m)g \tan\beta$ (2) $g \tan\beta$
(3) $mg \cos\beta$ (4) $(M + m)g \operatorname{cosec}\beta$

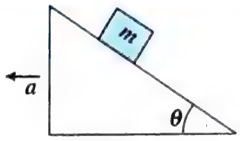
22. A man is raising himself as well as the crate on which he stands with an acceleration of $5\ \text{m s}^{-2}$ by a massless rope-and-pulley arrangement. Mass of the man is 100 kg and that of the crate is 50 kg. If $g = 10\ \text{m s}^{-2}$, then the tension in the rope is



(1) 2250 N (2) 1125 N
(3) 750 N (4) 375 N

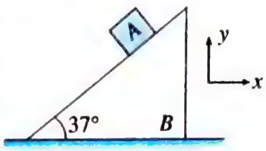
23. In the previous problem, the contact force between the man and the crate is
(1) 2250 N (2) 1125 N
(3) 750 N (4) 375 N

24. A small block of mass m rests on a smooth wedge of angle θ. With what horizontal acceleration a should the wedge be pulled, as shown in figure, so that the block falls freely?



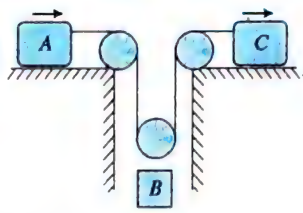
(1) $g\cos\theta$ (2) $g\sin\theta$
(3) $g\cot\theta$ (4) $g\tan\theta$

25. In the given figure, the acceleration of A is $\vec{a}_A = 15\hat{i}+15\hat{j}$. Then the acceleration of B is (A remains in contact with B)



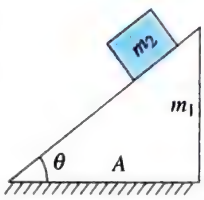
(1) $6\hat{i}$ (2) $-15\hat{i}$
(3) $-10\hat{i}$ (4) $-5\hat{i}$

26. Blocks A and C start from rest and move to the right with acceleration $a_A = 12t$ m s^{-2} and $a_C = 3$ m s^{-2}. Here t is in seconds. The time when block B again comes to rest is



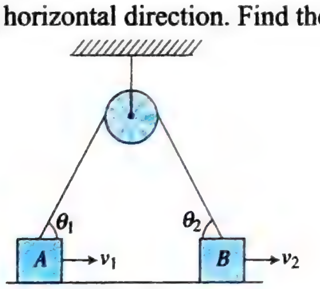
(1) 2 s (2) 1 s
(3) 3/2 s (4) 1/2 s

27. In the given figure, the mass m_2 starts with velocity v_0 and moves with constant velocity on the surface. During motion, the normal reaction between the horizontal surface and fixed triangle block m_1 is N. Then during motion



(1) $N=(m_1+m_2)g$ (2) $N=m_1g$
(3) $N<(m_1+m_2)g$ (4) $N>(m_1+m_2)g$

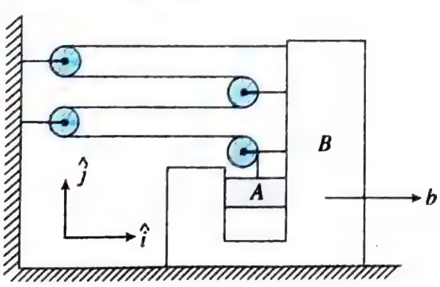
28. In the given figure, blocks A and B move with velocities v_1 and v_2 along horizontal direction. Find the ratio of v_1/v_2.



(1) $\dfrac{\sin\theta_1}{\sin\theta_2}$ (2) $\dfrac{\sin\theta_2}{\sin\theta_1}$

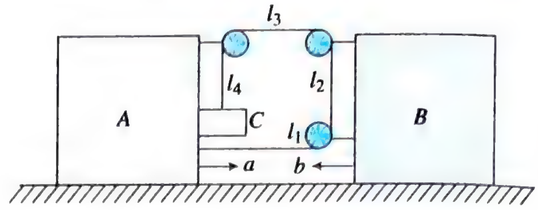
(3) $\dfrac{\cos\theta_2}{\cos\theta_1}$ (4) $\dfrac{\cos\theta_1}{\cos\theta_2}$

29. If block B moves towards right with acceleration b, find the net acceleration of block A.



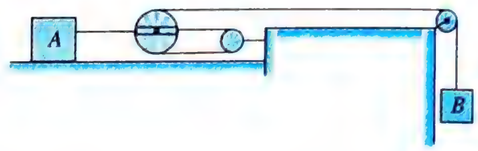
(1) $b\hat{i}+4b\hat{j}$ (2) $b\hat{i}+b\hat{j}$
(3) $b\hat{i}+2b\hat{j}$ (4) None of these

30. If the blocks A and B are moving towards each other with accelerations a and b as shown in figure, then find the net acceleration of block C.



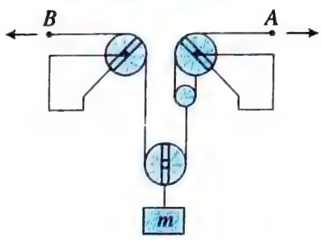
(1) $a\hat{i}-2(a+b)\hat{j}$ (2) $-(a+b)\hat{j}$
(3) $a\hat{i}-(a+b)\hat{j}$ (4) None of these

31. A block A has a velocity of 0.6 m s^{-1} to the right. Determine the velocity of cylinder B.



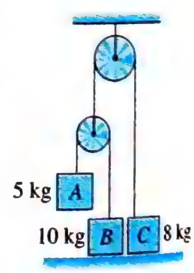
(1) 1.2 m s^{-1} (2) 2.4 m s^{-1}
(3) 1.8 m s^{-1} (4) 3.6 m s^{-1}

32. For the pulley system shown in figure, each of the cables at A and B is given a velocity of 2 m s^{-1} in the direction of the arrow. Determine the upward velocity v of the load m.



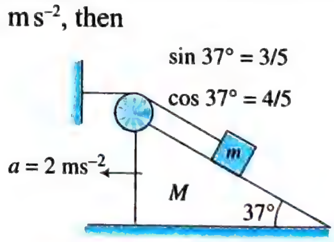
(1) 1.5 m s^{-1} (2) 3 m s^{-1}
(3) 6 m s^{-1} (4) 4.5 m s^{-1}

33. In the following arrangement, the system is initially at rest. The 5-kg block is now released. Assuming the pulleys and string to be massless and smooth, the acceleration of block C will be



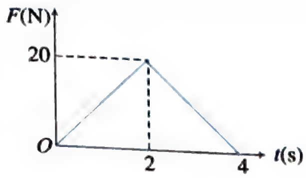
(1) Zero (2) 2.5 m s^{-2}
(3) 10/7 m s^{-2} (4) 5/7 m s^{-2}

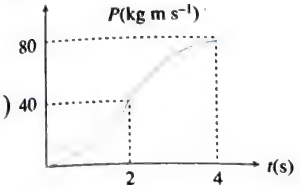
34. As shown in figure, if acceleration of M with respect to ground is 2 m s^{-2}, then

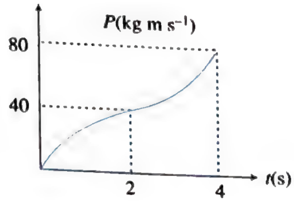


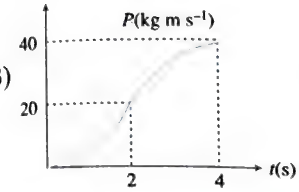
(1) Acceleration of m with respect to M is 5 m s^{-2}.
(2) Acceleration of m with respect to ground is 5 m s^{-2}
(3) Acceleration of m with respect to M is 2 m s^{-2}
(4) Acceleration of m with respect to ground is 10 m s^{-2}.

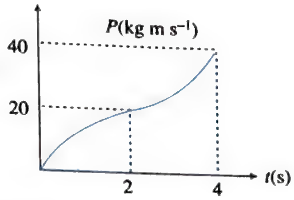
35. Figure shows the variation of force acting on a body with time. Assuming the body to start from rest, the variation of its momentum with time is best represented by which plot?



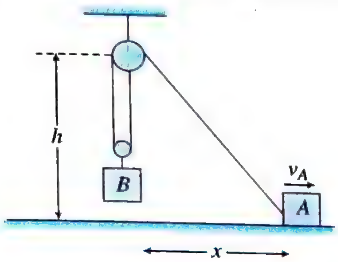
(1)


(2)


(3)


(4)


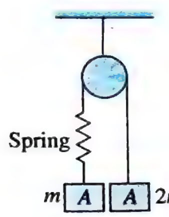
36. If block A is moving horizontally with velocity v_A, then find the velocity of block B at the instant as shown in Figure.



(1) $\dfrac{hv_A}{2\sqrt{x^2+h^2}}$ (2) $\dfrac{xv_A}{\sqrt{x^2+h^2}}$

(3) $\dfrac{xv_A}{2\sqrt{x^2+h^2}}$ (4) $\dfrac{hv_A}{\sqrt{x^2+h^2}}$

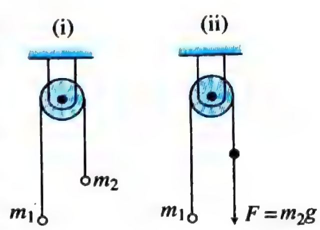
37. Two blocks A and B of masses m and $2\,m$, respectively, are held at rest such that the spring is in natural length. Find out the accelerations of both the blocks just after release.



(1) $g\downarrow, g\downarrow$ (2) $\dfrac{g}{3}\downarrow, \dfrac{g}{3}\uparrow$

(3) 0, 0 (4) $g\downarrow$, c

38. In two pulley–particle systems (i) and (ii), the acceleration and force imparted by the string on the pulley and tension in the strings are (a_1, a_2), (N_1, N_2), and (T_1, T_2), respectively. Ignoring friction in all contacting surfaces, study the following statements:



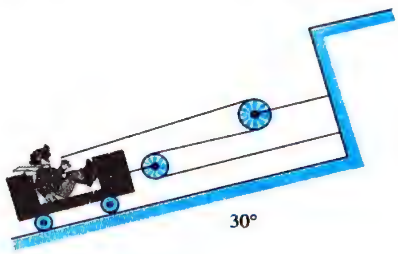
(1) $\dfrac{a_1}{a_2}=1$ (2) $\dfrac{T_1}{T_2}<1$

(3) $\dfrac{N_1}{N_2}>1$ (4) $\dfrac{a_1}{a_2}<1$

Now mark the correct answer:

(1) Relations (ii) and (iii) always follow.
(2) Relations (ii) and (iv) always follow.
(3) Only relation (i) always follows.
(4) Only relation (iv) always follows.

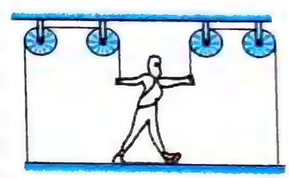
39. A man pulls himself up the 30° incline by the method shown in figure. If the combined mass of the man and cart is 100 kg, determine the acceleration of the cart if the man exerts a pull of 250 N on the rope. Neglect all friction and the mass of the rope, pulleys, and wheels.



(1) 4.5 m s^{-2} (2) 2.5 m s^{-2}

(3) 3.5 m s^{-2} (4) 1.5 m s^{-2}

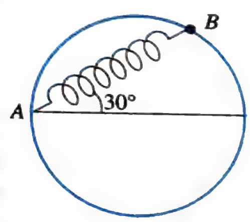
40. A painter of mass M stands on a platform of mass m and pulls himself up by two ropes which hang over pulley as shown in figure. He pulls each rope with force F and moves upward with a uniform acceleration a. Find a, neglecting the fact that no one could do this for long time.



(1) $\dfrac{4F+(2M+m)g}{M+2m}$ (2) $\dfrac{4F+(M+m)g}{M+2m}$

(3) $\dfrac{4F-(M+m)g}{M+m}$ (4) $\dfrac{4F-(M+m)g}{2M+m}$

41. A bead of mass m is attached to one end of a spring of natural length R and spring constant $K=\dfrac{(\sqrt{3}+1)mg}{R}$. The other end of the spring is fixed at a point A on a smooth vertical ring of radius R as shown in figure. The normal reaction at B just after it is released to move is



(1) $mg/2$ (2) $\sqrt{3}\,mg$

(3) $3\sqrt{3}\,mg$ (4) $\dfrac{3\sqrt{3}\,mg}{2}$

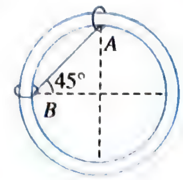
42. Two objects A and B, each of mass m, are connected by a light inextensible string. They are restricted to move on a frictionless ring of radius R in a vertical plane (as shown in figure). The objects are released from rest at the position shown. Then the tension in the cord just after release is

(1) Zero

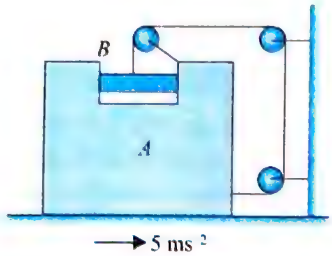
(2) mg

(3) $\sqrt{2}\, mg$

(4) $mg/\sqrt{2}$



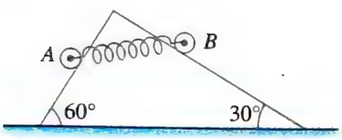
43. If block A is moving with an acceleration of 5 m s^{-2}, the acceleration of B w.r.t ground is



(1) 5 m s^{-2} (2) $5\sqrt{2}$ m s^{-2}

(3) $5\sqrt{5}$ m s^{-2} (4) 10 m s^{-2}

44. Two uniform solid cylinders A and B each of mass 1 kg are connected by a spring of constant 200 N m^{-1} at their axles and are placed on a fixed wedge as shown in the figure. There is no friction between cylinders and wedge. The angle made by the line AB with the horizontal, in equilibrium, is



(1) 0° (2) 15°

(3) 30° (4) None of these

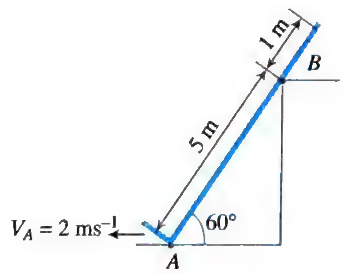
45. The velocity of point A on the rod is 2 m s^{-1} (leftwards) at the instant shown in figure. The velocity of the point B on the rod at this instant is

(1) $\dfrac{2}{\sqrt{3}}$ m s^{-1}

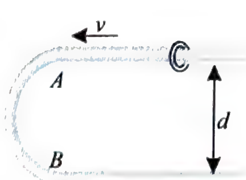
(2) 1 m s^{-1}

(3) $\dfrac{1}{2\sqrt{3}}$ m s^{-1}

(4) $\dfrac{\sqrt{3}}{2}$ m s^{-1}



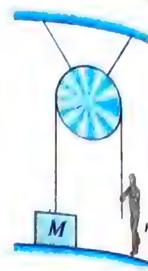
46. A fixed U-shaped smooth wire has a semi-circular bending between A and B as shown in figure. A bead of mass m moving with uniform speed v through the wire enters the semicircular bend at A and leaves at B. The average force exerted by the bead on the part AB of the wire is



(1) 0 (2) $\dfrac{4mv^2}{\pi d}$

(3) $\dfrac{2mv^2}{\pi d}$ (4) None of these

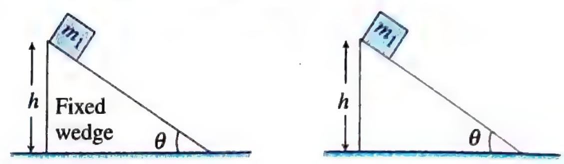
47. In figure, the block of mass M is at rest on the floor. The acceleration with which a boy of mass m should climb along the rope of negligible mass so as to lift the block from the floor is



(1) $\left(\dfrac{M}{m}-1\right)g$ (2) $\left(\dfrac{M}{m}-1\right)g$

(3) $\dfrac{M}{m}g$ (4) $>\dfrac{M}{m}g$

48. A block of mass m_1 lies on the top of fixed wedge as shown in Fig. (a) and another block of mass m_2 lies on top of wedge which is free to move as shown in Fig. (b). At time $t = 0$, both the blocks are released from rest from a vertical height h above the respective horizontal surface on which the wedge is placed as shown. There is no friction between block and wedge in both the figures. Let T_1 and T_2 be the time taken by the blocks, respectively, to just reach the horizontal surface. Then

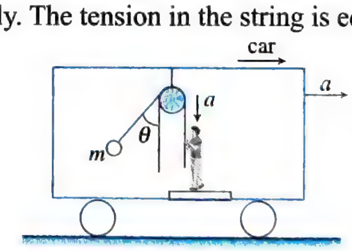


(a) Horizontal surface **(b) Smooth horizontal surface**

(1) $T_1 > T_2$ (2) $T_1 < T_2$

(3) $T_1 = T_2$ (4) Data insufficient

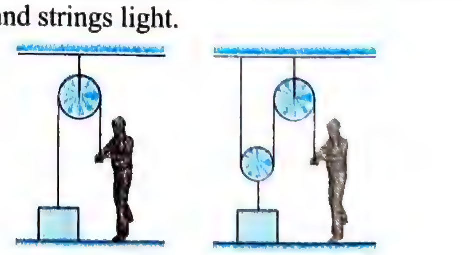
49. A bob is hanging over a pulley inside a car through a string. The second end of the string is in the hands of a person standing in the car. The car is moving with constant acceleration a directed horizontally as shown in Figure. The other end of the string is pulled with constant acceleration a vertically. The tension in the string is equal to



(1) $m\sqrt{g^2+a^2}$ (2) $m\sqrt{g^2+a^2}-ma$

(3) $m\sqrt{g^2+a^2}+ma$ (4) $m(g+a)$

50. In figure, a person wants to rise a block lying on the ground to a height h. In both the cases, if the time required is same, then in which case he has to exert more force? Assume pulleys and strings light.

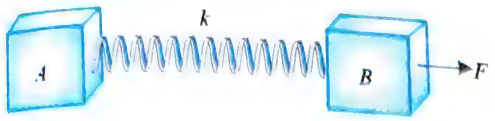


(i) **(ii)**

(1) (i) (2) (ii)

(3) Same in both (4) Cannot be determined

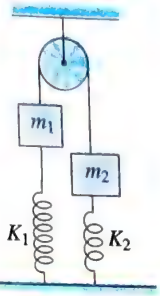
51. Two identical particles A and B, each of mass m, are interconnected by a spring of stiffness k. If particle B experiences a force F and the elongation of the spring is x, the acceleration of particle B relative to particle A is equal to



(1) $\frac{F}{2m}$ (2) $\frac{F-kx}{m}$

(3) $\frac{F-2kx}{m}$ (4) $\frac{kx}{m}$

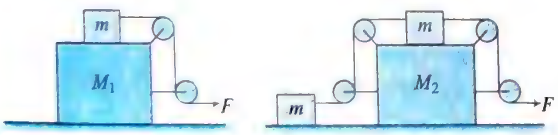
52. The system shown in figure is in equilibrium. Masses m_1 and m_2 are 2 kg and 8 kg, respectively. Spring constants k_1 and k_2 are 50 N m^{-1} and 70 N m^{-1}, respectively. If the compression in second spring is 0.5 m. What is the compression in first spring? (Both springs have natural length initially.)



(1) 1.3 m (2) – 0.5 m

(3) 0.5 m (4) 0.9 m

53. In the situation shown in figure, all the strings are light and inextensible and pullies are light. There is no friction at any surface and all blocks are of cuboidal shape. A horizontal force of magnitude F is applied to right most free end of string in both cases shown in the figure. At the instant shown, the tension in all strings are non-zero. Let the magnitudes of acceleration of large blocks (of mass M) in Fig.(a) and Fig. (b) be a_1 and a_2, respectively. Then,

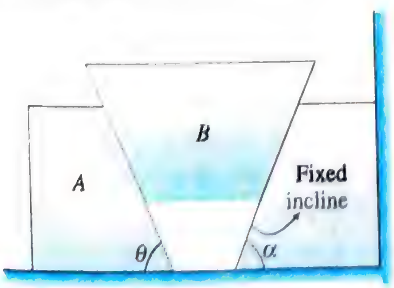


Smooth horizontal surface (a) **Smooth horizontal surface (b)**

(1) $a_1 = a_2 \neq 0$ (2) $a_1 = a_2 = 0$

(3) $a_1 > a_2$ (4) $a_1 < a_2$

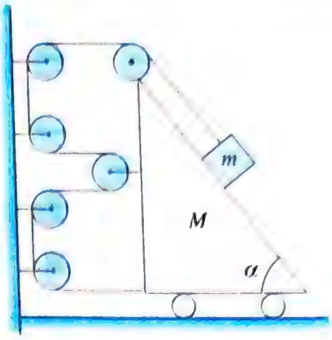
54. In the arrangement shown in figure, if the acceleration of B is $\vec{a}$, then find the acceleration of A.



(1) $a \sin \alpha$ (2) $a \cot \theta$

(3) $a \tan \theta$ (4) $a(\sin \alpha \cot \theta + \cos \alpha)$

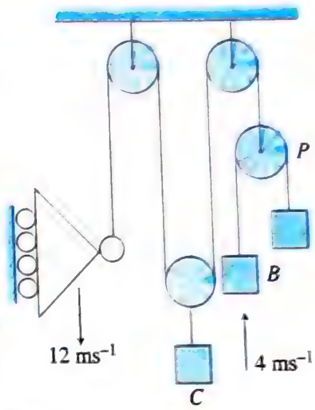
55. If the acceleration of wedge in the shown arrangement is $a\ \text{m s}^{-2}$ towards left, then at this instant, acceleration of the block (magnitude only) would be



(1) $4a\ \text{m s}^{-2}$ (2) $a\sqrt{17-8\cos\alpha}\ \text{m s}^{-2}$

(3) $(\sqrt{17})a\ \text{m s}^{-2}$ (4) $\sqrt{17}\cos\frac{\alpha}{2} \times a\ \text{m s}^{-2}$

56. In the arrangement shown in figure at a particular instant, the roller is coming down with a speed of 12 m s^{-1} and C is moving up with 4 m s^{-1}. At the same instant, it is also known that w.r.t. pulley P, block A is moving down with speed 3 m s^{-1}. Determine the motion of block B (velocity) w.r.t. ground.



(1) 4 m s^{-1} in downward direction

(2) 3 m s^{-1} in upward direction

(3) 7 m s^{-1} in downward direction

(4) 7 m s^{-1} in upward direction

57. A particle of mass 2 kg moves with an initial velocity of $(4\hat{i}+2\hat{j})\ \text{m s}^{-1}$ on the x-y plane. A force $\vec{F} = (2\hat{i}-8\hat{j})$ N acts on the particle. The initial position of the particle is (2 m, 3 m). Then for $y = 3$ m,

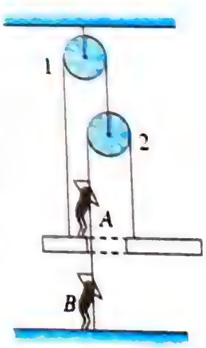
(1) Possible value of x is only $x = 2$ m

(2) Possible value of x is not only $x = 2$ m, but there exists some other value of x also

(3) Time taken is 2 s

(4) All of the above

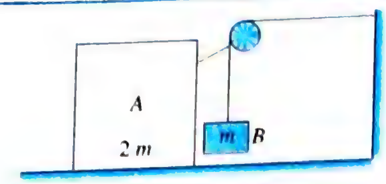
58. In figure, man A is standing on a movable plank while man B is standing on a stationary platform. Both are pulling the string down such that the plank moves slowly up. As a result of this the string slips through the hands of the men. Find the ratio of length of the string that slips through the hands of A and B.



(1) 3/2 (2) 3/4

(3) 4/3 (4) 2/3

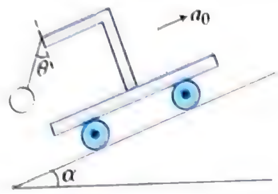
59. In the system shown, all the surfaces are frictionless while pulley and string are massless. The mass of block A is $2m$ and that of block B is m. The acceleration of block B immediately after system is released from rest is



(1) g/2 (2) g

(3) g/3 (4) none of these

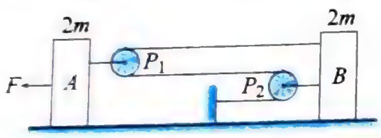
60. A pendulum of mass m hangs from a support fixed to a trolley. The direction of the string when the trolley rolls up a plane of inclination α with acceleration a_0 is



(1) $\theta = \tan^{-1} \alpha$ (2) $\theta = \tan^{-1}\left(\dfrac{a_0}{g}\right)$

(3) $\theta = \tan^{-1}\left(\dfrac{g}{a_0}\right)$ (4) $\theta = \tan^{-1}\left(\dfrac{a_0 + g\sin\alpha}{g\cos\alpha}\right)$

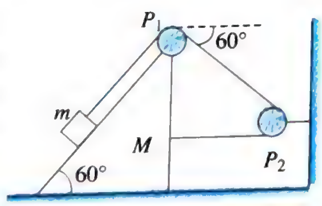
61. The acceleration of the block B in figure, assuming the surfaces and the pulleys P_1 and P_2 are all smooth, is



(1) $\dfrac{F}{4m}$ (2) $\dfrac{3F}{13m}$

(3) $\dfrac{F}{2m}$ (4) $\dfrac{3F}{17m}$

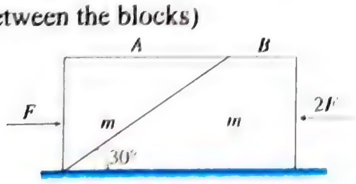
62. In the arrangement shown in figure the block of mass m = 2 kg lies on the wedge of mass M = 8 kg. The initial acceleration of the wedge, if the surfaces are smooth, is



(1) $\dfrac{\sqrt{3}g}{23}\,\mathrm{m\,s^{-2}}$ (2) $\dfrac{3\sqrt{3}g}{23}\,\mathrm{m\,s^{-2}}$

(3) $\dfrac{3g}{23}\,\mathrm{m\,s^{-2}}$ (4) $\dfrac{g}{23}\,\mathrm{m\,s^{-2}}$

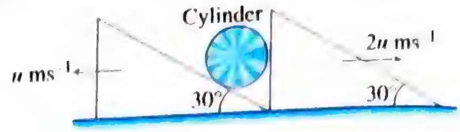
63. Two blocks A and B each of mass m are placed on a smooth horizontal surface. Two horizontal forces F and $2F$ are applied on blocks A and B, respectively, as shown in figure. Block A does not slide on block B. Then the normal reaction acting between the two blocks is (assume no friction between the blocks)



(1) F (2) $F/2$

(3) $\dfrac{F}{\sqrt{3}}$ (4) $3F$

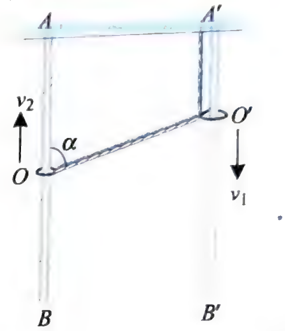
64. A system is shown in Figure. Assume that the cylinder remains in contact with the two wedges. Then the velocity of cylinder is



(1) $\sqrt{19 - 4\sqrt{3}}\ \mathrm{m\,s^{-1}}$ (2) $\dfrac{\sqrt{13}\,u}{2}\ \mathrm{m\,s^{-1}}$

(3) $\sqrt{3}\,u\ \mathrm{m\,s^{-1}}$ (4) $\sqrt{7}\ \mathrm{m\,s^{-1}}$

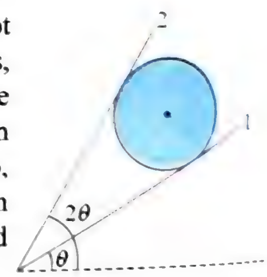
65. Two small rings O and O' are put on two vertical stationary rods AB and $A'B'$, respectively. One end of an inextensible thread is tied at point A'. The thread passes through ring O' and its other end is tied to ring O. Assuming that ring O' moves downwards at a constant velocity v_1, then velocity v_2 of the ring O, when $\angle AOO' = \alpha$, is



(1) $v_1\left[\dfrac{2\sin^2 \alpha/2}{\cos\alpha}\right]$ (2) $v_1\left[\dfrac{2\cos^2 \alpha/2}{\sin\alpha}\right]$

(3) $v_1\left[\dfrac{3\cos^2 \alpha/2}{\sin\alpha}\right]$ (4) None of these

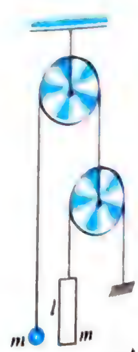
66. A sphere of mass m is kept between two inclined walls, as shown in the figure. If the coefficient of friction between each wall and the sphere is zero, then the ratio of normal reaction (N_1/N_2) offered by the walls 1 and 2 on the sphere will be



(1) $\tan\theta$ (2) $\tan 2\theta$

(3) $2\cos\theta$ (4) $\cos 2\theta$

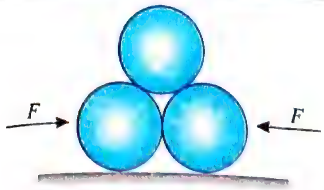
67. In the figure shown, all pulleys are massless and frictionless. The time taken by the ball to reach the upper end of the rod is:



(1) $\sqrt{\dfrac{10l}{3g}}$ (2) $\sqrt{\dfrac{5l}{3g}}$

(3) $\sqrt{\dfrac{3l}{4g}}$ (4) $\sqrt{\dfrac{3l}{10g}}$

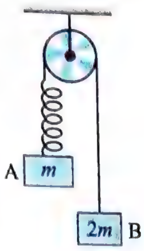
68. Two smooth cylindrical bars weighing W N each lie next to each other in contact. A similar third bar is placed over the two bars as shown in figure. Neglecting friction, the minimum horizontal force on each lower bar necessary to keep them together is



(1) $\frac{W}{2}$ (2) EW

(3) $\frac{W}{\sqrt{3}}$ (4) $\frac{W}{2\sqrt{3}}$

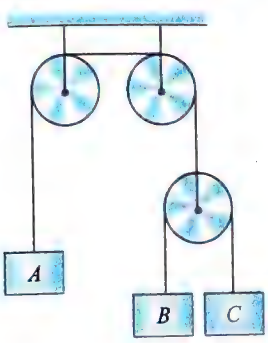
69. In the figure a block A of mass m is attached at one end of a light spring and the other end of the spring is connected to another block B of mass $2m$ through a light string. A is held and B has obtained equilibrium. Now A is released. The acceleration of A just after that instant is a. The same thing is repeated for B. In that case the acceleration of B is b. Then the value of a/b is



(1) 2 (2) 1

(3) 3 (4) 1.5

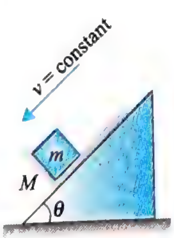
70. In the system shown, block A is of mass 4.0 kg and blocks B and C are of equal mass each of 3 kg. Find the acceleration in m/s² of block C, if the system is set free, [g = 10 m/s²]



(1) 1 m/s² (2) 2 m/s²

(3) 1.5 m/s² (4) 3 m/s²

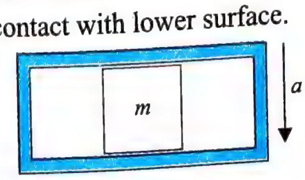
71. The given figure shows a block of mass m placed on a bracket of mass M. Bracket block system is moved downward with constant velocity on an incline. What is the magnitude of total force of bracket on block?

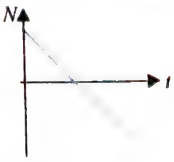


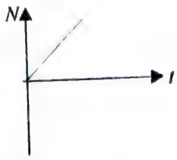
(1) zero (2) $mg \sin \theta$

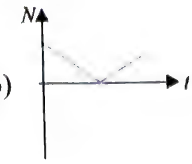
(3) $mg \cos \theta$ (4) mg

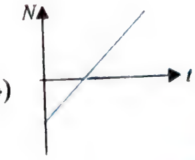
72. A block of mass m is placed in a horizontal tube. Block fits the tube with a very small gap. A time dependent force F acts on the system such that the resultant downward acceleration of system is given by its time. Which of the following graphs represents normal reaction of tube on block. Take downward direction as positive for N. Initially block is in contact with lower surface.



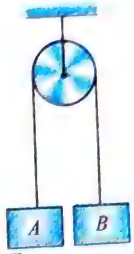
(1)


(2)


(3)


(4)


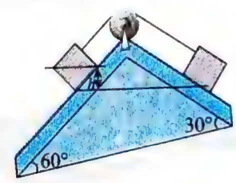
73. In the arrangement shown in the figure, the system is in equilibrium. Mass of the block A is M and that of the insect clinging to block B is m. Pulley and string are light. The insect loses contact with the block B and begins to fall. After how much time the insect and the block B will have a separation L between them.



(1) $\sqrt{\frac{M}{(2M-m)Lg}}$ (2) $\sqrt{\frac{(2M-m)L}{Mg}}$

(3) $\sqrt{\frac{(2M+m)L}{Mg}}$ (4) $\sqrt{\frac{M}{(2M+m)Lg}}$

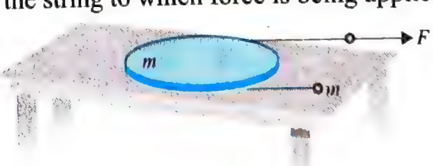
74. Two blocks of equal mass have been placed on two faces of a fixed wedge as shown in figure. The blocks are released from position where centre of one block is at a height h above the centre of the other block. Find the time after which the centre of the two blocks will be at same horizontal level. There is no friction anywhere.



(1) $2\sqrt{2}\sqrt{\frac{h}{g}}$ (2) $2\sqrt{2}\sqrt{\frac{h}{g}}$

(3) $2\sqrt{2}\sqrt{\frac{h}{g}}$ (4) $2\sqrt{2}\sqrt{\frac{h}{g}}$

75. A disc of mass m lies flat on a smooth horizontal table. A light string runs halfway around it as shown in figure. One end of the string is attached to a particle of mass m and the other end is being pulled with a force F. There is no friction between the disc and the string. Find acceleration of the end of the string to which force is being applied.

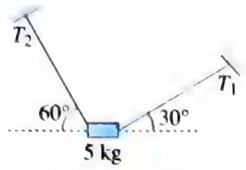


(1) $\frac{2F}{m}$ (2) $\frac{3F}{m}$

(3) $\frac{F}{m}$ (4) $\frac{5F}{m}$

Multiple Correct Answers Type

1. A body of mass 5 kg is suspended by the strings making angles 60° and 30° with the horizontal as shown in Figure ($g = 10\ \text{m s}^{-2}$). Then

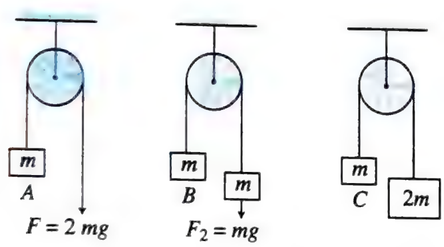


(1) $T_1 = 25$ N (2) $T_2 = 25$ N

(3) $T_1 = 25\sqrt{3}$ N (4) $T_2 = 25\sqrt{3}$ N

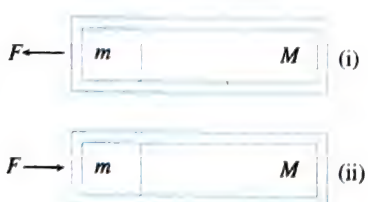
2. The accelerations of a particle as observed from two different frames S_1 and S_2 have equal magnitudes of $2\ \text{m s}^{-2}$.
   (1) The relative acceleration of the frame may either be zero or $4\ \text{m s}^{-2}$.
   (2) Their relative acceleration may have any value between 0 and $4\ \text{m s}^{-2}$.
   (3) Both the frames may be stationary with respect to earth.
   (4) The frames may be moving with same acceleration in same direction.

3. In the given figure, blocks A, B, and C of mass m each have acceleration a_1, a_2, and a_3, respectively. F_1 and F_2 are external forces of magnitude $2mg$ and mg, respectively. Then



(1) $a_1 \neq a_2 \neq a_3$ (2) $a_1 = a_2 \neq a_3$
(3) $a_1 > a_2 > a_3$ (4) $a_1 \neq a_2 = a_3$

4. A block of mass m is placed in contact with one end of a smooth tube of mass M. A horizontal force F acts on the tube in each case (i) and (ii). Then



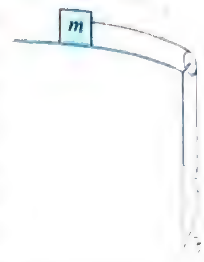
(1) $a_m = 0$ and $a_M = \dfrac{F}{M}$ in (i)

(2) $a_m = a_M = \dfrac{F}{M+m}$ in (i)

(3) $a_m = a_M = \dfrac{F}{M+m}$ in (ii)

(4) Force on m is $\dfrac{mF}{M+m}$ in (ii)

5. In the given figure, a man of true mass M is standing on a weighing machine placed in a cabin. The cabin is joined by a string with a body of mass m. Assuming no friction, and negligible mass of cabin and weighing machine, the measured mass of man is (normal force between the man and the machine is proportional to the mass)



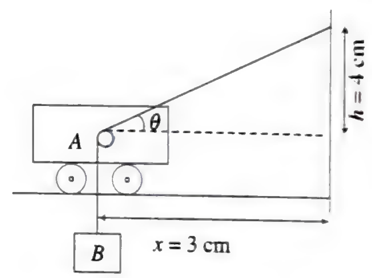
(1) The measured mass of man is $\dfrac{Mm}{(M+m)}$.

(2) The acceleration of man is $\dfrac{mg}{(M+m)}$.

(3) The acceleration of man is $\dfrac{Mg}{(M+m)}$.

(4) The measured mass of man is M.

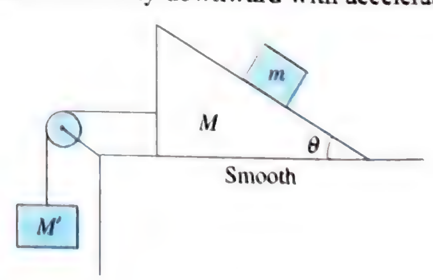
6. The string shown in figure is passing over small smooth pulley rigidly attached to trolley A. If the speed of trolley is constant and equal to v_A towards right, speed and magnitude of acceleration of block B at the instant shown in figure are



(1) $v_B = v_A$, $a_B = 0$ (2) $a_B = 0$

(3) $v_B = \dfrac{3}{5}v_A$ (4) $a_B = \dfrac{16v_A^2}{125}$

7. The given figure shows a block of mass m placed on a smooth wedge of mass M. Calculate the minimum value of M' and tension in the string, so that the block of mass m will move vertically downward with acceleration $10\ \text{m s}^{-2}$.



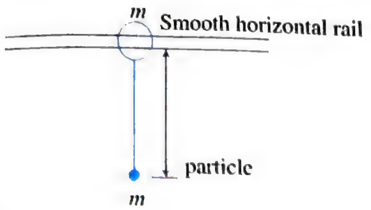
(1) The value of M' is $\dfrac{M\cot\theta}{1-\cot\theta}$.

(2) The value of M' is $\dfrac{M\tan\theta}{1-\tan\theta}$.

(3) The value of tension in the string is $\dfrac{Mg}{\tan\theta}$.

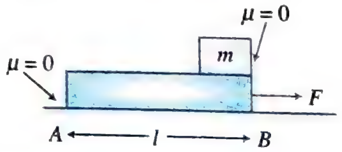
(4) The value of tension is $\dfrac{Mg}{\cot\theta}$.

8. The ring shown in figure is given a constant horizontal acceleration ($a_0 = g/\sqrt{3}$). The maximum deflection of the string from the vertical is θ_0. Then



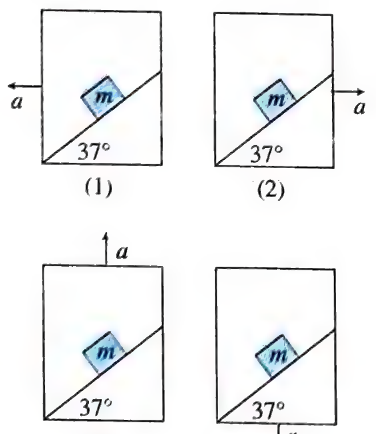
(1) $\theta_0 = 30°$
(2) $\theta_0 = 60°$
(3) At maximum deflection, tension in string is equal to mg.
(4) At maximum deflection, tension in string is equal to $\dfrac{2mg}{\sqrt{3}}$.

9. In the given figure, a small block is kept on m. Then



(1) The acceleration of m w.r.t. ground is F/m.
(2) The acceleration of m w.r.t. ground is zero.
(3) The time taken by m to separate from M is $\sqrt{\dfrac{2lm}{F}}$.
(4) The time taken by m to separate from M is $\sqrt{\dfrac{2lM}{F}}$.

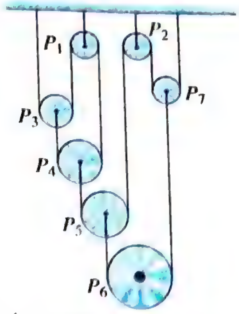
10. A block of mass m is placed on a wedge. The wedge can be accelerated in four manners marked as (1), (2), (3), and (4) as shown in figure. If the normal reactions in situations (1), (2), (3), and (4) are N_1, N_2, N_3, and N_4, respectively, and acceleration with which the block slides on the wedge in the situations are b_1, b_2, b_3, and b_4, respectively. Then



(3) (4)

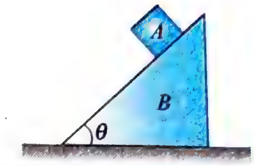
(1) $N_3 > N_1 > N_2 > N_4$
(2) $N_4 > N_3 > N_1 > N_2$
(3) $b_2 > b_3 > b_4 > b_1$
(4) $b_2 > b_3 > b_1 > b_4$

11. Seven pulleys are connected with the help of three light strings as shown in figure. Consider P_3, P_4, P_5 as light pulleys and pulleys P_6 and P_7 have masses m each. For this arrangement, mark the correct statement(s).



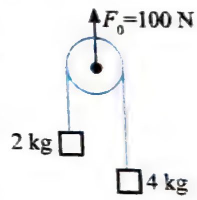
(1) Tension in the string connecting P_1, P_3, and P_4 is zero.
(2) Tension in the string connecting P_1, P_3 and P_4 is $mg/3$.
(3) Tensions in all the three strings are same and equal to zero.
(4) Acceleration of P_6 is g downwards and that of P_7 is g upwards.

12. In the figure shown, A and B are free to move. All the surfaces are smooth. Then ($0 < \theta < 90°$)



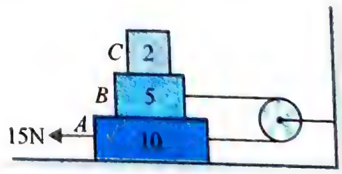
(1) the acceleration of A will be more than $g \sin\theta$
(2) the acceleration of A will be less than $g \sin\theta$
(3) normal force on A due to B will be more than $mg \cos\theta$
(4) normal force on A due to B will be less than $mg \cos\theta$

13. Two blocks of masses m_1 = 2 kg and m_2 = 4 kg hang over a massless pulley as shown in the figure. A force F_0 = 100 N acting at the axis of the pulley accelerates the system upwards. Then



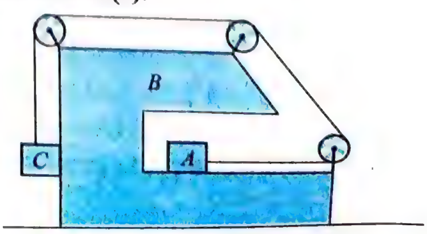
(1) acceleration of 2 kg mass is 15 m/s²
(2) acceleration of 4 kg mass is 2.5 m/s²
(3) acceleration of both the masses is same
(4) acceleration of both the masses is upward

14. In the figure below, all surfaces are smooth, string is light and pulley is frictionless. Mark the correct statement(s).



(1) Acceleration 01 block A is 1 m/s²
(2) Acceleration of block A and B will be same
(3) Acceleration of block C is zero
(4) Tension in string between pulley and wall is 10 N.

15. For given figure, m_A is 30 kg, $m_B = m_C$ = 5 kg, respectively. All contact surfaces are smooth (g = 10 m/s²). Select the correct statement(s).



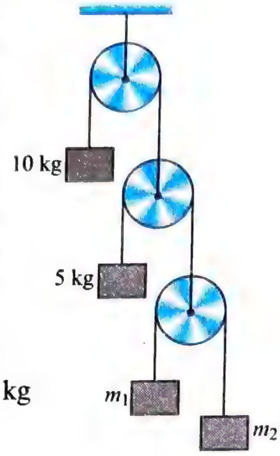
(1) Acceleration of block A, B and C are 1 m/s², 3 m/s² and 5 m/s² respectively

(2) Contact force between B and ground is 350 N

(3) Tension in string is 30 N

(4) Contact force between B and C is 15 N

16. For the system shown in figure, all the pulleys are light and frictionless and strings are ideal. Initially the system is held at rest by some external means. Now, the system is released i.e., external effect has been removed, but still it has been observed that 10 kg and 5 kg blocks are not moving. This is possible when



(1) m_1 and m_2 are both equal to $\frac{5}{2}$ kg

(2) sum of m_1 and m_2 is 5 kg

(3) m_1 and m_2 have many possible values

(4) harmonic mean of m_1 and m_2 is $\frac{5}{2}$ kg

17. A block of weight 9.8 N is placed on a table. The table surface exerts an upward force of 10 N on the block. Assume $g = 9.8$ m/s². Select the correct statement(s).

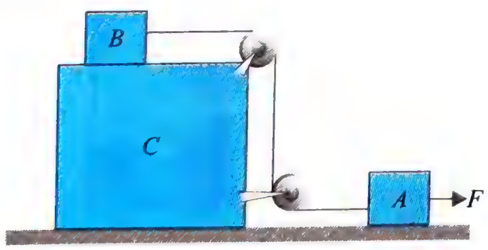
(1) The block exerts a force of 10 N on the table

(2) The block exerts a force of 19.8 N on the table

(3) The block exerts a force of 9.8 N on the table

(4) The block has an upward acceleration

18. Three blocks A, B, and C of masses 10 kg, 10 kg and 20 kg are arranged as shown in figure. All the surfaces are frictionless and string is inextensible. Pulleys and strings are light. A constant force $F = 20$ N is applied on block A as shown. Part of the string connecting both pulleys is vertical and part of the strings connecting pulleys with blocks A and B are horizontal. Then



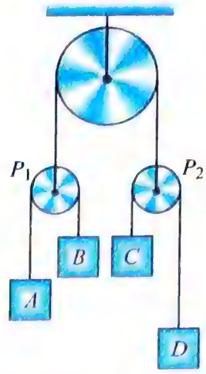
(1) Acceleration of mass blocks A, B and C is 0.5 m/s².

(2) Acceleration of mass block B is 1.

(3) Tension in the string is 10 N.

(4) Acceleration of mass block C is 0.5.

19. In the system shown in figure, masses of the blocks are such that when system is released, the acceleration of pulley P_1 is a upward and acceleration of block A is a_1 upward. It is found that the acceleration of block C is same as that of A both in magnitude and direction ($a_1 > a > a_1/2$). Then



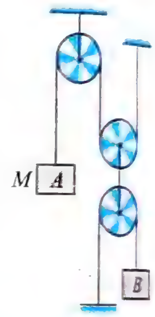
(1) Acceleration of B is $(2a - a_1)$ upward

(2) Acceleration of D is $(2a + a_1)$ downward

(3) Acceleration of B with respect to C is downward

(4) Acceleration of B with respect to D is upward

20. In the arrangement shown in the figure, all pulleys are massless and the strings are inextensible and light. Block A has mass M. Select the correct statement(s).



(1) If the system stays at rest after it is released, then the mass of the block B should be M.

(2) If the system stays at rest after it is released, then the mass of the block B should be $M/2$.

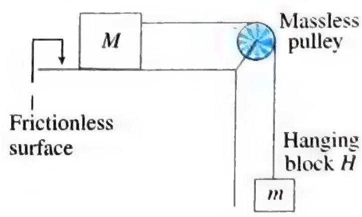
(3) If mass of the block B is twice the value required to keep the system in equilibrium, then the acceleration of block A should be $g/3$.

(4) If mass of the block B is twice the value required to keep the system in equilibrium, then the acceleration of block A should be $g/5$.

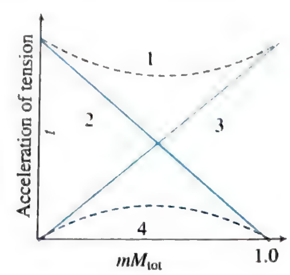
Linked Comprehension Type

For Problems 1 and 2

Two containers of sand are arranged like the block as shown in Figure. The containers alone have negligible mass; the sand in them has a total mass M_{tot}; the sand in the hanging container H has mass m.



To measure the magnitude a of the acceleration of the system, a large number of experiments carried out where m varies from experiment to experiment but M_{tot} does not; that is, sand is shifted between the containers before each trial.



1. Which of the curves in graph correctly gives the acceleration magnitude as a function of the ratio m/M_{tot} (vertical axis is for acceleration)?

(1) 1 (2) 2

(3) 3 (4) 4

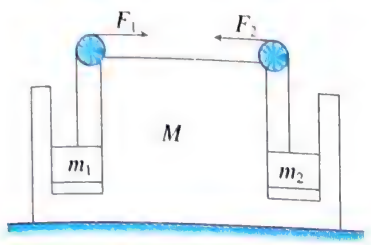
2. Which of them gives the tension in the connecting cord (vertical axis is for tension)?

(1) 1 (2) 2

(3) 3 (4) 4

For Problems 3–5

For the system shown in figure, there is no friction anywhere. Masses m_1 and m_2 can move up or down in the slots cut in mass M. Two non-zero horizontal forces F_1 and F_2 are applied as shown. The pulleys are massless and frictionless. Given $m_1 \neq m_2$.



3. According to the above passage, which is correct?

(1) It is not possible for the entire system to be in equilibrium.

(2) For some values of F_1 and F_2, it is possible that the entire system is in equilibrium.

(3) It is possible that F_1 and F_2 are applied in such a way that m_1 and m_2 remain in equilibrium but M does not.

(4) None of the above.

4. Let F_1 and F_2 be applied in such a way that m_1 and m_2 do not move w.r.t. M. Then what is the magnitude of the acceleration of M? Let $m_1 > m_2$.

(1) $\dfrac{(m_1+m_2)g}{M+m_1+m_2}$ (2) $\dfrac{(m_1-m_2)g}{M}$

(3) $\dfrac{(m_1-m_2)g}{M+m_1+m_2}$ (4) $\dfrac{F_1-F_2}{M}$

5. Let F_1 and F_2 be applied in such a way that accelerations of both masses m_1 and m_2 are same both in magnitude and direction. Then

(1) $\dfrac{F_1}{m_1}-\dfrac{m_2 g}{m_1}=\dfrac{F_2}{m_2}-\dfrac{m_1 g}{m_2}$ (2) $\dfrac{F_1}{m_2}=\dfrac{F_2}{m_1}$

(3) $\dfrac{F_1}{m_1}+\dfrac{m_2 g}{m_1}=\dfrac{F_2}{m_2}+\dfrac{m_1 g}{m_2}$ (4) $\dfrac{F_1}{m_1}=\dfrac{F_2}{m_2}$

For Problems 6–7

A time-varying force $F = 6t - 2t^2$ N, at $t = 0$ starts acting on a body of mass 2 kg initially at rest, where t is in second. The force is withdrawn just at the instant when the body comes to rest again. We can see that at $t = 0$, the force $F = 0$. Now answer the following:

6. Find the duration for which the force acts on the body.

(1) 2 s (2) 3 s

(3) 3.5 s (4) 4.5 s

7. Find the time when the velocity attained by the body is maximum.

(1) 2 s (2) 3 s

(3) 3.5 s (4) 4.5 s

8. Mark the correct statement:

(1) Velocity of the body is maximum when force acting on the body is maximum for the first time.

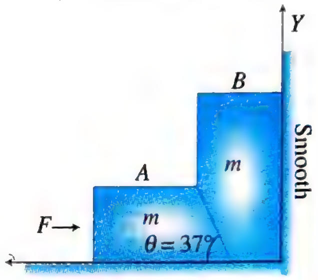
(2) The velocity of the body becomes maximum when force acting on the body becomes zero again

(3) When force becomes zero again, velocity of the body also becomes zero at that instant.

(4) All of the above

For Problems 9–11

Two smooth blocks are placed at a smooth corner as shown in figure. Both the blocks are having mass m. We apply a force F on the small block m. Block A presses block B in the normal direction, due to which pressing force on vertical wall will increase, and pressing force on the horizontal wall decreases, as we increases F ($\theta = 37°$ with horizontal).



As soon as the pressing force on the horizontal wall by block B becomes zero, it will lose contact with ground. If the value of F further increases, block B will accelerate in the upward direction and simultaneously block A will move towards right.

9. What is the minimum value of F to lift block B from ground?

(1) $\dfrac{25}{12}mg$ (2) $\dfrac{5}{3}mg$

(3) $\dfrac{3}{4}mg$ (4) $\dfrac{4}{3}mg$

10. If both the blocks are stationary, the force exerted by ground on block A is

(1) $mg+\dfrac{3F}{4}$ (2) $mg-\dfrac{3F}{4}$

(3) $mg+\dfrac{4F}{3}$ (4) $mg-\dfrac{4F}{3}$

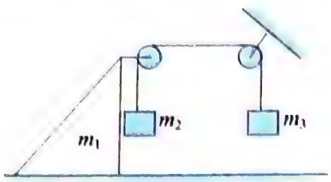
11. If the acceleration of block A is a rightwards, then the acceleration of block B will be

(1) $\dfrac{3a}{4}$, upwards (2) $\dfrac{4a}{3}$, upwards

(3) $\dfrac{3a}{5}$, upwards (4) $\dfrac{4a}{5}$, upwards

For Problems 12–14

In Figure, both pulleys and the string are massless and all the surfaces are frictionless. Given $m_1 = 1$ kg, $m_2 = 2$ kg, $m_3 = 3$ kg.



12. Find the tension in the string.

(1) $\frac{120}{7}$ N (2) $\frac{240}{7}$ N

(3) $\frac{130}{7}$ N (4) None of these

13. The acceleration of m_1 is

(1) $\frac{40}{7}$ m s^{-2} (2) $\frac{30}{7}$ m s^{-2}

(3) $\frac{20}{7}$ m s^{-2} (4) $\frac{\sqrt{17}g}{7}$ m s^{-2}

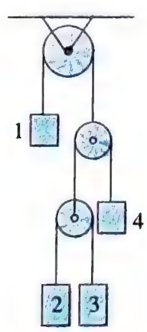
14. The acceleration of m_3 is

(1) $\frac{40}{7}$ m s^{-2} (2) $\frac{30}{7}$ m s^{-2}

(3) $\frac{20}{7}$ m s^{-2} (4) None of these

For Problems 15–18

In the arrangement shown in figure, all pulleys are smooth and massless. When the system is released from the rest, acceleration of blocks 2 and 3 relative to 1 are 1 m s^{-2} downwards and 5 m s^{-2} downwards, respectively. Acceleration of block 3 relative to 4 is zero.



15. Find the absolute acceleration of block 1.

(1) 2 m s^{-2} upwards

(2) 1 m s^{-2} downwards

(3) 3 m s^{-2} upwards

(4) 1.5 m s^{-2} downwards

16. Find the absolute acceleration of block 2.

(1) 2 m s^{-2} downwards (2) 1 m s^{-2} upwards

(3) 3 m s^{-2} upwards (4) 1.5 m s^{-2} downwards

17. Find the absolute acceleration of block 3.

(1) 2 m s^{-2} upwards (2) 1 m s^{-2} downwards

(3) 3 m s^{-2} downwards (4) 1.5 m s^{-2} upwards

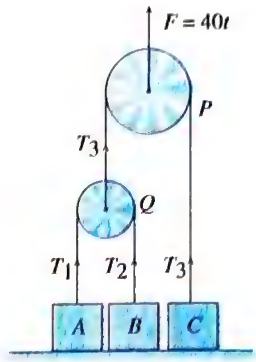
18. Find the absolute acceleration of block 4.

(1) 2 m s^{-2} upwards (2) 1 m s^{-2} downwards

(3) 3 m s^{-2} downwards (4) 1.5 m s^{-2} upwards

For Problems 19–21

Three blocks A, B, and C having masses 1 kg, 2 kg, and 3 kg, respectively, are arranged as shown in figure. The pulleys P and Q are light and frictionless. All the blocks are resting on a horizontal floor and the pulleys are held such that strings remain just taut. At moment $t = 0$, a force $F = 40t$ N starts acting on pulley P along vertically upward direction as shown in the figure. Take $g = 10$ m s^{-2}.



19. Regarding the times when the blocks lose contact with ground, which is correct?

(1) A loses contact at $t = 2$ s.

(2) C loses contact at $t = 1.5$ s.

(3) A and B lose contact at the same time.

(4) All three blocks lose contact at the same time.

20. Find the velocity of A when B loses contact with ground.

(1) 5 m s^{-1} (2) 5/4 m s^{-1}

(3) 4 m s^{-1} (4) 7/3 m s^{-1}

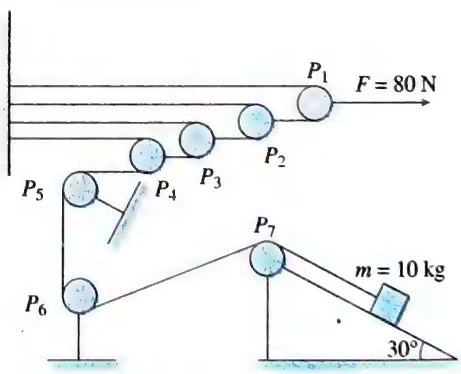
21. What is the magnitude of relative acceleration of A with respect to B just after both have lost contact with ground?

(1) 15 m s^{-1} (2) 5 m s^{-1}

(3) 20 m s^{-1} (4) 10 m s^{-1}

For Problems 22–24

In figure all the pulleys and strings are massless and all the surfaces are frictionless. A small block of mass m is placed on fixed wedge (take $g = 10$ m s^{-2}).



22. The tension in the string attached to m is

(1) 40 N (2) 10 N

(3) 20 N (4) 5 N

23. The acceleration of m is

(1) 4.5 m s^{-2} down the incline

(2) 4.5 m s^{-2} up the incline

(3) 5 m s^{-2} down the incline

(4) 5 m s^{-2} up the incline

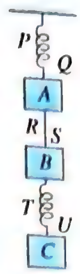
24. The acceleration of pulley p_4 is

(1) 2.25 m s^{-2} towards left (2) 2.25 m s^{-2} towards right

(3) 9 m s^{-2} towards left (4) 9 m s^{-2} towards right

For Problems 25–27

Three blocks A, B, and C of masses $3M$, $2M$, and M are suspended vertically with the help of springs PQ and TU, and a string RS as shown in figure. If the acceleration of blocks A, B, and C is a_1, a_2, and a_3, respectively, then



25. The value of acceleration a_3 at the moment spring PQ is cut is

(1) g, downward
(2) g, upwards
(3) More than g, downwards
(4) Zero

26. The value of acceleration a_1 at the moment string RS is cut is

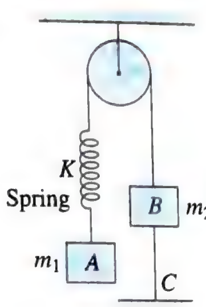
(1) g, downward (2) g, upwards
(3) More than g, downwards (4) Zero

27. The value of acceleration a_2 at the moment spring TU is cut is

(1) g/5, upwards (2) g/5, downwards
(3) g/3, upwards (4) Zero

For Problems 28–30

In the system shown in figure, $m_1 > m_2$. The system is held at rest by thread BC. Now thread BC is burnt. Answer the following:



28. Before burning the thread, what are the tensions in spring and thread BC, respectively?

(1) m_1g, m_2g (2) m_1g, $m_1g - m_2g$
(3) m_2g, m_1g (4) m_1g, $m_1g + m_2g$

29. Just after burning the thread, what is the tension in the spring?

(1) m_1g (2) m_2g
(3) Zero (4) Cannot say

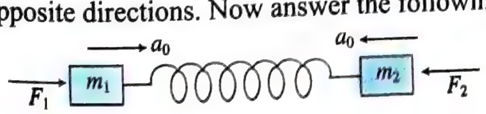
30. Just after burning the thread, what is the acceleration of m_2?

(1) $\left(\dfrac{m_2 - m_1}{m_2}\right)g$ (2) $\left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)g$

(3) Zero (4) $\left(\dfrac{m_1 - m_2}{m_2}\right)g$

For Problems 31–33

Two blocks of masses m_1 and m_2 are connected with a light spring of force constant k and the whole system is kept on a frictionless horizontal surface. The masses are applied forces F_1 and F_2 as shown in figure. At any time, the blocks have same acceleration a_0 but in opposite directions. Now answer the following:



31. The value of a_0 is

(1) $\dfrac{F_1 - F_2}{m_1 + m_2}$ (2) $\dfrac{F_1 - F_2}{m_1 - m_2}$

(3) $\dfrac{F_1 + F_2}{m_1 - m_2}$ (4) $\dfrac{F_1 + F_2}{m_1 + m_2}$

32. The value of spring force is

(1) $\dfrac{m_1F_2 + F_1m_2}{m_1 - m_2}$ (2) $\dfrac{m_1F_2 - F_1m_2}{m_1 + m_2}$

(3) $\dfrac{m_1F_2 + F_1m_2}{m_1 + m_2}$ (4) $\dfrac{m_1F_2 - F_1m_2}{m_1 - m_2}$

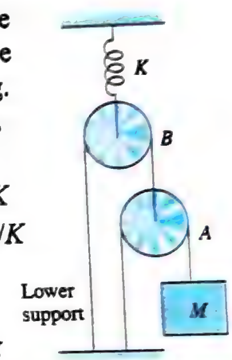
33. If F_2 is removed at this moment, then just after this, the acceleration of m_2 is

(1) $\dfrac{F_1}{m_2} - a_0$ (2) $a_0 + \dfrac{F_1}{m_2}$

(3) $\dfrac{F_2}{m_2} - a_0$ (4) $a_0 + \dfrac{F_2}{m_2}$

For Problems 34–36

A mass M is suspended as shown in figure. The system is in equilibrium. Assume pulleys to be massless. K is the force constant of the spring.



34. The extension produced in the spring is given by

(1) $4Mg/K$ (2) Mg/K
(3) $2Mg/K$ (4) $3Mg/K$

35. Find the net tension force acting on the lower support.

(1) Mg (2) $2Mg$
(3) $3Mg$ (4) $4Mg$

36. If each of the pullies A and B has mass M, then find the net tension force acting on the lower support. Assume pulleys to be frictionless.

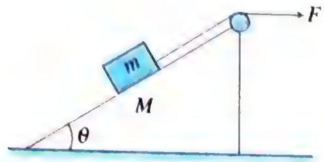
(1) $2Mg$ (2) $6Mg$
(3) $3Mg$ (4) $4Mg$

Matrix Match Type

1. Column I describes the motion of the object and one or more of the entries of Column II may be the cause of motions described in Column I. Match the entries of Column I with the entries of Column II.

Column I	Column II
i. An object is moving towards east.	a. The net force acting on the object must be towards east.
ii. An object is moving towards east with constant acceleration.	b. At least one force must act towards east.
iii. An object is moving towards east with varying acceleration.	c. No force may act towards east.
iv. An object is moving towards east with constant velocity.	d. No force may act on the object.

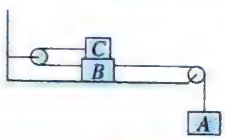
2. There is no friction anywhere in the system shown in figure. The pulley is light. The wedge is free to move on a frictionless surface. A horizontal force F is applied on the system in such a way that m does not slide on M or both move together with some common acceleration. Given $M > \sqrt{2}\,m$.



Match the entries of Column I with that of Column II.

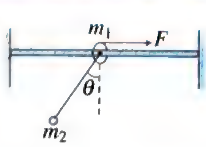
Column I	Column II
i. Pseudo force acting on m as seen from the frame of M is	a. Equal to $\dfrac{mF}{m+M}$
ii. Pseudo force acting on M as seen from the frame of m is	b. Greater than $\dfrac{mF}{m+M}$
iii. Normal force (for $\theta = 45°$) between m and M is	c. Less than $mg \sin\theta$
iv. Normal force between ground and M is	d. Greater than $mg \sin\theta$

3. When the system shown in figure is released, A accelerates downwards. Match the entries of Column I with Column II.



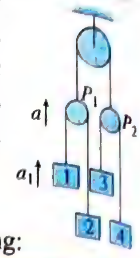
Column I	Column II
i. Acceleration of B	a. Towards left
ii. Acceleration of C w.r.t. B	b. Towards right
iii. Acceleration of A w.r.t. C	c. At some angle θ with horizontal $(0 < \theta < 90°)$
iv. Acceleration of B w.r.t. A	d. At some angle θ with vertical $(0 < \theta < 90°)$

4. A horizontal force F pulls a ring of mass m_1 such that θ remains constant with time. The ring is constrained to move along a smooth rigid horizontal wire. A bob of mass m_2 hangs from m_1 by an inextensible light string. Then match the entries of Column I with that of Column II.



Column I	Column II
i. F	a. $(m_1 + m_2)g$
ii. Force acting on m_2 is	b. $m_2 g \sec\theta$
iii. Tension in the string is	c. $m_2 \dfrac{F}{m_1 + m_2}$
iv. Force acting on m_1 by the wire is	d. $(m_1 + m_2)g \tan\theta$

5. In the system shown in figure, masses of the blocks are such that when system is released, the acceleration of pulley P_1 is a upwards and the acceleration of block 1 is a_1 upwards. It is found that the acceleration of block 3 is same as that of 1 both in magnitude and direction.



Given that $a_1 > a > a_1/2$. Match the following:

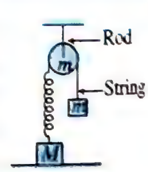
Column I	Column II
i. Acceleration of 2	a. $2a + a_1$
ii. Acceleration of 4	b. $2a - a_1$
iii. Acceleration of 2 w.r.t. 3	c. Upwards
iv. Acceleration of 2 w.r.t. 4	d. Downwards

6. A block is attached to an unstretched vertical spring and released from rest. As a result of this, the block comes down due to its weight, stops momentarily, and then bounces back. Finally, the block starts oscillating up and down.

During oscillations, match Column I with Column II:

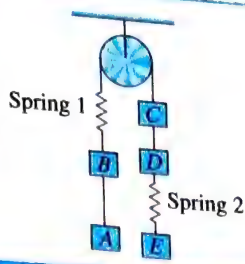
Column I	Column II
i. When the block is at its maximum downward displacement position (may be known as extreme position)	a. Acceleration is in upward direction.
ii. When the block is at its equilibrium position	b. Acceleration is in downward direction.
iii. When the block is somewhere between equilibrium position and downward extreme position	c. Acceleration is zero.
iv. When the block is above equilibrium position but below the initial unstretched position	d. Velocity may be in upward or in downward direction.

7. In figure, a block of mass m is released from rest when spring was in its natural length. The pulley also has mass m but it is frictionless. Suppose the value of m is such that finally it is just able to lift block M up after releasing it.



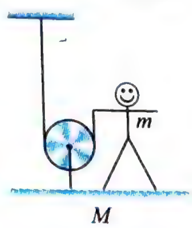
Column I	Column II
i. Weight of m required to just lift M	a. $3\dfrac{M}{2}g$
ii. Tension in the rod, when m is in equilibrium	b. Mg
iii. Normal force acting on M when m is in equilibrium.	c. $\dfrac{M}{2}g$
iv. Tension in the string when displacement of m is maximum possible	d. $2\,mg$

8. The system shown figure is initially in equilibrium. Masses of the blocks A, B, C, D, and E are, respectively, 3 kg, 3 kg, 2 kg, 2 kg and 2 kg. Match the conditions in Column I with the effect in Column II.



Column I	Column II
i. After spring 2 is cut, tension in string AB	a. Increases
ii. After spring 2 is cut, tension in string CD	b. Decreases
iii. After string between C and pulley is cut, tension in string AB	c. Remains constant
iv. After string between C and pulley is cut, tension in string CD	d. Zero

9. A person stands on a platform-and-pulley system, as shown in figure. The masses of the platform and person are M and m respectively. The rope and pulley is massless. The man can pull on the rope so as to produce one of the situations shown in column I. In column II, T is tension & N is normal reaction between the man and the platform. Match the possible situations.



Column I	Column II
i. At equilibrium	a. $N = (2m + M)g$
ii. When the system falls down freely	b. $T = (m + M)g$
iii. When accelerating upwards	c. $N < (2m + M)g$
iv. When accelerating downwards	d. $N > (2m + M)g$
	e. $T < (m + M)g$

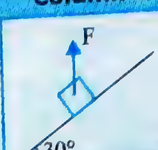
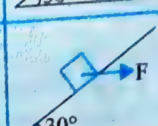
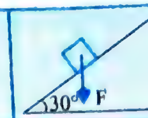
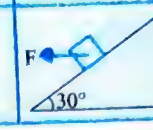
(1) i → a,c; ii → c,d; iii → e; iv → c,e

(2) i → a,b; ii → c,e; iii → d; iv → c,e

(3) i → a,d; iii → c,e; iii → b; iv → c,d

(4) i → a,b; iv → c,d; iii → a; iv → c,e

10. A force of 20 N is acting on a block of mass 2 kg kept on a smooth inclined plane. The direction of this force can be any of the four as indicated in column II. Match with the description given in column I.

Column I		Column II
i. Normal force cannot be zero	a.	
ii. Body slides down the inclined plane.	b.	
iii. Acceleration of the body is greater than 10 m/s^2	c.	
	d.	

(1) i → a,c,d; ii → c,d; iii → d

(2) i → a,b,d; ii → b,c; iii → c

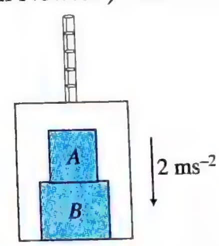
(3) i → c,d; ii → b,d; iii → b

(4) i → b,c,d; ii → c,d; iii → d

Numerical Value Type

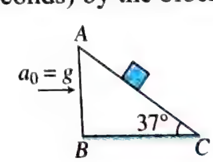
1. You are designing an elevator for a hospital. The force exerted on a passenger by the floor of the elevator is not to exceed 1.60 times the passenger's weight. The elevator accelerates upward with constant acceleration for a distance of 3.0 m and then starts to slow down. What is the maximum speed (in m s^{-1}) of the elevator?

2. The elevator shown in figure is descending with an acceleration of 2 m s^{-2}. The mass of the block $A = 0.5$ kg. Find the force (in Newton) exerted by block A on block B.

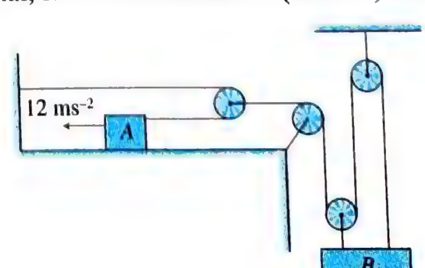


3. Figure represents a painter in a crate which hangs alongside a building. When the painter of mass 100 kg pulls the rope, the force exerted by him on the floor of the crate is 450 N. If the crate weighs 25 kg, find the acceleration (in m s^{-2}) of the painter.

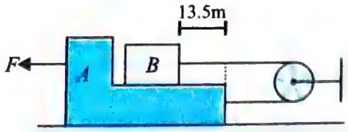
4. A block is placed on an inclined plane moving towards right horizontally with an acceleration $a_0 = g$. The length of the plane $AC = 1$ m. Friction is absent everywhere. Find the time taken (in seconds) by the block to reach from C to A.



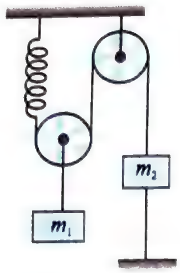
5. Block A is given an acceleration 12 m s^{-2} towards left as shown in figure. Assuming block B always remains horizontal, find the acceleration (in m s^{-2}) of B.



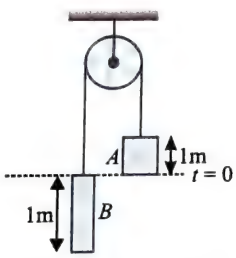
6. A 1-kg block 'B' rests as shown on a frictionless bracket 'A' of same mass. Constant force $F = 3$N starts to act at time $t = 0$, when the distance of block B from the end of bracket is 13.5 m. Find time (in sec), when block B falls off the bracket.



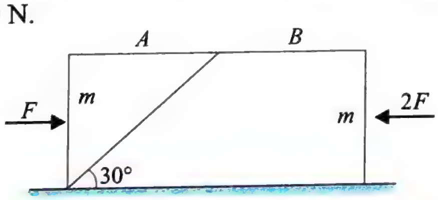
7. In figure shown, pulleys are ideal. Initially the system is in equilibrium and string connecting m_2 to rigid support below is cut. What is the initial acceleration (in m/s²) of m_2? (Given $m_1 = 9$ kg; $m_2 = 3$ kg)



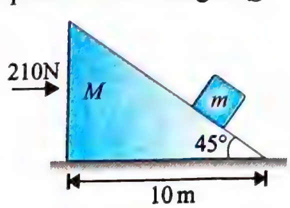
8. At $t = 0$, the lower end of the bar A is just above the upper end of bar B (mass of bar $A = 3$ kg, mass of bar $B = 11/3$ kg. Find time when upper end of block A just crosses the lower end of B. (Assume the system was released at $t = 0$). [$g = 10$ m/s²]



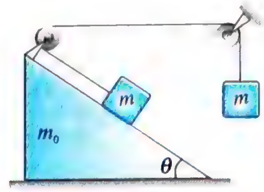
9. Two blocks A and B each of mass m are placed on a smooth horizontal surface. Two horizontal force F and $2F$ are applied on both the blocks A and B, respectively, as shown in figure. The block A does not slide on block B. Find the normal reaction acting between the two blocks (in N) if $F = 10$ N.



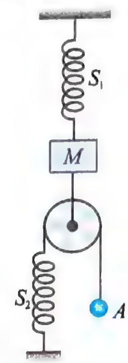
10. A small block of mass $m = 2$kg is kept at rest at the bottom of the incline plane of a wedge of mass $M = 9$ kg. A force of 210 N is applied on the wedge horizontally as shown in diagram. All the surfaces are smooth. Determine the time taken in seconds by the mass m to reach the highest point of the incline plane of the wedge? ($g = 10$m/s²)



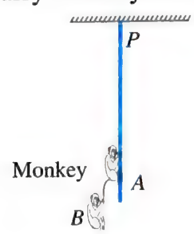
11. In the system shown in figure all surfaces are smooth and string and pulleys are light. Angle of wedge $\theta = \sin^{-1}\left(\frac{3}{5}\right)$. When released from rest it was found that the wedge of mass m_0 does not move. Find $\frac{m}{M}$



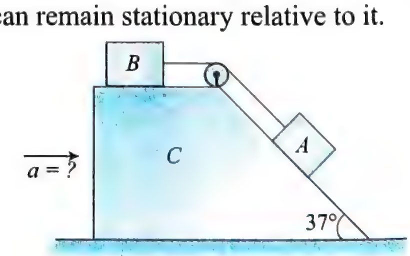
12. In the system shown in figure, the two springs S_1 and S_2 have force constant k each. Pulley, springs and strings are all massless. Initially, the system is in equilibrium with spring S_1 stretched and S_2 relaxed. The end A of the string is pulled down slowly through a distance L =10 cm. By what distance(in cm) does the block of mass M move?



13. A monkey A (mass = 6 kg) is climbing up a rope tied to a rigid support. The monkey B (mass = 2 kg) is holding on the tail of monkey A. If the tail can tolerate a maximum tension of 30 N, what maximum force should monkey A apply on the rope in order to carry monkey B with it? ($g = 10$ ms^{-2})

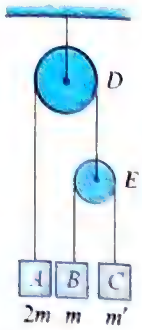


14. The upper surface of block C is horizontal and its right part is inclined to the horizontal at angle 37°. The mass of blocks A and B are $m_1 = 1.4$ kg and $m_2 = 5.5$ kg, respectively. Neglect friction and mass of the pulley. Calculate acceleration a with which block C should be moved to the right so that A and B can remain stationary relative to it.



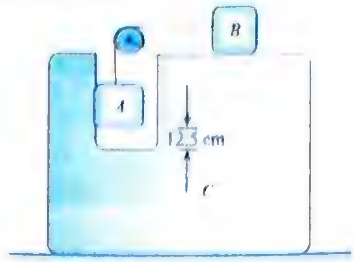
15. In the arrangement shown in figure, pulley D and E are small and frictionless. They do not rotate but threads slip over them without friction and their masses being 4 kg and

11.25 kg, respectively, while masses of block A, B and C are 2m, m and m' respectively. When the system is released from rest, downward accelerations of blocks B and C relative to A are found to be 5 ms^{-2} and 3 ms^{-2} respectively.

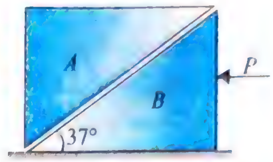


Find the ratio of the acceleration of blocks B and C, relative to the ground.

16. A small, light pulley is attached with a block C of mass 4 kg as shown in figure. Block B of mass 1.5 kg is placed on the top horizontal surface of C. Another block A of mass 2 kg is hanging from a string, attached with B and passing over the pulley. Taking $g = 10$ m s^{-2} and neglecting friction, find the acceleration of block B (in m/s^2) when the system is released from rest.



17. Blocks A and B each have same mass $m = 1$ kg. Determine the largest horizontal force P (in newton) which can be applied to B so that A will not slip up on B. Neglect any friction.

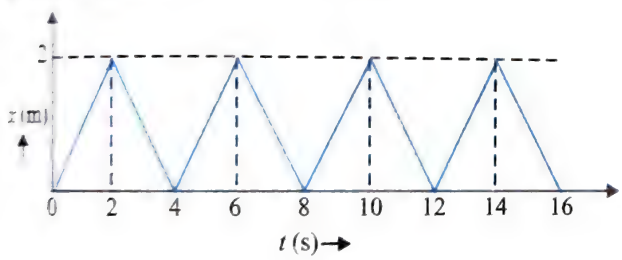


Archives

JEE MAIN

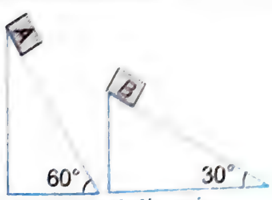
Single Correct Answer Type

1. The figure shows the position–time (x–t) graph of one-dimensional motion of a body of mass 0.4 kg. The magnitude of each impulse is



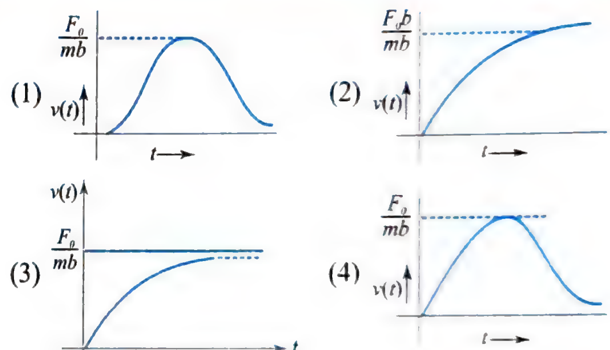
(1) 0.4 Ns (2) 0.8 Ns
(3) 1.6 Ns (4) 0.2 Ns **(AIEEE 2010)**

2. Two fixed frictionless inclined planes making angles 30° and 60° with the vertical are shown in the figure. Two blocks A and B are placed on the two planes. What is the relative vertical acceleration of A with respect to B?



(1) 4.9 m/s^2 in horizontal direction
(2) 9.8 m/s^2 in vertical direction
(3) zero
(4) 4.9 m/s^2 in vertical direction **(AIEEE 2010)**

3. A particle of mass m is at rest at the origin at time $t = 0$. It is subjected to a force $F(t) = F_0e^{-bt}$ in the x direction. Its speed $v(t)$ is depicted by which of the following curves?



(AIEEE 2012)

JEE ADVANCED

Single Correct Answer Type

1. A piece of wire is bent in the shape of a parabola $y = kx^2$ (y-axis vertical) with a bead of mass m on it. The bead can slide on the wire without friction. It stays at the lowest point of the parabola when the wire is at rest. The wire is now accelerated parallel to the x-axis with a constant acceleration a. The distance of the new equilibrium position of the bead, where the bead can stay at rest with respect to the wire, from the y-axis is

(1) $\frac{a}{gk}$ (2) $\frac{a}{2gk}$

(3) $\frac{2a}{gk}$ (4) $\frac{a}{4gk}$

(IIT-JEE 2009)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	**2.** (1)	**3.** (1)	**4.** (2)	**5.** (2)
6. (3)	**7.** (2)	**8.** (4)	**9.** (3)	**10.** (3)
11. (3)	**12.** (2)	**13.** (4)	**14.** (4)	**15.** (4)
16. (4)	**17.** (2)	**18.** (3)	**19.** (4)	**20.** (2)
21. (1)	**22.** (2)	**23.** (4)	**24.** (3)	**25.** (4)
26. (4)	**27.** (1)	**28.** (3)	**29.** (1)	**30.** (1)
31. (3)	**32.** (1)	**33.** (4)	**34.** (3)	**35.** (3)
36. (3)	**37.** (1)	**38.** (4)	**39.** (2)	**40.** (3)
41. (4)	**42.** (4)	**43.** (3)	**44.** (3)	**45.** (2)
46. (2)	**47.** (2)	**48.** (1)	**49.** (3)	**50.** (1)
51. (3)	**52.** (2)	**53.** (2)	**54.** (4)	**55.** (2)
56. (4)	**57.** (2)	**58.** (3)	**59.** (3)	**60.** (4)
61. (2)	**62.** (2)	**63.** (4)	**64.** (4)	**65.** (1)
66. (3)	**67.** (1)	**68.** (4)	**69.** (1)	**70.** (2)
71. (4)	**72.** (4)	**73.** (2)	**74.** (1)	**75.** (4)

Multiple Correct Answers Type

1. (1),(4)	**2.** (2),(3),(4)	**3.** (1),(3)
4. (1),(3),(4)	**5.** (1),(3)	**6.** (3),(4)
7. (1),(3)	**8.** (1),(4)	**9.** (1),(4)
10. (1),(3)	**11.** (1),(3)	**12.** (1),(4)
13. (1),(2),(4)	**14.** (1),(3),(4)	**15.** (1),(3),(4)
16. (1),(3),(4)	**17.** (1),(4)	**18.** (2),(3)
19. (1),(2),(3),(4)	**20.** (1),(3)	

Linked Comprehension Type

1. (1)	**2.** (4)	**3.** (1)	**4.** (3)	**5.** (4)
6. (4)	**7.** (2)	**8.** (2)	**9.** (3)	**10.** (3)
11. (1)	**12.** (1)	**13.** (4)	**14.** (2)	**15.** (1)
16. (2)	**17.** (3)	**18.** (3)	**19.** (2)	**20.** (1)
21. (4)	**22.** (4)	**23.** (1)	**24.** (1)	**25.** (4)
26. (2)	**27.** (1)	**28.** (2)	**29.** (1)	**30.** (4)
31. (2)	**32.** (4)	**33.** (3)	**34.** (1)	**35.** (3)
36. (4)				

Matrix Match Type

1. i → c, d; ii → a, c; iii → a, c; iv → c, d.
2. i → a, c; ii → b, c; iii → b, c; iv → b, d.
3. i → b; ii → a; iii → c, d; iv → c, d.
4. i → d; ii → c; iii → b; iv → a.
5. i → b, c; ii → a, d; iii → d; iv → c.
6. i → a; ii → c, d; iii → a, d; iv → b, d.
7. i → c; ii → c; iii → c; iv → b, d.
8. i → c; ii → b; iii → b, d; iv → b.
9. (2)
10. (4)

Numerical Value Type

1. (6)	**2.** (4)	**3.** (2)	**4.** (1)	**5.** (2)
6. (3)	**7.** (5)	**8.** (2)	**9.** (30)	**10.** (2)
11. (5)	**12.** (4)	**13.** (120)	**14.** (1.82)	**15.** (3)
16. (5)	**17.** (15)			

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2) **2.** (4) **3.** (3)

JEE Advanced

Single Correct Answer Type

1. (2)

7 Newton's Laws of Motion (With Friction)

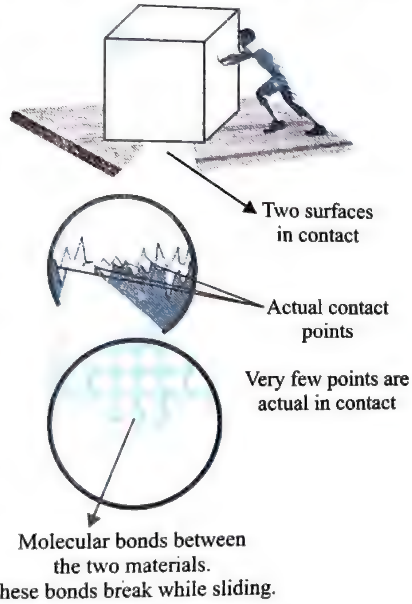
INTRODUCTION

In the previous chapter, we introduced Newton's law of motion and applied them to situations in which we ignored friction. In this section, we shall expand our investigation to objects moving in the presence of friction, which will allow us to model situations more realistically.

When an object moves either on a surface or through a viscous medium such as air or water, there is resistance to the motion because the object interacts with its surroundings. We call such resistance a force of friction.

FRICTION AND FRICTIONAL FORCE

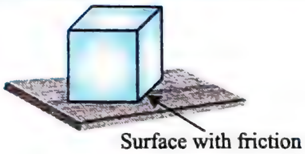
Whenever an object moves or tends to move on a rough surface, the component of contact force parallel to surface resists the relative motion of the object with respect to surface in contact. This property of the surface is friction and the resistive force is the frictional force. Experiments show that the frictional force arises from the nature of the two surfaces: because of their roughness, contact is made only at a few locations where peaks of the material touch. The microscopic area of contact for these spots is substantially less than the apparent macroscopic area of contact between the surfaces—perhaps thousands of times less. At these contact points, the molecules of different bodies are close enough together to exert strong attractive intermolecular forces on one another, leading to what are known as "cold welds." At these locations, the frictional force arises in part because one peak physically blocks the motion of a peak from the opposing surface and in part from chemical bonding ("spot welds") of opposing peaks as they come into contact. Although the details of friction are quite complex at the atomic level, this force ultimately involves an electrical interaction between atoms or molecules. Hence, we can say that frictional forces oppose the tendency of contacting surfaces to slip one relative to the other.



STATIC AND KINETIC FRICTION

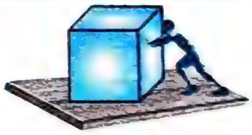
The force of friction comes into action only when there is a relative motion or there is a tendency of relative motion between the two contact surfaces. When two surfaces are in contact, frictional forces oppose relative motion or impending motion.

The direction of frictional forces are parallel to the surfaces in contact and oppose relative motion between two contact surfaces. Let us consider one situation: a block resting on a rough surface [Figure (a)], is there any friction force acting on the block? Answer is 'no', because it is not having any motion tendency with respect to surface on which it is placed. If we try to slide the block and the block is not sliding [Figure (b)], then again consider the same question. Is there any frictional force acting? In this case, answer is 'yes', because this frictional force is resisting the relative motion of the block. As the box is at rest, the frictional force must exactly balance the pushing force. The nature of friction acting at this stage is called **static friction**.



The block is resting on rough surface

Fig. (a)



The man tries to slide the block but the block is not sliding

Fig. (b)

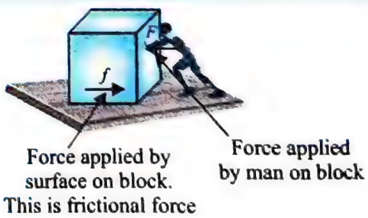


Fig. (c)

As you push harder and harder, you reach a point where the block finally slides across the floor. The nature of friction acting at this stage is called **kinetic friction**. From this experiment, it is clear that we need to distinguish between the case where an object is at rest relative to its supporting surface (static friction) and the case where an object is moving across the surface (kinetic friction).

PROPERTIES OF FRICTION

From experiments it is found that when a dry and unlubricated body presses against a surface in the same condition and a force attempts to slide the body along the surface, the resulting frictional force can have three properties:

Property 1: For any external force acting on an object that remains at rest, the frictional force is exactly equal in magnitude and opposite in direction to the external force or the component of that external force that acts along the contact surface between the object and its supporting surface. If the body does not move, then the static frictional force and the force or the component of force which is parallel to the surface balance each other.

Property 2: If the applied force however continues to increase, there comes a point when the block finally "breaks away" and begins to slide. The magnitude f_s of the static frictional force can have any value from zero up to a maximum value of depending on the applied force.

In other words, $f_s \le f_{s,\max}$ ...(i)

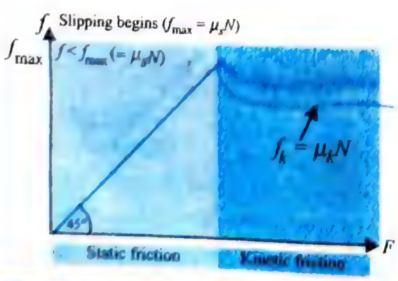
The equality in Eq. (i) holds when the surfaces are on the verge of slipping, that is, when $f = f_{s,\max} = \mu_s N$, where μ_s is the **coefficient of static friction** and N is the magnitude of the normal force on the body from the surface. This situation is called **impending motion.** The inequality holds when the component of the applied force parallel to the surfaces is less than this value. If the magnitude of the applied force or the component of force which is parallel to the surface exceeds $f_{s,\max}$, then the body begins to slide along the surface.

Property 3: If the body begins to slide along the surface, the magnitude of the frictional force rapidly decreases to a value f_k given by

$$f_k = \mu_k N \quad \text{...(ii)}$$

The magnitude N of the normal force appears in Eq. (i) and (ii) as a measure of how firmly the body presses against the surface; μ_k is the **coefficient of kinetic friction**. This coefficient is always equal to or greater than zero. (The case where $\mu_k = 0$ corresponds to a frictionless approximation. In practice, however, it can never be reached perfectly.) In almost all cases, μ_k is also less than 1. (Some special tire surfaces used for car racing, though, have a coefficient of friction with the road that can significantly exceed 1.)

The direction of the kinetic frictional force is *always opposite to the direction of motion* of the object relative to the surface it moves on.



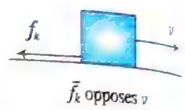
Important Points:

- The magnitude of the frictional force acting on a moving object is proportional to the magnitude of the normal force.
- The frictional force is independent of the size of the contact area between object and surface.
- The frictional force depends on the roughness of the surfaces; that is, a smoother interface generally provides less friction force than a rougher one.
- The frictional force is independent of the velocity of the object.

- We define the static friction as tangential contact force which prevents relative sliding between the points of contact.
- Static friction can have any value between zero and its maximum (limiting) value $\mu_s N$. We can write $0 \le f_s \le \mu_s N$
- *Static friction is of self-adjusting nature and can change its magnitude and direction as per requirement.*
- The values of μ_k and μ_s depend on the nature of the surfaces, but μ_k is generally less than μ_s.
- The direction of the friction force on an object is opposite to the actual motion (kinetic friction) or the impending motion (static friction) of the object relative to the surface with which it is in contact.

Direction of Frictional Force

- Consider a block placed on a horizontal floor and give an initial push (figure). The block will stop after travelling some distance.



According to Newton's second law, a retarding force must be acting on the block. This opposing force is called *kinetic frictional force.* The direction of frictional force is opposite to relative motion. This gives us an idea that kinetic friction of the block is opposite its motion (velocity).

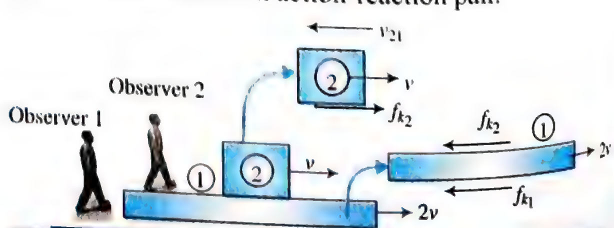
- Let us understand the direction of kinetic friction through an example.

If a plank (1) moves on rough horizontal surface with some velocity (say, $2v$) and we project a block (2) with a velocity v in same direction. The top surface of plank is rough. To know the direction of friction we need to know the relative motion tendencies of different surfaces. Let us discuss how to find the direction of kinetic friction.

At first, let us climb onto the plank and observe the block. It seems as if the block moves relative to plank with a velocity $\vec{v}_{21}$.

$$\vec{v}_{21} = \vec{v}_2 - \vec{v}_1 = v\hat{i} - 2v\hat{i} = -v\hat{i}$$

Hence, the block is observed to move towards left as seen by observer (2) or w.r.t. plank. That means frictional force $\vec{f}_{k_2}$ acts forward on the block in order to oppose relative sliding (relative velocity $\vec{v}_{21}$), as shown in figure. On the plank equal and opposite kinetic friction $\vec{f}_{k_1}$ acts in backward direction because the two frictional forces are an action-reaction pair.



f_{k_2} on the block opposes $\vec{v}_{21}$ and f_{k_2} on this plane opposes $\vec{v}_{12}$. f_{k_1} on bottom surface of the plank opposes the relative motion of plank w.r.t. ground.

As plank moves towards right as seen from observer (1) (or w.r.t ground). The direction of the kinetic friction $\vec{f}_{k_1}$ on plank will be towards left.

- Take one more example to find the direction of static friction. Consider a block placed at rest in an accelerating cart as shown in figure. The block, in fact, is accelerating along with the cart. Which force causes the acceleration of the block? It is clear that the only force in the horizontal direction is the frictional force. The frictional force on block will be static nature and should act in forward direction.

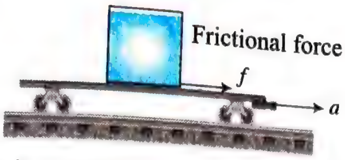
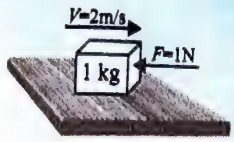


ILLUSTRATION 7.1

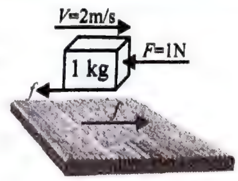
A block of mass 1 kg is sliding on a rough platform with velocity 2m/s. An external force F = 1 N is acting on the block as shown in figure. Find the direction of kinetic friction force

(a) on the block exerted by the ground.

(b) on the ground exerted by the block.



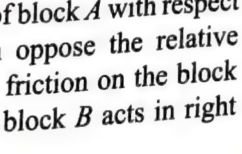
Sol. The direction of kinetic friction force is opposite to relative motion of the surface. The motion of point of contact of the block with platform moves in right direction.



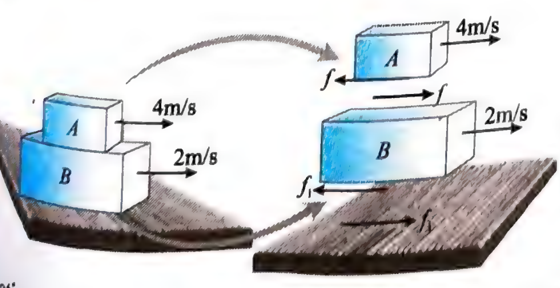
Friction opposes the relative motion of point of contact. Hence, kinetic friction on the block acts in left direction and kinetic friction on the ground acts in right direction.

ILLUSTRATION 7.2

Two blocks are moving on a rough platform with different velocities as shown in figure. All surfaces are rough. Find the direction of kinetic friction forces on each block at this instant.



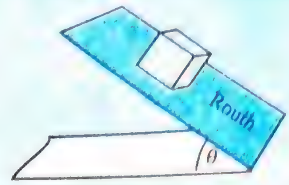
Sol. The direction of frictional force is opposite to relative motion of the surface. The relative motion of block A with respect to block B is in right direction. Friction oppose the relative motion of point of contact. Hence, kinetic friction on the block A acts in left direction and friction on the block B acts in right direction.



The relative motion of block B relative to platform is in right direction. Hence, kinetic friction on lower surface of block B should be in left direction and on platform in right direction.

ILLUSTRATION 7.3

A block of mass m is placed at rest on rough inclined plane. Is there any friction force acting on the block? Draw the free body diagram of block and calculate the frictional force acting on the block.



(Given: The coefficient of static friction = μ_s and the coefficient of kinetic friction = μ_k)

Sol. Let us analyze the first part of question. In this situation, block is at rest but component of gravity parallel to the inclined surface is acting on block. If there is no friction, the block will slide down the incline. There is motion tendency of block in downward direction. Friction opposes the relative motion tendency. Hence, the frictional force will act on the block in upwards direction and the nature of friction should be static.

Static friction is of self-adjusting nature. To make the block at rest, this friction should be equal to the component of weight along the inclined plane, i.e, $mg \sin \theta$.

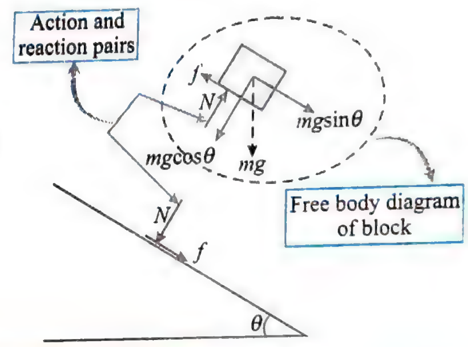
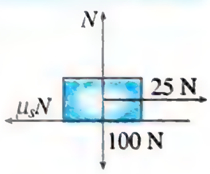


ILLUSTRATION 7.4

A block of weight 100 N lying on a horizontal surface just begins to move when a horizontal force of 25 N acts on it. Determine the coefficient of static friction.

Sol. As the 25-N force brings the block to the point of sliding, the frictional force = $\mu_s N$. From force diagram, N = 100 N

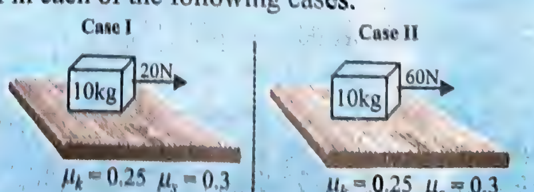


$$\mu_s N = 25$$

$$\Rightarrow \quad \mu_s = 0.25$$

ILLUSTRATION 7.5

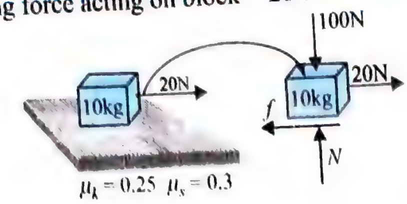
Determine the magnitude of frictional force and acceleration of block in each of the following cases.



Sol. The force parallel to surface on which the block can move is called driving force acting on block.

Case I

Here driving force acting on block = 20 N.



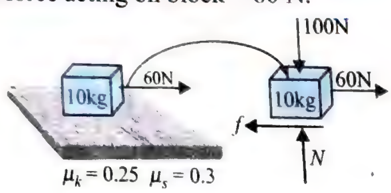
Maximum resisting force on the block = Maximum possible frictional force on the block

$$f_{\max} = f_{\lim} = \mu_s N = 0.3 \times 100 = 30\text{ N}$$

Here maximum resisting force on the block (30N) is greater than driving force acting on block (20N). Hence the block will not move. Friction will be of static nature. Static friction is of self-adjusting nature. The value of static friction will be such that to keep the block in static condition. Hence, the frictional force $f = 20$ N.

Case II

Here driving force acting on block = 60 N.



Maximum resisting force on the block = Maximum possible frictional force on the block

$$f_{\max} = f_{\lim} = \mu_s N = 0.3 \times 100 = 30\text{ N}$$

Here maximum resisting force on the block (30N) is less than driving force acting on block (60N)

Hence, there will be relative motion between block and ground. Friction will be of kinetic nature.

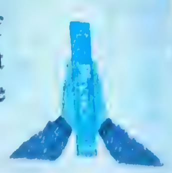
$$f = f_k = \mu_k N = 0.25 \times 100 = 25\text{ N}$$

For calculating acceleration: From free body diagram of the block, $a = \dfrac{60-25}{10} = 3.5\text{ m/s}$

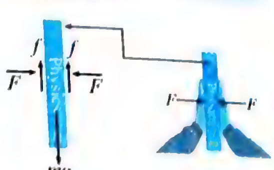
ILLUSTRATION 7.6

If you want to press your physics book of mass m between your vertical palms so that it remains in equilibrium, with how much force you should press the book?

μ_s = coefficient of static friction between the palms and book.



Sol. Let us draw the free body diagram of the book. The frictional force which is static in nature will act on both surfaces of book in vertical up direction as book has tendency to slid down with respect to hands. The force applied by palms on the book will be equal to normal reaction acting on it. As book is in equilibrium, net force acting on book should be zero.



From F.B.D. of book: $2f = mg \Rightarrow f = \dfrac{mg}{2}$

For friction to be static $f \le \mu_s N \Rightarrow \dfrac{mg}{2} \le \mu_s F$ or $F \ge \dfrac{mg}{2\mu_s}$

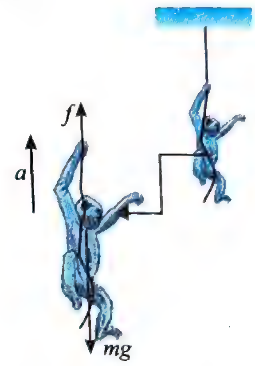
Minimum force applied should be $\dfrac{mg}{2\mu_s}$.

ILLUSTRATION 7.7

A monkey of mass m is climbing a rope hanging from the roof with acceleration a. The coefficient of static friction between the body of the monkey and the rope is μ. Find the direction and value of friction force on the monkey and tension in the string.

Sol. There is no sliding between monkey and rope. Hence, friction will be of static nature.

From F.B.D of monkey,



$$f - mg = ma \Rightarrow f = m(g+a)$$

This friction is equal to tension in the string

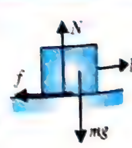
$$f = T = m(g+a)$$

ILLUSTRATION 7.8

A block of mass m = 1 kg is at rest on a rough horizontal surface having coefficient of static friction $\mu_s = 0.2$ and kinetic force μ_k = 0.15. Discuss the frictional forces if a horizontal force F is applied on the block.

S.No.	Value of F
1.	1 N
2.	2 N
3.	2.5 N

Sol. Assume no friction, the motion of the block will be in the direction of applied force. The friction opposes relative motion between two surfaces. The direction of friction will be opposite to relative motion, i.e., in leftward direction.



Now let us find the maximum frictional force.

$$f_{\max} = f_{\lim} = \mu_s N = 0.2 \times 1 \times 10 = 2\text{ N}$$

(a) If F = 1.0 N

If the block is at rest friction acting on the block, $f = F$ = 1.0 N.

As $f < f_{\lim}$, hence, friction will be of static nature and acceleration of the block will be zero.

(b) If $F = 2.0$ N

Here the applied force equals limiting friction. The block is on the verge of slipping.

or $f = f_{\lim} = 2.0$ N

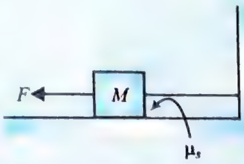
(c) If $F = 2.5$ N

Here the applied force is greater than the magnitude of the maximum nature of static friction. The nature of friction between the block and ground is of kinetic nature.

or $f = f_k = \mu_k N = 0.15 \times 1 \times 10 = 1.5$ N

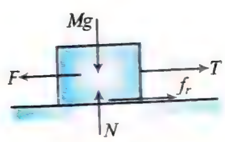
ILLUSTRATION 7.9

In the given figure, force F is gradually increased from zero. Draw the graph between applied force F and tension T in the string. The coefficient of static friction between the block and the ground is μ_S.



Sol. As the external force F is gradually increased from zero, it is compensated by the friction and the string bears no tension. When the limiting friction is achieved by increasing force F to a value till $\mu_s mg$, the further increase in F is transferred to the string.

$$\text{When}\begin{cases} F < \mu_s mg; \text{ the friction is static and } T = 0 \\ F \geq \mu_s mg;\ T = F - \mu_s mg \end{cases}$$



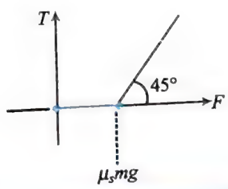
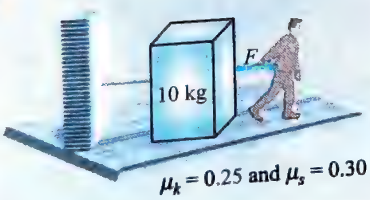


ILLUSTRATION 7.10

Calculate the tension in the string connecting the wall and block if

(i) $F = 20$ N

(ii) $F = 60$ N



Sol.

Case (i)

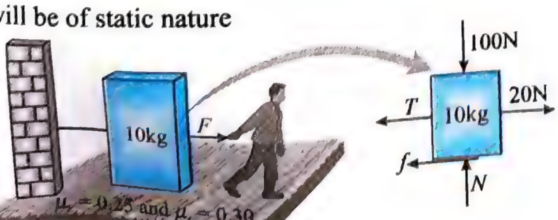
Here driving force acting on block = 20 N

Maximum resisting force on the block = Maximum possible frictional force on the block

$= f_{max} = f_{\lim} = \mu_s N = 0.3 \times 100 = 30$ N

Here maximum resisting force on the block (30N) is greater than driving force acting on block (20N)

Friction will be of static nature



Static friction is of self-adjusting nature.

The value of static friction will be the block stays in static condition.

$f = 20$ N

Hence tension in string will be zero.

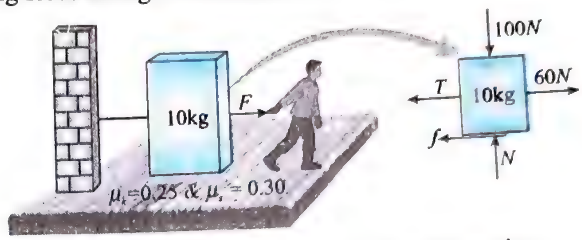
Case (ii)

Here driving force acting on block = 60 N

Maximum resisting force on the block = Maximum possible friction force on the block

$= f_{max} = f_{\lim} = \mu_s N = 0.3 \times 100 = 30$ N

Here maximum resisting force on the block (30N) is less than driving force acting on block (60N)



The block cannot move as it is connected with a string.

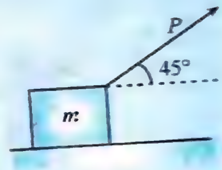
The friction will be static nature $f = 30$ N

From free body diagram of the block, $T + 30 = 60$ N

$\Rightarrow \quad T = 60 - 30 = 30$ N

ILLUSTRATION 7.11

Find the least pulling force which acting at an angle of 45° with the horizontal, will slide a body weighing 5 kg along a rough horizontal surface. The coefficient of friction $\mu_s = \mu_k = 1/3$. If a force of double this is applied along the same direction, find the resulting acceleration of the block.



Sol. When the block is about to start sliding, the frictional force acting on block reaches to its limiting value $f = \mu_S N$

In vertical direction: $N + P \sin 45° = mg$

$\Rightarrow \quad N = mg - P \sin 45°$

In horizontal direction:

$P \cos 45° = \mu_s N$

or $\quad P \cos 45° = \mu_s(mg - P \sin 45°)$

or $$P = \frac{\mu_s\, mg}{\cos 45° + \mu_s \sin 45°} = \frac{25}{\sqrt{2}}\text{ N}$$

If applied force is 2P: Friction will be kinetic nature. The block will move $f = \mu_k N$.

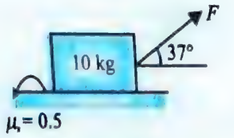
$N = mg - 2P \sin 45°$ and $2P \cos 45° - \mu_k N = ma$

$2P \cos 45° - \mu_k(mg - 2P \sin 45°) = ma$

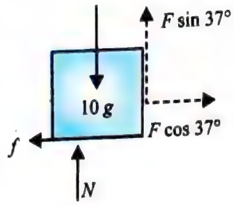
$$a = \frac{2P}{m\sqrt{2}}(\mu_k + 1) - \mu_k g = \frac{10}{3}\text{ ms}^{-2}$$

ILLUSTRATION 7.12

Force F is gradually increased from zero. Determine whether the block will first slide or lift up?



Sol. There is minimum magnitude of forces required both in horizontal and vertical directions, either to slide or lift up the block. The block will first slide or lift up will depend upon which minimum magnitude of force is lesser.



For vertical direction to start lifting up,

$$F \sin 37° + N \geq Mg$$

N becomes zero just lifting condition.

$$F \geq \frac{Mg}{\sin 37°}$$

$$F_{\text{lift}} \geq \frac{10g}{3/5} \quad \text{or} \quad F_{\text{lift}} \geq \frac{500}{3}\text{N}$$

For horizontal direction to start sliding

$$F \cos 37° \geq \mu_s N$$

$$F \cos 37° > 0.5\,[10g - F \sin 37°]$$

$(\because N = 10g - F \sin 37°)$

$$\text{Hence } F_{\text{slide}} > \frac{50}{\cos 37° + 0.5 \sin 37°}$$

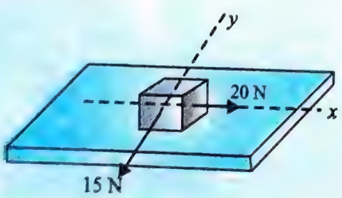
$$F_{\text{slide}} > \frac{500}{11}\text{N}$$

$$F_{\text{lift}} > \frac{500}{3}\text{N} \Rightarrow F_{\text{slide}} < F_{\text{lift}}$$

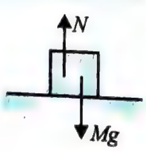
Therefore, the block will begin to slide before lifting.

ILLUSTRATION 7.13

In the given figure, an object of mass $M = 10$ kg is kept on a rough table as seen from above. Forces are applied on it as shown. Find the direction of static friction if the object does not move. (Take $\mu = 0.4$)



Sol. Here limiting value of frictional force



$$f_{\text{lim}} = \mu_s N = 0.4 \times 10 \times 10 = 40 \text{ N}$$

The resultant of external forces acting on the block (figure)

$$F_{\text{net}} = \sqrt{(15)^2 + (20)^2} = 25\text{N}$$

If the block is at rest, $f = F_{\text{net}} = 25$ N

As $f < f_{\text{lim}}$.

Here actual frictional force acting on the block is less than f_{lim}. The friction in this case is of static nature.

For the direction of frictional force, we draw the free body diagram of find the resultant force.

The direction of static friction is opposite to the direction of the resultant force F_{net}. Its magnitude is equal to 25 N at angle,

$$\tan\theta = \frac{15}{20} = \frac{3}{4} \text{ or } \theta = 37° \text{ as shown in figure.}$$

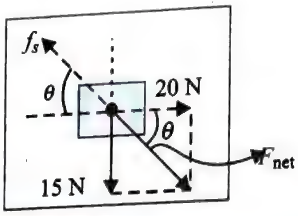
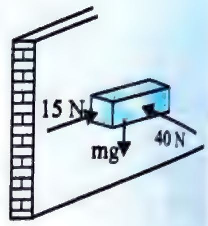
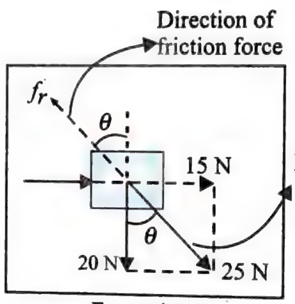


ILLUSTRATION 7.14

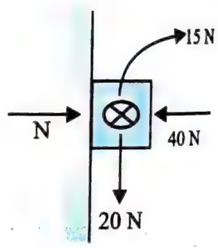
A block of mass 2 kg is pushed normally against a rough vertical wall with a force of 40 N, co-efficient of static friction being 0.5. Another horizontal force of 15 N is applied on the block in a direction parallel to the wall. Will the block move? If yes, in which direction and with what acceleration? If no, find the frictional force exerted by the wall on the block.



Sol. The force which may cause the tendency of motion or motion in the body is its own weight and the applied horizontal force of 15 N.



Front view



Side view

The resultant of the forces $F = \sqrt{20^2 + 15^2} = 25\text{N}$

Hence, net driving force $F_{\text{driving}} = 25$ N in a direction $\tan^{-1}\left(\frac{15}{20}\right) = 37°$ with the vertical.

The friction opposing the tendency of relative motion act in a direction opposite to the resultant force.

Maximum resisting force = limiting friction

$$f_{\text{lim}} = \mu N = 0.5 \times 40 = 20 \text{ N}$$

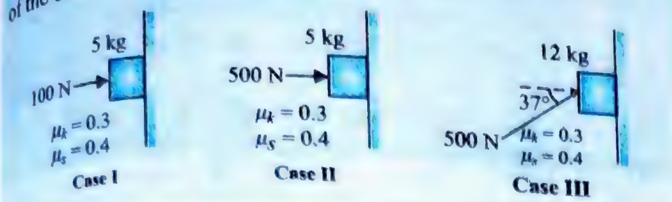
As driving force is greater than maximum resisting force, hence the block will move and friction will be kinetic nature and $f_r = 20$ N

The direction of frictional force will be opposite to the direction of net force.

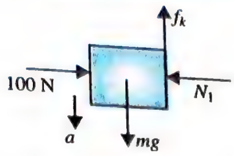
$$\text{Acceleration of block, } a = \frac{F_{\text{net}} - f_r}{m} = \frac{25 - 20}{2} = 2.5 \text{ ms}^2$$

ILLUSTRATION 7.15

Determine the magnitude of frictional force and acceleration of the block in each of the following cases:



Sol. Case I



$N_1 = 100$ N, $mg = 50$ N

$f_l = \mu_s N_1 = 0.4 \times 100 = 40$ N,

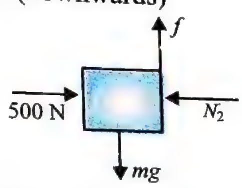
$f_k = \mu_k N_1 = 0.3 \times 100 = 30$ N

Here mg (driving force) is greater than maximum friction $f_l = 40$ N. Hence, the block will not be able to stay at rest. It will accelerate downwards. But when it starts slipping, then kinetic friction will come into play. Now

$$a = \frac{mg - f_k}{m} = \frac{50 - 30}{5} = 4 \text{ m s}^{-2}$$

So, in this case, $f = f_k = 30$ N and $a = 4$ m s^{-2} (downwards)

Case II



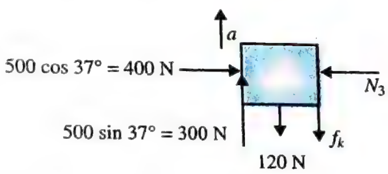
$N_2 = 500$ N, $mg = 50$ N

$f_l = \mu_s N_2 = 200$ N,

$f_k = \mu_k N_2 = 150$ N

Here f_l is greater than mg (driving force). Hence, block will not move. So, in this case, $a = 0$, $f = mg = 50$ N.

Case III



$N_3 = 400$ N,

$f_l = \mu_s N_3 = 160$ N,

$f_k = \mu_k N_3 = 120$ N

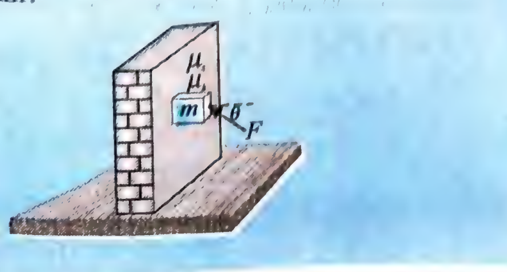
Here driving force = 300 – 120 = 180 N in upward direction; hence, friction will act downwards. Driving force is more than f_l. So the block will accelerate upwards.

$$a = \frac{180 - f_k}{m} = \frac{180 - 120}{12} = 5 \text{ m s}^{-2} \text{(upwards)}$$

So, in this case, $f = f_k = 120$ N and $a = 5$ m s^{-2} (upwards)

ILLUSTRATION 7.16

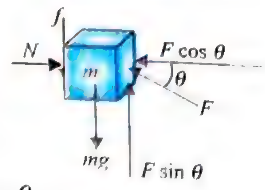
A block of mass m is supported on a rough wall by applying a force F as shown in the figure. Coefficient of static friction and kinetic friction between block and wall are μ_s and μ_k, respectively. For what range of values of F, the block remains in static equilibrium?



Sol. We can make components of F in vertical (up) and horizontal (right).

The block under the influence of $F \sin \theta$ may have a tendency to move upward or it may be assumed that $F \sin \theta$ just prevents downward fall of the block. Therefore, there are two possibilities.

Case (i) For maximum value of F



The motion tendency of block will be in upward direction.

The friction will act in downward direction.

Hence driving force acting on block $f \sin \theta - mg$

If block is at rest $f = F \sin \theta = mg$

Here normal reaction $N = F \cos\theta$

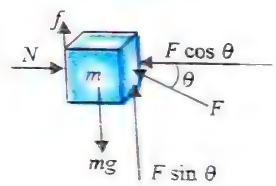
If friction is static $f \le \mu_s N \Rightarrow (F \sin\theta - mg) \le \mu_s (F \cos\theta)$

$$F \sin\theta - \mu_s F \cos\theta \le mg \Rightarrow F(\sin\theta - \mu_s \cos\theta) \le mg$$

or $$F \le \frac{mg}{(\sin\theta - \mu_s \cos\theta)}$$

Hence maximum value of applied force, $F_{max} = \dfrac{mg}{(\sin\theta - \mu_s \cos\theta)}$

Case (ii) For minimum value of F



The motion tendency of block will be in downward direction. The friction will act in upward direction.

Hence driving force acting on block

$= mg - F \sin \theta$

If block is at rest $f = mg - F \sin \theta$

Here normal reaction $N = F \cos \theta$

If friction is static $f \le \mu_s N \Rightarrow (mg - F \sin\theta) \le \mu_s (F \cos\theta)$

$$F \sin\theta + \mu_s F \cos\theta \ge mg \Rightarrow F(\sin\theta + \mu_s \cos\theta) \ge mg$$

or $$F \ge \frac{mg}{(\sin\theta + \mu_s \cos\theta)}$$

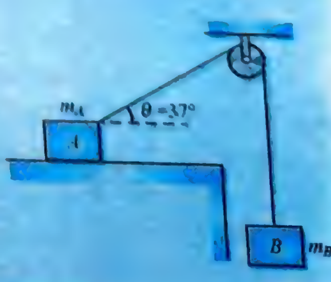
Hence minimum value of applied force, $F_{min} = \dfrac{mg}{(\sin\theta + \mu_s \cos\theta)}$

Therefore, the block will be in static equilibrium for

$$\frac{mg}{\sin\theta + \mu \cos\theta} \le F \le \frac{mg}{\sin\theta - \mu \cos\theta}$$

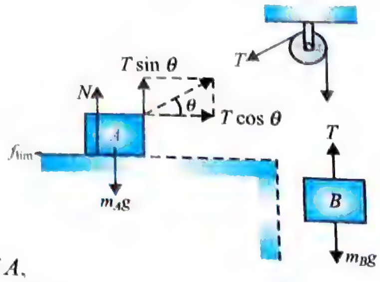
ILLUSTRATION 7.17

Two blocks A and B are connected by a light inextensible string passing over a fixed smooth pulley as shown in the figure. The coefficient of friction between the block A and B the horizontal table is $\mu = 0.5$. If the block A is just to slip, find the ratio of the masses of the blocks.



Sol. From FBD of A and B, shown in the figure, block A is just to slip. The friction will reach to its limiting value.

$$f_{\lim} = \mu N \quad \text{...(i)}$$



From FBD of A,

$$N + T\sin\theta = m_Ag \quad \text{...(ii)}$$

$$T\cos\theta = f_{\max} = \mu N \quad \text{...(iii)}$$

From (ii), $N = m_Ag - T\sin\theta$ ...(iv)

From (iii) and (iv),

$$T\cos\theta = \mu m_Ag - \mu T\sin\theta \quad \text{...(v)}$$

$$T(\cos\theta + \mu\sin\theta) = \mu m_Ag \quad \text{...(vi)}$$

From FBD of B, $T = m_Bg$ ...(vii)

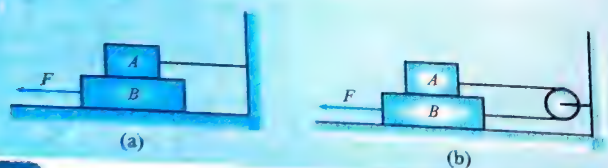
Taking ratio of (vi) and (vii),

$$\frac{\mu}{\cos\theta + \mu\sin\theta} = \frac{m_B}{m_A}$$

$$\therefore \quad \frac{m_A}{m_B} = \frac{\cos\theta + \mu\sin\theta}{\mu} = \frac{\cos 37° + (0.5)\sin 37°}{(0.5)} = \frac{11}{5}$$

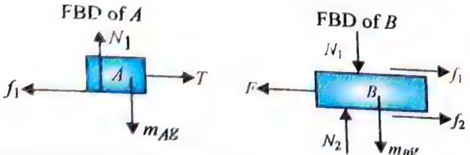
ILLUSTRATION 7.18

Block A weighs 4 N and blocks weight 8 N. The coefficient of kinetic friction is 0.25 for all surfaces. Find the force F to slide B at a constant speed when (a) A is held at rest and (b) A and B are connected by a light cord passing over a smooth pulley as shown in figure (a-b), respectively.



Sol.

(a) If A is held at rest. The frictional force on top and bottom surface of block B will be kinetic.



Friction at the top surface of block B,

$$f_1 = \mu N_1 = \mu m_Ag$$

Friction at the bottom surface of block B,

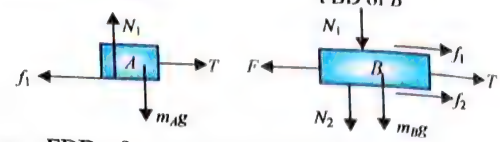
$$f_2 = \mu N_2 = \mu(m_A + m_B)g$$

If block B moves with constant speed,

$$F = f_1 + f_2 = \mu m_Ag + \mu(m_A + m_B)g$$

$$F = \mu(2m_A + m_B)g = 0.25\,(2 \times 4 + 8) = 4\text{N}$$

(b) Now A and B are connected by light string. If force is applied on block B, block B will move towards left. As blocks A and B are connected by string, block A will move towards right. The direction of frictional force will be opposite to relative motion and is shown in figure.



From FBD of A: $T = f_1 = \mu N_1 = \mu m_Ag$

From FBD. of B: $F = f_1 + f_2 + T$

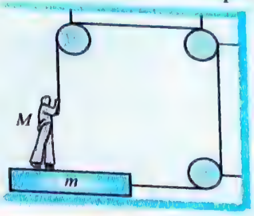
$$= 2f_1 + f_2$$

$$= 2\mu m_Ag + \mu(m_A + m_B)g$$

$$\Rightarrow \quad F = \mu(3m_A + m_B)g$$

ILLUSTRATION 7.19

The friction coefficient between the board and the floor shown in figure is μ. Find the maximum force that the man can exert on the rope so that the board does not slip on the floor.



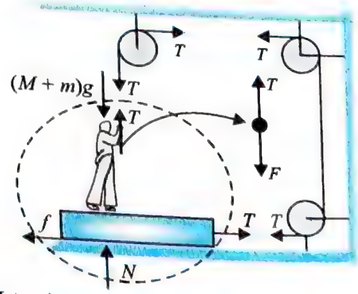
Sol. Let F is the force applied by man on rope. Hence, tension in string, $T = F$.

The man is equilibrium in vertical direction

Along vertical direction,

$$\Sigma F_v = 0:$$

or $\quad N + T = (M + m)g$



or $\quad N = (M + m)g - T$

If horizontal direction of the board is not sliding on floor, then $f = T$ and friction should static nature or $f \le f_{\max}$.

The broad will not slip over the floor, if $T \le f$.

For maximum value of T, we have

$$f = \mu N = \mu[(M + m)g - T] = \mu(M + m)g - \mu T$$

$$\Rightarrow \quad T = \left[\frac{\mu(M + m)g}{1 + \mu}\right]$$

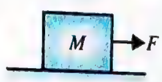
But $F = T$. Hence, $F = \left[\dfrac{\mu(M + m)g}{1 + \mu}\right]$

CONCEPT APPLICATION EXERCISE 7.1

1. A block weighing 20 N rests on a horizontal surface. The coefficient of static friction between the block and surface is 0.4 and the coefficient of kinetic friction is 0.20.
   (a) How much is the friction force exerted on the block?
   (b) How much will the friction force be if a horizontal force of 5 N is exerted on the block?
   (c) What is the minimum force that will start the block in motion ?
   (d) What is the minimum force that will keep the block in motion once it has been started?
   (e) If the horizontal force is 10 N, what is the friction force?

2. A block of mass m rests on a rough floor. The coefficient of friction between the block and the floor is μ.
   (a) Two boys apply force P at an angle θ to the horizontal. One of them pushes the block; the other one pulls. Which one would require less efforts to cause impending motion of the block?
   (b) What is the minimum force required to move the block by pulling it?
   (c) Show that if the block is pushed at a certain angle θ_0, it cannot be moved for whatever the value of P be.

3. What is the value of friction f for the following value of applied force F?

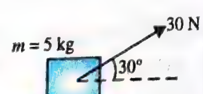


   (a) 1 N (b) 2 N (c) 3 N
   (d) 4 N (e) 20 N
   Assume the coefficient of friction to be $\mu_s = 0.3$; $\mu_k = 0.25$. Mass of the body is $m = 1$ kg. (Assume $g = 10\ \text{m s}^{-2}$)

4. A block of mass 5 kg rests on a rough horizontal surface. It is found that a force of 10 N is required to make the block just move. However, once the motion begins, a force of only 8 N is enough to maintain the motion. Find the coefficients of kinetic and static friction between the block and the horizontal surface.

5. A body of mass m is kept on a rough horizontal surface of friction coefficient μ. A force P is applied horizontally, but the body is not moving. Find the net force F exerted by the surface on the body.

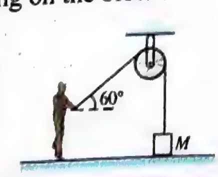
6. A 5-kg box is moving straight across the floor at a constant speed by a force of 30 N, as shown in figure.



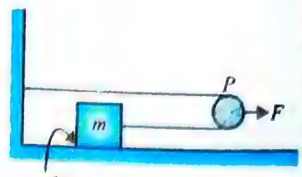
   (a) How large a friction force impedes the motion of the box?
   (b) Find μ_K between the box and the floor.

7. A block of mass $m = 2$ kg is kept on a rough horizontal surface. A horizontal force $F = 5$ N is just able to slide the block. Find the coefficient of static friction. If $F = 4$ N, then what is the frictional force acting on the block?

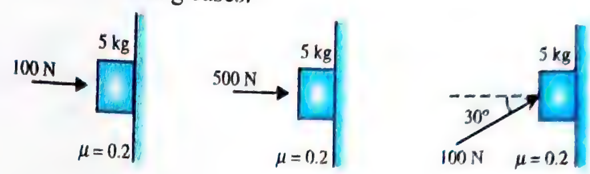
8. A man of mass 60 kg is pulling a mass M by an inextensible light rope passing through a smooth and massless pulley as shown in figure. The coefficient of friction between the man and the ground is $\mu = 1/2$. Find the maximum value of M that can be pulled by the man without slipping on the ground.



9. A block of mass $m = 2$ kg is accelerating by a force $F = 20$ N applied on a smooth light pulley as shown in figure. If the coefficient of kinetic friction between the block and the surface is $\mu = 0.3$, find its acceleration.

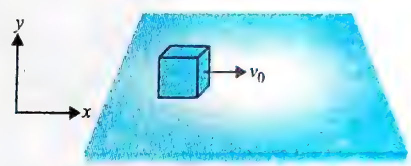


10. Determine the magnitude of frictional force f in each of the following cases:

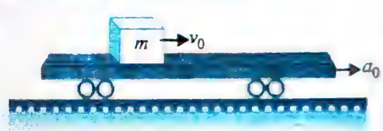


(a) (b) (c)

11. A block is pushed with a velocity on a horizontal plane which moves with a velocity (a) $2v_0\hat{i}$ (b) $-v_0\hat{j}$.
    If the coefficient of kinetic friction between the particle and plane is μ, find the force of friction acting on the particle in each case.



12. A block of mass m is projected at $t = 0$ with a horizontal velocity v_0 on a stationary plank which starts moving with a constant horizontal acceleration a_0. If μ_k = coefficient of kinetic friction between the block and plank, find



    (a) the magnitude and direction of the frictional force on the block at $t = 0$.
    (b) time t_0 after with the block comes to rest relative to the plank.

ANSWERS

1. (a) 0 (b) 5N (c) 8N (d) 4N (e) 4N

2. (a) In first case, less effort is required
   (b) $P_{\min} = \dfrac{\mu mg}{\sqrt{1+\mu^2}}$

3. (a) 1 N (b) 2N (c) 3N (d) 2.5N (e) 2.5N

4. $f_K = 8$, $\mu_K = 0.16$ 5. $\sqrt{P^2 + m^2g^2}$

6. (a) $15\sqrt{3}$ N (b) $3\sqrt{3}/7$ 7. 0.25, 4N 8. $\dfrac{120}{\sqrt{3}+2}$ kg

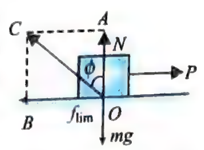
9. 2 ms^{-2} 10. (a) 20 N (b) 50 N (c) 0

11. (a) μmg, acts in forward direction (b) $-\dfrac{\mu mg}{\sqrt{2}}(\hat{i}+\hat{j})$

12. (a) μmg, acts in backward direction
    (b) $t_0 = \dfrac{v_0}{(\mu g + a_0)}$

ANGLE OF FRICTION

The angle of friction between any two surfaces in contact is defined as the angle which the resultant of the force of limiting friction F and normal reaction R makes with the direction of normal reaction R. It is represented by ϕ.



In the figure, OA represents the normal reaction R which balances the weight mg of the body. OB represents F, the limiting force of sliding friction, when the body tends to move to the right. Complete the parallelogram $OACB$. Join OC. This represents the resultant of N and F_{lim}. By definition, $\angle AOC = \phi$ is the angle of friction between the two bodies in contact.

The value of angle of friction depends on the material and nature of the surfaces in contact.

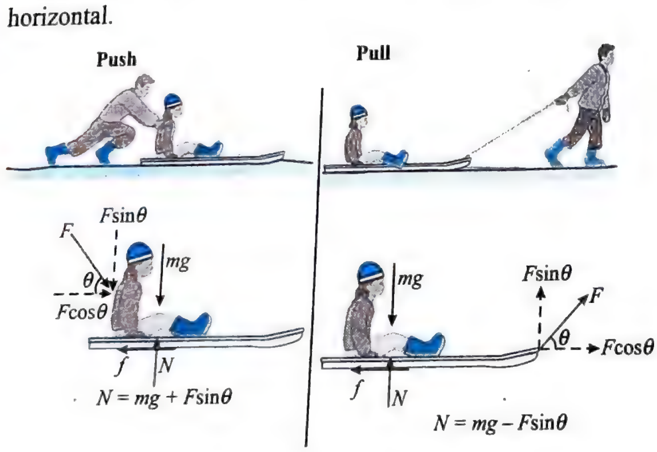
Relation between μ_s and ϕ

In ΔAOC, $\tan\phi = \dfrac{AC}{OA} = \dfrac{OB}{OA} = \dfrac{f_{lim}}{N} = \mu_s$

Hence, $\mu_s = \tan\phi$, i.e., coefficient of limiting friction between any two surfaces in contact is equal to tangent of the angle of friction between them.

PULL IS EASIER THAN PUSH

Push: Consider a block of mass m placed on a rough horizontal surface. The coefficient of static friction between the block and surface is μ. Let a push force F is applied at an angle θ with the horizontal.



As the block is in equilibrium along y-axis, thus, we have

$$\sum f_y = 0 \quad \text{or} \quad N = mg + F\sin\theta$$

To just move the block along x-axis, we have

$$F\cos\theta = \mu N = \mu(mg + F\sin\theta)$$

or
$$F = \frac{\mu mg}{\cos\theta - \mu\sin\theta} \qquad \text{...(i)}$$

Pull: Along y-axis, we have

$$\sum f_y = 0 \quad \text{or} \quad N = mg - F\sin\theta$$

To just move the block along x-axis, we have

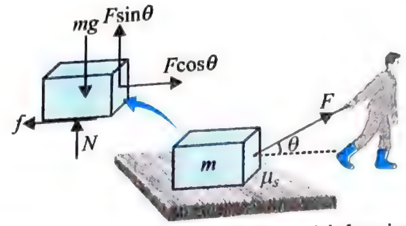
$$F\cos\theta = \mu N = \mu(mg - F\sin\theta)$$

or
$$F = \left(\frac{\mu mg}{\cos\theta + \mu\sin\theta}\right) \qquad \text{...(ii)}$$

It is clear from above discussion that pull force is smaller than push force.

MINIMUM FORCE REQUIRED TO MOVE A BLOCK

Consider a block of mass m placed at rest on a rough horizontal surface. The coefficient of static friction between the block and the surface is μ_s. It is pulled by a force F at an angle θ as shown in figure.



We wish to calculate the value of θ at which minimum force is required to move the block.

From the free body diagram of the block, we can easily understand that the applied force has two effects:

- It reduces the normal reaction thus, reduces the frictional force.
- It tends to move the block along the surface.

The minimum value of F occurs at an angle θ at which the normal reaction is reduced such that the limiting friction becomes just equal to the horizontal component of the applied force.

$$F\cos\theta = f \text{ or } F\cos\theta = \mu_s N$$

$$F\cos\theta = \mu_s(mg - F\sin\theta)$$

$$F(\cos\theta + F\sin\theta) = \mu_s mg \Rightarrow F = \frac{\mu_s mg}{(\cos\theta + \mu_s\sin\theta)} \qquad \text{...(i)}$$

For minimum value of F, $(\cos\theta + \mu_s\sin\theta)$ should be maximum.

Let $y = \cos\theta + \mu_s\sin\theta$

For minimum or maximum $\dfrac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = -\sin\theta + \mu_s\cos\theta = 0 \Rightarrow \tan\theta = \mu_s$$

$$\frac{d^2y}{d\theta^2} = -\cos\theta - \mu_s\sin\theta$$

As $\theta < 90°$, Hence $\dfrac{d^2y}{d\theta^2}$ will be negative. Hence, for $\theta = \tan^{-1}(\mu_s)$, y will be maximum or F will be minimum. But $\tan^{-1}(\mu_s)$ is angle of friction. It means if we pull the block at angle equal to angle of friction, the applied force will be minimum.

The magnitude of minimum force is obtained by putting the value of θ in equation, $F = \dfrac{\mu_s mg}{(\cos\theta + \mu_s\sin\theta)}$

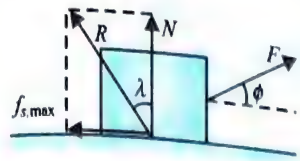
Since $\mu_s = \tan\theta$, hence $F_{\min} = \dfrac{(\tan\theta).mg}{(\cos\theta + (\tan\theta).\sin\theta)}$

$$= \frac{\left(\dfrac{\sin\theta}{\cos\theta}\right).mg}{\left(\cos\theta + \left(\dfrac{\sin\theta}{\cos\theta}\right).\sin\theta\right)} = \frac{\left(\dfrac{\sin\theta}{\cos\theta}\right).mg}{\dfrac{(\cos^2\theta + \sin^2\theta)}{\cos\theta}}$$

$$\Rightarrow \quad F_{\min} = mg\sin\theta$$

Important Points:

- The block can be moved with least effort on a rough surface (μ) if the force is applied at the angle of friction (λ). That is, $\phi = \tan^{-1}\mu_s = \lambda$



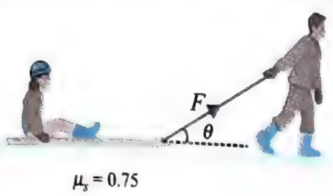
- The minimum magnitude of force is given by $F_{min} = mg \sin \lambda$

ILLUSTRATION 7.20

A sledge of mass m = 10 kg is to be pulled on a horizontal rough surface with the minimum force.

(i) The sledge should be pulled at an angle θ = ...

(ii) The magnitude of the force F is equal to ....



Sol.

(i) The sledge can be moved with least effort on a rough surface (μ_s) if the force is applied at the angle of friction. That is, $\theta = \tan^{-1}\mu_s$

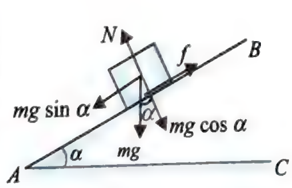
Angle of friction $\theta = \tan^{-1}(0.75) = 37°$

(ii) The magnitude of external force is

$$f_{\min} = mg\sin\theta = (10)(10)\sin 37° = 10 \times 10 \times \frac{3}{5} = 60\,\text{N}$$

ANGLE OF REPOSE OR ANGLE OF SLIDING

The angle of repose or angle of sliding is defined as the minimum angle of inclination of a plane with the horizontal such that a body placed on the plane just begins to slide down. Its value depends on the material and nature of the surfaces in contact.



In figure, AB is an inclined plane such that a body placed on it just begins to slide down. $\angle BAC = \alpha$ = angle of repose. The various forces involved are:

1. Weight, mg, of the body acting vertically downwards
2. normal reaction, N, acting perpendicular to AB
3. Force of friction, f, acting upon the plane AB

Now, mg can be resolved into two rectangular components: $mg \cos \alpha$ opposite to N and $mg \sin \alpha$ opposite to F. In equilibrium,

$$f = mg \sin \alpha \quad \text{...(i)}$$

$$N = mg \cos \alpha \quad \text{...(ii)}$$

Dividing (i) by (ii), we get $\dfrac{f}{N} = \dfrac{mg\sin\alpha}{mg\cos\alpha} = \mu_s$

i.e., $\mu = \tan \alpha$ ...(iii)

Hence, the coefficient of limiting friction between any two surfaces in contact is equal to the tangent of the angle of repose between them.

ILLUSTRATION 7.21

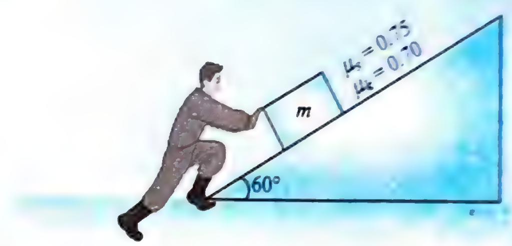
A block lying on an inclined plane has a weight of 50 N. It just beings to slide down when inclination of plane with the horizontal is 30°. Find the coefficient of static friction.

Sol. The block reaches the point of sliding when the plane makes an angle of 30° with the horizontal. Hence, 30° is the angle of repose; so $\mu_s = \tan 30° = 1/\sqrt{3}$.

Note: Minimum angle for which a block starts sliding down an inclined plane is known as angle of repose. Angle of repose is independent of mass of the object.

ILLUSTRATION 7.22

A block of mass m is placed on a rough inclined plane. If the block is released from rest. the block will slide or not after release.



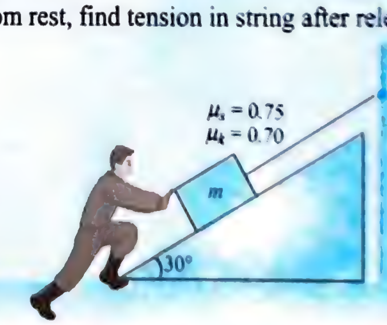
Sol. Here angle of repose $\theta = \tan^{-1}\mu_s = \tan^{-1}(0.75) = 37°$

and angle of inclination = 60°

Here angle of inclination of plane is greater than angle of repose. Hence, the object will start sliding when released.

ILLUSTRATION 7.23

A block is connected with a string whose other end is connected with wall. Initially there is no tension in string. If the block is released from rest, find tension in string after release.



Sol. Let us first find angle of repose i.e., the angle of inclination of inclined plane with horizontal such when a body placed on it just begins to slide down. Here angle of repose $\theta = \tan^{-1}\mu_s = \tan^{-1}(0.75) = 37°$

And angle of inclination = 30°

Here angle of inclination of plane is less than angle of repose. Hence, the object will not slide when released. The friction will be static nature

$$f = mg\sin 30° = \frac{mg}{2}$$

From F.B.D. of the block, $T + f = mg \sin 30°$

or $\quad T + mg \sin 30° = mg \sin 30° \Rightarrow T = 0$

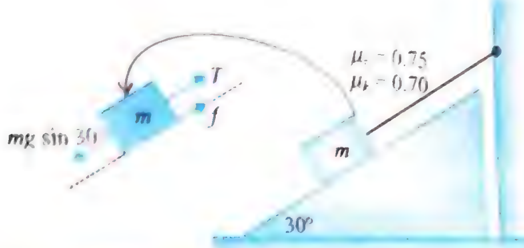
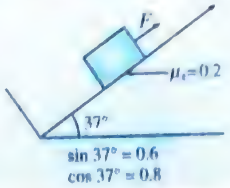


ILLUSTRATION 7.24

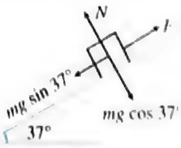
An object of mass 10 kg is to be kept at rest on an inclined plane making an angle of 37° to the horizontal by applying a force F along the plane upwards as shown in figure. The coefficient of static fiction between the object and the plane is 0.2. Find the magnitude of force F. [Take $g = 10\ \text{m s}^{-2}$.]



Sol. Component of weight parallel to inclined plane $mg \sin 37° = 10 \times 10 \times 3/5 = 60$ N

Limited friction

$f_l = \mu N = 0.2\, mg \cos 37° = 0.2 \times 10 \times 10 \times 4/5 = 16$ N



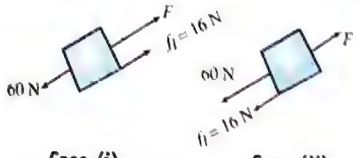
Frictional force alone is unable to balance $mg \sin 37°$. Hence, F is required for balancing. Here two cases arise.

Case I: If the block has tendency to slide down, friction on block will act upwards.

$F + 16 = 60$

$\Rightarrow \quad F = 44$ N

This is the minimum force required for balancing.



Case (i) Case (ii)

Case II: If the block has tendency to slide up, friction on block will act downwards.

$F = 60 + 16 \quad \Rightarrow \quad F = 76$ N

This is the maximum force required for balancing.

Therefore, for balancing $44\ \text{N} \le F \le 76$ N

ILLUSTRATION 7.25

A 5-kg block slides down a plane inclined at 30° to the horizontal. Find

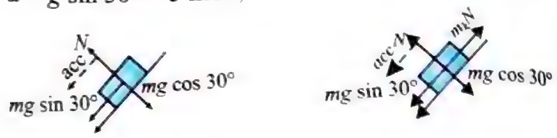
(a) The acceleration of the block if the plane is frictionless.

(b) The acceleration if the coefficient of kinetic friction is $1/2\sqrt{3}$.

Sol.

(a) $mg \sin 30° = ma$

$a = g \sin 30° = 5\ \text{m s}^{-2}$, down the plane if plane is smooth.



(b) $mg \sin 30° - \mu_k N = ma$

$a = g \sin 30° - \mu_k g \cos 30° = 5/2\ \text{m s}^{-2}$

ILLUSTRATION 7.26

A 5-kg block is projected upwards with an initial speed of $10\ \text{m s}^{-1}$ from the bottom of a plane inclined at 30° with horizontal. The coefficient of kinetic friction between the block and the plane is 0.2.

(a) How far does the block move up the plane?

(b) How long does it move up the plane?

(c) After what time from its projection does the block again come back to the bottom? With what speed does it arrive?

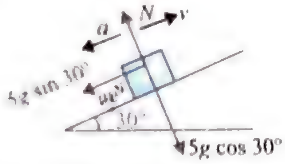
Sol. If we consider the forces acting on the block parallel to inclined plane.

$5g \sin 30°$ and $\mu_k N$ both act downwards, so acceleration a will be downward. Therefore,

$$a = \frac{5g \sin 30° + \mu_k N}{5}$$

$$= g \sin 30° + \mu_k g \cos 30°$$

$$= 6.7\ \text{m s}^{-2} \qquad (\because N = 5g \cos 30°)$$



(a) Final velocity is zero. Let the block move up to distance l before stopping: $0^2 = u^2 - 2al$

$$\Rightarrow \quad l = \frac{u^2}{2a} = \frac{10^2}{2 \times 6.7} = 7.46\ \text{m}$$

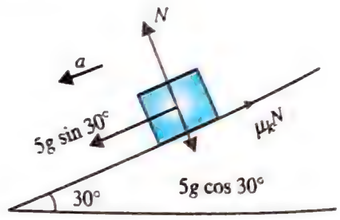
(b) On applying $v = u + at \Rightarrow 0 = 10 - 6.7t$

$\Rightarrow \quad t = 10/6.7 = 1.5$ s

(c) In this case, friction will act in upward direction

$$a = \frac{5g \sin 30° - \mu_k N}{5}$$

$$= 3.26\ \text{m s}^{-2}$$



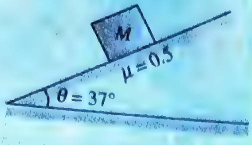
Let the time taken be t. Block will start from rest.

$$l = \frac{1}{2}at^2 \Rightarrow t = \sqrt{2l/a} = \sqrt{\frac{2 \times 7.46}{3.26}} = 2.14\ \text{s}$$

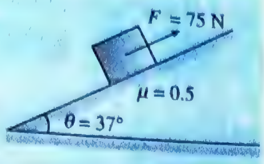
Total time $= 1.5 + 2.14 = 3.64$ s

ILLUSTRATION 7.27

A block of mass $M = 10$ kg is placed on an inclined plane, inclined at angle $\theta = 37°$ with horizontal. The coefficient of friction between the block and inclined is $\mu = 0.5$.



(a) Calculate the acceleration of the block when it is released.



(b) Now a force $F = 75$ N is applied on block as shown. Find out the acceleration of the block. If the block is initially at rest.

(c) In case (b), how much force should be added to 75 N force so that block the starts to move up the incline.

(d) In case (b), what is the minimum force by which 75 N force should be replaced with so that the block does not move?

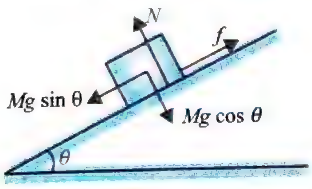
Sol.

(a) Here angle of repose $\alpha = \tan^{-1}(\mu_s)$ or $\mu_s = \tan \alpha$

$$\text{or } \tan \alpha = \frac{1}{2}$$

But inclination of plane is 37° and $\tan 37° = \frac{3}{4}$; it means $\alpha < 37°$.

Hence the block will slide.



The angle of inclination is greater than the angle of repose. The frictional force on the block will act in upward direction.

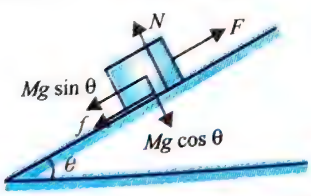
For the acceleration of block,

$$Mg \sin \theta - \mu N = Ma$$

$$\Rightarrow \quad a = g \sin \theta - \mu g \cos \theta$$

$$= 10(\sin 37° - 0.5 \cos 37°)$$

(b) If external force $F = 75$ N is applied on the block.



Let us find net driving force acting on block. Parallel to inclined two external forces are acting one in upward direction F and other is the component of weight in the direction downward the plane, $Mg \sin \theta$.

$$\text{Net driving force } f_{driving} = F - Mg \sin \theta$$

$$\Rightarrow \quad F_{driving} = 75 - 10 \times 10 \times \sin 37° = 75 - 60 = 15 \text{ N}$$

Maximum resisting force that oppose relative motion is maximum frictional force (or f_{lim})

$$f_{lim} = \mu_s Mg \cos \theta$$

$$= 0.5 \times 10 \times 10 \times \cos 37° = 40 \text{ N}$$

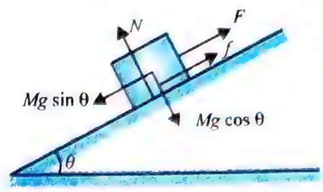
Here $F_{driving} < f_{resisting}$. Hence, the block will not move and friction will be static and will act in the direction opposite to driving force, i.e., in downward direction.

(c) To move the block, the least value of driving force should be 40 N. But in above case, driving force is 15 N (up).

Hence, if we add $\Delta F = 25$ N in upward direction, the block will overcome maximum resistance force (or friction) and starts moving up.

$$\therefore \quad 60 + 40 = 75 + \Delta F \quad \Rightarrow \Delta F = 25 \text{ N}$$

(d) As resisting force which is maximum frictional force is 40 N and the component of weight parallel to incline is 60 N and acting downward. If we remove F, then the driving forces will be the only component of the weight in the direction downward the incline plane.



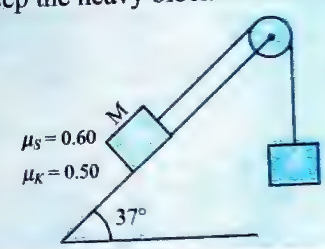
In this case, friction will act in upward direction. Hence, the required value of F to make block in equilibrium,

$$F + 40 = 60$$

$$\text{or} \quad F = 20 \text{ N}$$

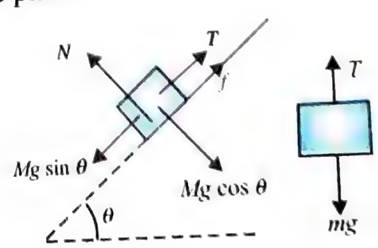
ILLUSTRATION 7.28

Two blocks M and m are arranged as shown in figure. If $M = 50$ kg, then determine the minimum and maximum values of mass of block m to keep the heavy block M stationary.



Sol. Since the angle of inclination of the plane is more than the angle of repose, i.e., $37° > \tan^{-1} 0.5$ or $\tan 37° > 0.5$, therefore, the block M has a tendency to slide down. In order to keep it stationary, the necessary force is applied by the tension in the string.

If the block M also has the tendency to slide down, the friction will act in up the plane.



For equilibrium of M: $f = Mg \sin \theta - T$ ...(i)

For equilibrium of m: $T = mg$ ...(ii)

From (i) and (ii),

$$f = Mg \sin \theta - mg \quad \text{...(iii)}$$

For friction to be static, $f < f_{max}$

$$Mg \sin \theta - mg < \mu_s (Mg \cos \theta)$$

$$\Rightarrow \quad M(\sin \theta - \mu_s \cos \theta) < m$$

Hence, $\quad m > M(\sin \theta - \mu_s \cos \theta)$ ...(i

$\Rightarrow \quad m > 6$

Hence, minimum value of mass, $m_{min} = 6$ kg.

Block A has the tendency to slide up, the friction will act in downward direction. By this relation, we will get the maximum value of m. This value of m can be obtained by simply substituting $-\mu_s$ in place of μ_s in Eq. (iv). Finally, we will get the relation

$$m < M(\sin\theta - (-\mu_s)\cos\theta)$$

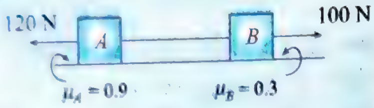
$$\Rightarrow \quad m < M(\sin\theta + \mu_s \cos\theta)$$

$$m < 50\left(\frac{3}{5} + \frac{3}{5}\times\frac{4}{5}\right) \Rightarrow m < 54 \text{ kg}$$

Hence, the maximum value of mass, $m_{max} = 54$ kg.

ILLUSTRATION 7.29

Two blocks A and B of mass $m_A = 10$ kg and $m_B = 20$ kg are place on rough horizontal surface. The blocks are connected with a string. If the coefficient of friction between block A and ground is $\mu_A = 0.9$ and between block B and ground is $\mu_B = 0.3$, find the tension in the string in situation as shown in figure. Forces 120 N and 100 N start acting when the system is at rest?

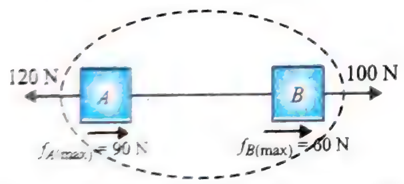


Sol. Let us assume that the system moves towards left. Then as it is clear from FBD,

$$F_{driving} = 120 - 100 = 20 \text{ N}$$

$$f_{resisting} = \mu_A N_A + \mu_B N_B$$

$$= 90 + 60 = 150 \text{ N}$$



As $F_{resisting} > F_{driving}$, therefore, it can be concluded that the system is stationary. Now there may be two possibilities.

- The friction between both blocks and ground should be static.
- The friction between one block and ground is static and between the other block and ground is limiting.

If friction between both blocks and ground is static, the tension in string should be zero.

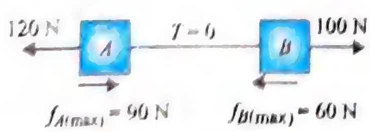
The friction on block A, $f_A = 120$ N

The friction on block B, $f_B = 100$ N

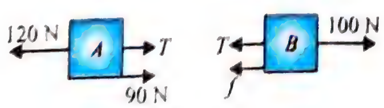
But $(f_A)_{max} = 90$ N and $(f_B)_{max} = 60$ N

Hence, the friction on both blocks cannot be static.

Hence, friction on one block is static and on other block should be limiting.



Assuming that the 10 kg block reaches limiting friction first then using FBD's.



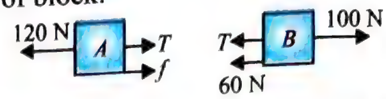
If block A is at rest

$$120 = T + 90 \Rightarrow T = 30 \text{ N}$$

From FBD of B: $T + f = 100$

$\therefore \quad 30 + f = 100$

Thus, $f = 70$ N which is not possible as the limiting value is 60 N for this surface of block.



Therefore, our assumption is wrong and now taking the 20-kg surface to be limiting, we have

From FBD of B,

$$T + 60 = 100 \quad \Rightarrow T = 40 \text{ N}$$

From FBD of A,

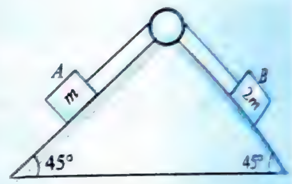
Also $\quad f + T = 120 \quad \Rightarrow f = 80$ N

This is acceptable as the static friction at this surface should be less than 90 N.

Hence, the tension in the string is $T = 40$ N.

ILLUSTRATION 7.30

Block A of mass m and block B of mass $2m$ are placed on a fixed triangular wedge by means of a light and inextensible string and a frictionless pulley as shown in figure. The wedge is inclined at 45° to the horizontal on both sides. The coefficient of friction between the block A and the wedge is 2/3 and that between the block B and the wedge is 1/3. If the system of A and B is released from rest, then find,



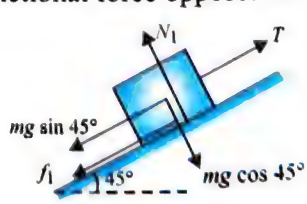
(a) the acceleration of A

(b) tension in the string

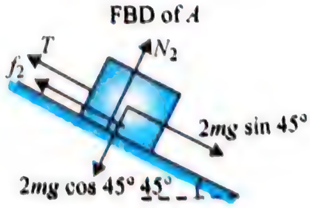
(c) the magnitude and direction of the frictional force acting on A

Sol.

(a) For finding the direction of friction, first assume there is no friction anywhere. In the absence of friction, block B will move down the plane and the block A will move up the plane. Frictional force opposes this motion.



FBD of A



FBD of B

FBD of block A,

$$\Rightarrow \quad T - mg\sin 45° - f_1 = ma \qquad \text{...(i)}$$

From FBD of B

$$2\,mg\sin 45° - f_2 - T = 2ma \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$mg \sin 45° - (f_1 + f_2) = 3ma$$

For a to be non-zero $mg \sin 45°$ must be greater than the maximum value of $(f_1 + f_2)$. Therefore,

$$(f_1 + f_2)_{max} = \mu_1 N_1 + \mu_2 N_2$$
$$= (\mu_1 m_1 + 2\mu_2 m_2) g \cos 45°$$
$$= \frac{4}{3} mg \cos 45°$$

$$\Rightarrow \quad mg \sin 45° < (f_1 + f_2)_{max}$$

Hence, blocks will remain stationary.

(b) FBD of block B

$$\mu_2 N_2 = \frac{1}{3} 2\, mg \cos 45° = \frac{2}{3\sqrt{2}} mg$$

Component of weight of B parallel to inclined plane

$$(W_{||})_B = 2mg \sin 45° = \frac{2mg}{\sqrt{2}}$$

Because $2\, mg \sin 45° > f_{2(max)}$, therefore block B has the tendency to slide down the plane.
For block B to be at rest,

$$T + f_{2(max)} = 2 \text{ mg} \sin 45°$$

$$\Rightarrow \quad T = \frac{mg}{\sqrt{2}}\left(2 - \frac{2}{3}\right) = \frac{4mg}{3\sqrt{2}} = \frac{2\sqrt{2}}{3} mg$$

(c) Component of weight of A parallel to inclined plane

$$(W_{||})_A = mg \sin 45° = \frac{mg}{\sqrt{2}}$$

As $T > (W_{||})_A$, hence block A has the tendency to move up the plane. Therefore, frictional force on the block A will be down the plane.

For A to be at rest

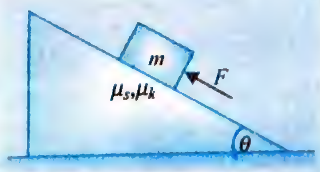
FBD of A $\quad mg \sin 45° + f = T$

$$\Rightarrow \quad f = T - mg \sin 45°$$

$$= \frac{2\sqrt{2}mg}{3} - \frac{mg}{\sqrt{2}} = \frac{mg}{3\sqrt{2}}$$

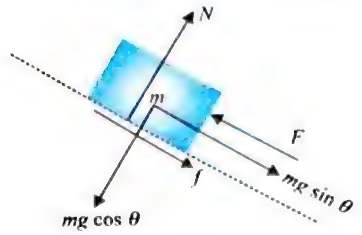
ILLUSTRATION 7.31

A block of mass m is supported on a rough inclined plane by applying a force F as shown in figure. The coefficient of static friction and kinetic friction between block and plane are μ_s and μ_k, respectively. For what range of values of F, the block remains in static equilibrium?



Sol. The block under the influence of F may have a tendency to move upward or it may be assumed that F just prevents downward sliding of the block. Therefore, there are two possibilities.

Case (i) For maximum value of F, the motion tendency of block will be in upward direction. The friction will act in downward direction.



The driving force acting on block $F_{driving} = F - mg \sin\theta$

If block is at rest $f = F - mg \sin\theta$

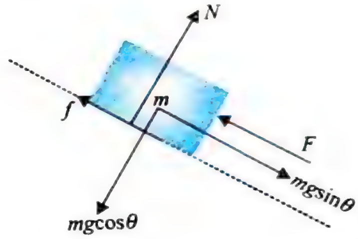
Here normal reaction $N = mg \cos\theta$

If friction is static $f \le \mu_s N \Rightarrow (F - mg \sin\theta) \le \mu_s (mg \cos\theta)$

$$F \le mg \sin\theta + \mu_s mg \cos\theta$$

Hence, maximum value of F is $mg(\sin\theta + \mu_s \cos\theta)$.

Case (ii) For minimum value of F, the motion tendency of block will be in downward direction. The friction will act in upward direction.



The driving force acting on block $F_{driving} = mg \sin\theta - F$

If block is at rest $f = mg \sin\theta - F$

Here normal reaction $N = mg \cos\theta$

If friction is static $f \le \mu_s N \Rightarrow (mg \sin\theta - F) \le \mu_s (mg \cos\theta)$

$$F \ge mg \sin\theta - \mu_s mg \cos\theta$$

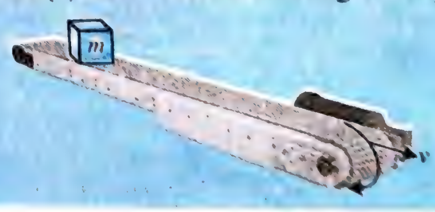
Hence minimum value of F is $mg(\sin\theta - \mu_s \cos\theta)$

The block remains stationary if $F_{min} \le F \le F_{max}$

$$mg(\sin\theta - \mu_s \cos\theta) \le F \le mg(\sin\theta + \mu_s \cos\theta)$$

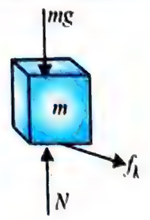
ILLUSTRATION 7.32

A packet is dropped from rest onto a conveyor belt which is running with a velocity v. If the coefficient of kinetic friction between the package and belt is μ, find the (a) maximum distance of sliding of the packet relative to (i) the ground, (ii) the belt and (b) time of relative sliding of the packet.



Sol. When block is placed onto the conveyor belt, the relative sliding of the block will be in backward direction. Hence, friction on block will act in forward direction till it slides.

Finally velocity of block also becomes v. After this the friction disappears.



Acceleration of the block, $a = \dfrac{\mu mg}{m} = \mu g$

Acceleration of the block with respect to conveyor belt.

$$\vec{a}_{\text{block,belt}} = \vec{a}_{\text{block}} - \vec{a}_{\text{plank}} = \mu mg - 0 = \mu mg$$

Analyzing block with respect to ground

Initial velocity of block $u = 0$

Final velocity of block $= v$

Using $v^2 = u^2 + 2as \Rightarrow v^2 = 0 + 2(\mu g)s \Rightarrow s = \dfrac{v^2}{2\mu g}$

Analyzing block with respect to conveyor belt.

Acceleration of the block with respect to conveyor belt.

$$\vec{a}_{\text{block,belt}} = \vec{a}_{\text{block}} - \vec{a}_{\text{belt}} = \mu g - 0 = \mu g$$

Initial velocity of block with respect to conveyor belt $u = 0 - v = -v$

Final velocity of block with respect to conveyor belt $v = 0$

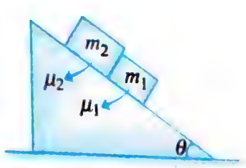
Using $v^2 = u^2 + 2as$

$$\Rightarrow \quad 0 = (-v)^2 + 2(\mu g)s' \Rightarrow s' = -\frac{v^2}{2\mu g}$$

Using $v = u + at \Rightarrow 0 = -v + (\mu g)t \Rightarrow t = \dfrac{v}{\mu g}$

Two Blocks in Contact Moving on an Inclined Plane

Two blocks are kept on an incline in contact with each other. Masses of blocks are m_1 and m_2 and coefficient of friction between inclined planes and blocks are μ_1 and μ_2, respectively. The angle of inclination is θ. If the blocks are released from rest, we need to find the acceleration of blocks.

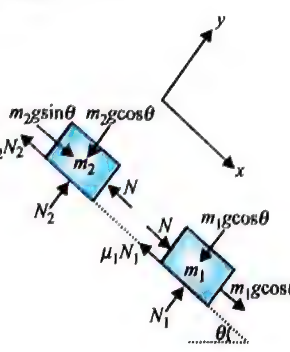


Here two cases are possible:

(i) both blocks move separately with respective accelerations (say a_1 and a_2).

(ii) both blocks move together with a common acceleration (say a).

Let us assume that both the blocks accelerate downwards. The contact force between m_1 and m_2 is N; it should not be negative or zero. Contact force between two bodies reduces to zero when the bodies are separated.



Now consider F.B.D. of the blocks. From Newton's second law:

Block 1:

$$\sum F_x = m_1 g\sin\theta + N - \mu_1 N_1 = m_1 a \quad \text{...(i)}$$

$$\sum F_y = N_1 - m_1 g\cos\theta = 0 \Rightarrow N_1 = m_1 g\cos\theta \quad \text{...(ii)}$$

Block 2:

$$\sum F_x = m_2 g\sin\theta - N - \mu_2 N_2 = m_2 a \quad \text{...(iii)}$$

$$\sum F_y = N_2 - m_2 g\cos\theta = 0 \Rightarrow N_2 = m_2 g\cos\theta \quad \text{...(iv)}$$

Now we substitute N_1 and N_2 from equations (ii) and (iv) in equations (i) and (iii), respectively. Now add equations (i) and (ii) to obtain,

$$a = \frac{(m_1 + m_2)g\sin\theta - (\mu_1 m_1 + \mu_2 m_2)g\cos\theta}{m_1 + m_2}$$

From equation (i), we obtain $N = \dfrac{(\mu_1 - \mu_2)m_1 m_2 g\cos\theta}{m_1 + m_2}$

which shows that if $\mu_2 > \mu_1$, then reaction N comes out to be negative, which is impossible. It also implies that blocks have separated.

The acceleration of m: $a_1 = \dfrac{m_1 g\sin\theta - \mu_1 N_1}{m_1}$

$$a_1 = \frac{m_1 g\sin\theta - \mu_1 m_1 g\cos\theta}{m_1} = g(\sin\theta - \mu_1\cos\theta)$$

Similarly acceleration of m_2: $a_2 = \dfrac{m_2 g\sin\theta - \mu_2 N_2}{m_2}$

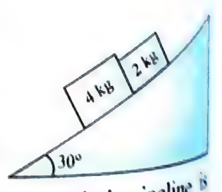
$$a_2 = \frac{m_2 g\sin\theta - \mu_2 m_2 g\cos\theta}{m_2} = g(\sin\theta - \mu_2\cos\theta)$$

Important Points:

- If the coefficient of friction between the upper block and the inclined plane is greater than the coefficient of friction between lower block and inclined plane, both the blocks will move separately with different acceleration. The acceleration of the lower block will be greater than the acceleration of upper block.
- If $\mu_1 = \mu_2$, both the blocks will move separately with same acceleration.
- If $\mu_1 = \mu_2$, both the blocks will move in contact with each other with common acceleration.

Illustration 7.33

The given figure shows two blocks in contact sliding down an inclined surface of inclination 30°. The friction coefficient between the block of mass 4 kg and the incline is $\mu_1 = 0.30$ and that between the block of mass 2 kg and the incline is $\mu_2 = 0.20$. Find the acceleration of 2.0 kg block. $g = 10\ \text{ms}^{-2}$.



Sol. Since, $\mu_1 > \mu_2$, acceleration of 2-kg block down the plane will be more than the acceleration of 4-kg block, if allowed to move separately.

In this case, both blocks are treated as a system of mass $(4 + 2) = 6$ kg and will move down with the same acceleration.

Net force down the plane is

$$F = (m_1 + m_2)g \sin\theta - \mu_1 m_1 g \cos\theta - \mu_2 m_2 g \cos\theta$$

$$= (4+2)g \sin 30^\circ - (0.2)(2)g \cos 30^\circ - (0.3)(4)g \cos 30^\circ$$

$$= (6)(10)\left(\frac{1}{2}\right) - (0.4)(10)\left(\frac{\sqrt{3}}{2}\right) - (1.2)(10)\left(\frac{\sqrt{3}}{2}\right)$$

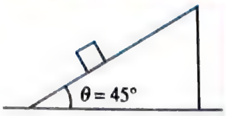
$$= 30 - 13.76 = 16.24 \text{ N}$$

Therefore, acceleration of both the blocks down the plane will be

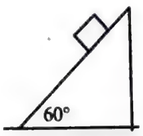
$$a = \frac{F}{m_1 + m_2} = \frac{16.24}{4+2} = 2.7 \text{ m s}^{-2}$$

CONCEPT APPLICATION EXERCISE 7.2

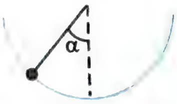
1. A block of mass $m = 3$ kg slides on a rough inclined plane of coefficient of friction 0.2. Find the resultant force offered by the plane on the block.



2. A block slides down an inclined plane (angle of inclination 60°) with an acceleration g/2. Find the coefficient of kinetic (dynamic) friction.

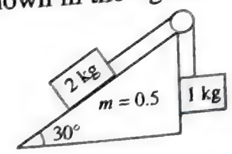


3. An insect crawls up a hemispherical surface very slowly. The coefficient of friction between the insect and the surface is 1/3. If the line joining the center of the hemispherical surface to the insect makes an angle α with the vertical, find the maximum possible value of α.

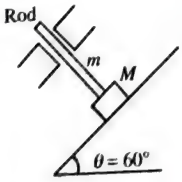


4. In the given figure, if f_1, f_2, and T are the frictional forces on 2 kg block, 3 kg block, and tension in the string, respectively, then find their values. Initially before applying the forces, tension in string was zero.

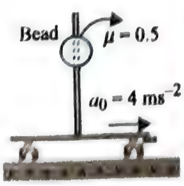
5. Find the frictional force on the 2-kg block in the arrangement shown in the figure.



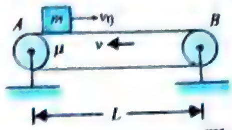
6. If the coefficient of friction between M and the inclined surface is $\mu = 1/\sqrt{3}$, find the minimum mass m of the rod so that the block of mass $M = 10$ kg remains stationary on the inclined plane.



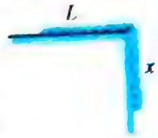
7. A thin rod of length 1 m is fixed in a vertical position inside a train, which is moving horizontally with constant acceleration 4 m s^{-2}. A bead can slide on the rod, and friction coefficient between them is 1/2. If the bead is released from rest at the top of the rod, find the time when it will reach at the bottom.



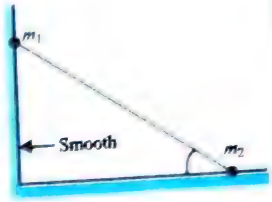
8. With what minimum velocity should block be projected from left end A towards end B such that it reaches the other end B of conveyer belt moving with constant velocity v. The friction coefficient between block and belt is μ.



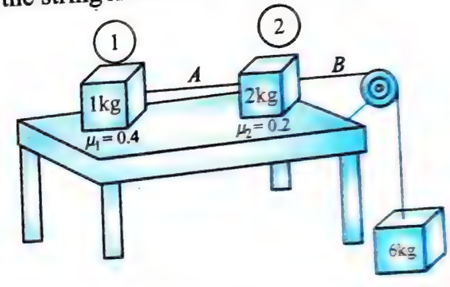
9. A uniform chain is just at rest over a rough horizontal table with its $(1/\eta)^{th}$ part of length hanging vertically. Find the co-efficient of static friction between the chain and the table.



10. Two small spheres of masses m_1 and m_2 are connected by a light rigid rod. The system is placed between a rough floor and smooth vertical wall as shown in figure. The co-efficient of friction between the floor and the sphere of mass m_2 is μ. Find the minimum value of θ so that the system of masses does not slip.

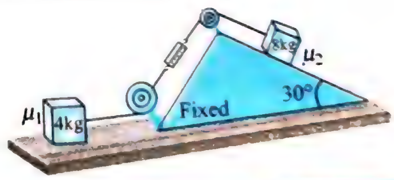


11. Three blocks are arranged as shown in the figure:
   (a) Calculate the accelerations of the blocks and the tension in the string A and B.



   (b) If the 6-kg block is replaced by a 0.3-kg block, find the new accelerations and tension in the strings A and B.

12. The reading of spring balance is 32 N and the accelerations of both the blocks is 0.5 m/s^2. Find μ_1 and μ_1.



ANSWERS

1. $6\sqrt{13}$ N **2.** $\sqrt{3}-1$ **3.** $\cot^{-1} 3$ **4.** $f_1 = 2$N, $f_2 = 6$N, $T = 2$N

5. No friction force is required.

6. 20 kg **7.** $\frac{1}{2}$ s **8.** $\sqrt{2\mu g L}$ **9.** $\mu = \frac{\eta}{1-\eta}$

10. $\theta = \cot^{-1}\left[\mu\left(1 + \frac{m_2}{m_1}\right)\right]$

11. (a) $\frac{52}{9}$ m/s^2

(b) Acceleration of the system will be zero. $T_B = 3$N, $T_A = 0$

12. $\mu_1 = \frac{3}{4}$, $\mu_2 = \frac{1}{10\sqrt{3}}$

Analysis of Frictional Force Between Two Blocks in Contact: Condition of Existence of Static Friction

In many situations, we need to analyze whether friction is present between the surface, and, if present what the nature of friction, kinetic or static, is? Following are some important points to analyze this situation.

When two surfaces tend to slide (but do not slide) relative to each other, static friction comes into play.

Then, how do we confirm the tendency of relative sliding between any two surfaces in contact? Let us find a process for it.

First of all, mentally eliminate the friction between the contact points of the surfaces where we want to detect the presence of friction. Then, considering all other forces, we find the accelerations a_1 and a_2 of these points. If the relative acceleration $a_{rel} (= |\vec{a}_1 - \vec{a}_2|)$ between the surfaces is not zero, then surfaces tend to slide relative to each other.

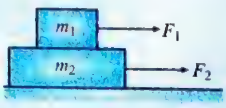
In general, when two bodies are in contact, if the net forces acting on them (ignoring the frictional forces between the contacting surface under consideration) are $\vec{F}_1$ and $\vec{F}_2$, respectively, their accelerations are given as

$$\vec{a}_1 = \frac{\vec{F}_1}{m_1} \text{ and } \vec{a}_2 = \frac{\vec{F}_2}{m_2}$$

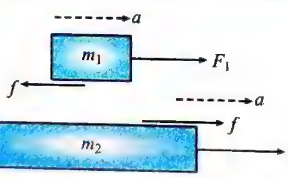
If $|\vec{a}_1 - \vec{a}_2| \neq 0$, static friction exists between the surface. If $\vec{a}_1 = \vec{a}_2$, we can say that there is no tendency of relative sliding. Hence, no friction exists; ($f_s = 0$) when $\vec{a}_1 = \vec{a}_2$.

ILLUSTRATION 7.34

Two blocks m_1 and m_2 are acted upon by the forces F_1 and F_2 as shown in figure. If there is no relative sliding between the blocks and the ground is smooth, find the static friction between the blocks. Make necessary assumptions and discuss different cases.



Sol. As there is no sliding between the blocks, the friction will be static. Assuming the frictional forces on m_1 as $f(\leftarrow)$ and m_2 as $f(\rightarrow)$, we have drawn the FBD without normal reactions and weights.



Referring to FBD as shown in figure, we have the following equations of motion:

For m_1: $\quad F_1 - f = m_1 a \quad$ ...(i)

For m_2: $\quad F_2 + f = m_2 a \quad$ ...(ii)

Since the friction is static, $a_1 = a_2$.

Substituting a_1 from Eq. (i) a_2 from Eq. (ii) in (iii), we have

$$\frac{F_1 - f}{m_1} = \frac{F_2 + f}{m_2}$$

This gives $f = \dfrac{m_2 F_1 - m_1 F_2}{m_1 + m_2}$

From Eq. (i) and (ii), we have

$$a = \frac{F_1 - f}{m_1} = \frac{F_2 + f}{m_2} \quad \text{...(iii)}$$

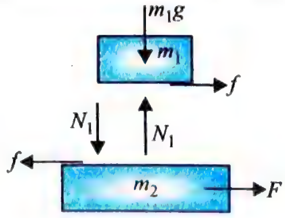
This gives $f = \dfrac{m_2 F_1 - m_1 F_2}{m_1 + m_2}$ ...(iv)

From (iii) and (iv) $a = \dfrac{(F_1 + F_2)}{m_1 + m_2}$

From equation (iv),

- If $m_2 F_1 > m_1 F_2$, then the direction of frictional force on m_1 will be in backward and on m_2 will be in forward direction.
- If $m_2 F_1 < m_1 F_2$, then friction on upper block will be in forward direction and on lower block, it will be in backward direction.
- If $F_1 = 0$ and $F_2 = F$, then $f = \dfrac{-m_1 F}{(m_1 + m_2)}$

The frictional force on upper block will be in forward direction and lower block it will be in backward direction.



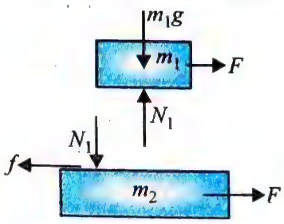
As friction is static in nature.

Hence $f \le \mu m_1 g$ or $\dfrac{m_1 F}{(m_1 + m_2)} \le \mu m_1 g \Rightarrow F \le \mu (m_1 + m_2) g$

Hence the maximum force for both the blocks will move together will be $F_{max} = \mu (m_1 + m_2) g$

- If $F_2 = 0$ and $F_1 = F$, then $f = \dfrac{m_2 F}{(m_1 + m_2)}$

The frictional force on upper block will be in backward direction and upper block it will be in forward direction.



As friction is static in nature. Hence

$$f \le \mu N, \frac{m_2 F}{(m_1 + m_2)} \le \mu m_1 g$$

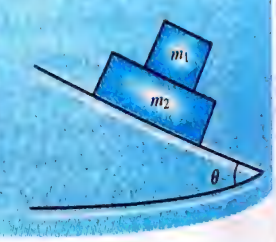
$$\Rightarrow \quad F \le \mu \frac{m_1}{m_2} (m_1 + m_2) g$$

Hence the maximum force that can be applied for both the blocks move together will be equal to

$$F_{max} = \mu \frac{m_1}{m_2} (m_1 + m_2) g$$

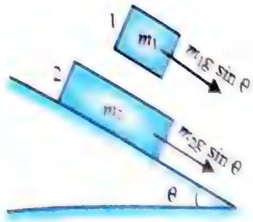
ILLUSTRATION 7.35

In the given figure, Block m_2 is loaded on block m_1. If there is no relative sliding between the blocks and the inclined plane is smooth, find the frictional force between the blocks.



Assuming no friction between the surfaces, acceleration of block (1),

$$a_1 = \frac{m_1 g\sin\theta}{m_1} = g\sin\theta$$



Acceleration of block (2),

$$a_2 = \frac{m_2 g\sin\theta}{m_2} = g\sin\theta$$

Value of $|\vec{a}_1 - \vec{a}_2| = |g\sin\theta - g\sin\theta| = 0$

So, we can say there is no tendency of relative sliding. Hence, no friction is present between (1) and (2).

ILLUSTRATION 7.36

Two blocks of masses M and m are arranged as shown in figure. There is no friction between ground and block M. The coefficients of static and kinetic friction between M and m are μ_s and μ_k, respectively.

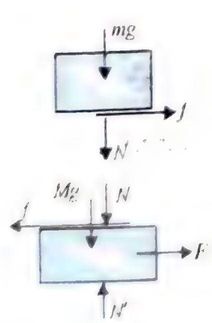
(a) Calculate the maximum possible value of F so that both the bodies move together.

(b) Find the accelerations of the blocks if F is greater than that found in part (a).

Sol.

(a) If both blocks move together, i.e., no sliding between M and m, the friction between m and M will be static nature. Static friction force is a self-adjusting force $0 \le f \le \mu_s N$

Acceleration of system in this case, $a = \dfrac{F}{(M+m)}$ ...(i)



FBD of m and M:

Equation of motion of m:

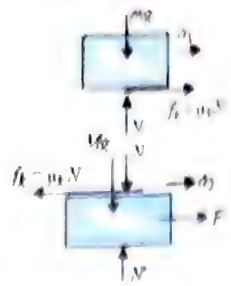
$$f = ma = m\frac{F}{(M+m)} \quad \text{(ii)}$$

If there is no sliding between M and m, then $f \le f_{s\text{-max}}$

$$\Rightarrow \quad \frac{mF}{(M+m)} \le \mu_s(mg)$$

$$\Rightarrow \quad F \le \mu_s (M+m)g$$

(b) If $F > \mu_s (M+m)g$, then there will be relative sliding between M and m. When relative sliding between M and m starts.



The frictional force reaches limiting value the frictional force becomes $f_k = \mu_k N$

Free-body diagrams of m and M.

Equation of motion of m: $\mu_k(mg) = ma_1$

Acceleration of m, $a_1 = \mu_k g$

Equation of motion of M: $F - \mu_k mg = Ma_2$

Acceleration of M, $a_2 = \dfrac{F - \mu_k mg}{M}$

ILLUSTRATION 7.37

Let us consider the next case in the previous illustration when there is no friction between ground and M. The coefficients of static and kinetic friction are μ_s and μ_k, respectively, and force F acts on upper block as shown in the figure.

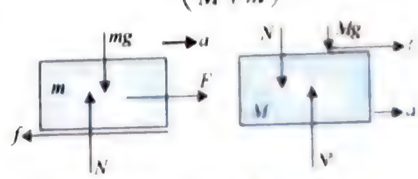
(a) What is the maximum possible value of F so that the system moves together?

(b) If there is relative sliding between M and m, then calculate acceleration of M and m.

Sol. Let the systems move together, then $a = \dfrac{F}{(M+m)}$

FBD of m and M

From FBD of M: $f = Ma = M\left(\dfrac{F}{M+m}\right)$ ...(i)

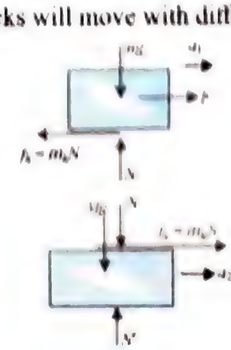


As there is no sliding between M and m,

$$M\left(\frac{F}{M+m}\right) \le \mu_s mg$$

$$\Rightarrow \quad F \le \mu_s \frac{m}{M}(M+m)g$$

If $F > \mu_s \dfrac{m}{M}(M+m)g$, the relative sliding between the blocks starts. The frictional force between the blocks will be of kinetic nature and both blocks will move with different accelerations.

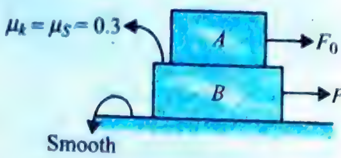


Acceleration of m, $a_1 = \dfrac{F - \mu_k mg}{m}$

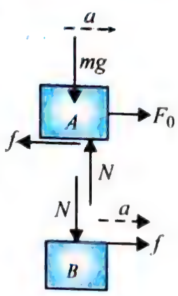
Acceleration of M, $a_2 = \dfrac{\mu_k mg}{M}$

ILLUSTRATION 7.38

Two blocks A and B of mass 10 kg and 20 kg respectively, are arranged as shown in figure. In the figure given a constant force F_0 = 120 N acts on block A and a force F applied horizontally on block B is gradually increased from zero. Discuss the direction and nature of friction force and the accelerations of the block for different values of F.



Sol. In the above situation, we see that the maximum possible value of friction between the blocks is $f_{max} = \mu_s m_A g = 0.3 \times 10 \times 10 = 30$ N.



Case I: When $F = 0$.

Considering that there is no slipping between the blocks the acceleration of system will be

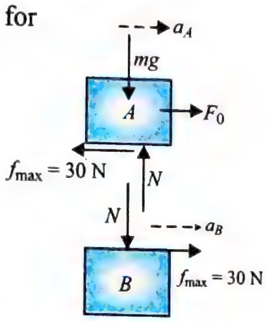
$$a = \frac{120}{20+10} = 4\,\text{m s}^{-2}$$

From FBD of B, $f = m_B a = 20 \times 4 = 80$ N

But $f_{max} = 30$ N

We can conclude that the blocks do not move together.

Now drawing the FBD of each block, for finding out individual accelerations.



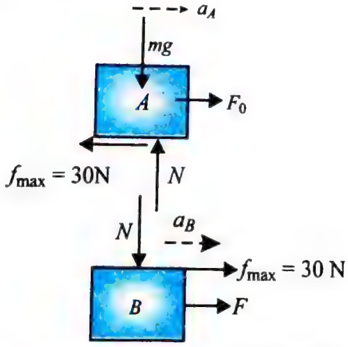
Acceleration of A,

$$a_A = \frac{120-30}{10} = 9\,\text{m s}^{-2} \text{ towards right}$$

Acceleration of B,

$$a_B = \frac{30}{20} = 1.5\,\text{m s}^{-2} \text{ towards right.}$$

Case II: F is increased from zero till the two blocks just start moving together.



As the two blocks move together, the friction is static in nature and its value is limiting. FBD in this case will be

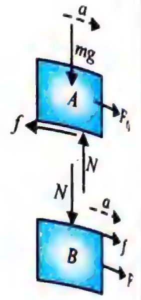
$$a_A = \frac{120-30}{10} = 9\,\text{m s}^{-2}$$

$$\Rightarrow \quad a_B = \frac{F+30}{20} = a_A \quad \Rightarrow \quad \frac{F+30}{20} = 9 \quad \text{or} \quad F = 150 \text{ N}$$

Hence, when $0 < F < 150$ N, the blocks do not move together and the friction is kinetic. As F increases, acceleration of block B increases from 1.5 m s^{-2} to 9 m s^{-2}.

At F = 150 N, limiting static friction starts acting and the two blocks start moving together.

Case III: When F is increased above 150 N.



In this scenario, the static friction adjusts itself so as to keep the blocks moving together. The value of static friction starts reducing but the direction still remains same. This happens continuously till the value of friction becomes zero. In this case, the FBD is as follows:

$$a_A = a_B = \frac{120-f}{10} = \frac{F+f}{20}$$

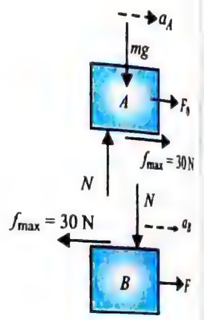
Therefore, when the frictional force f gets reduced to zero, the above accelerations become

$$a_A = \frac{120}{10} = 12\,\text{m s}^{-2} \quad \Rightarrow \quad a_B = \frac{F}{20} = a_A = 12\,\text{m s}^{-2}$$

$\therefore \quad F = 240$ N

Hence, when 150 N $\le F \le$ 240 N, the static friction force continuously decreases from maximum to zero at F = 240 N. The accelerations of the blocks increase from 9 m s^{-2} to 12 m s^{-2} during the change of force F.

Case IV: When F is increased again from 240 N, the direction of frictional force on the block reverses but it is still static. F can be increased till this reversed static friction reaches its limiting value. FBD at this juncture will be:



The blocks move together, therefore,

$$a_A = \frac{120+30}{10} = 15\,\text{m s}^{-2}$$

$$\Rightarrow \quad a_B = \frac{F-30}{20} = a_A = 15\,\text{m s}^{-2}$$

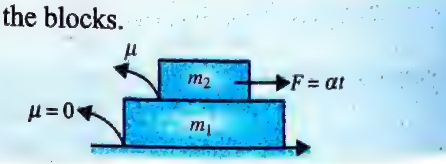
$$\therefore \quad \frac{F-30}{20} = 15\,\text{m s}^{-2}$$

Hence, F = 330 N.

Case V: When F is increased beyond 330 N. In this case, the limiting friction is achieved and slipping takes place.

ILLUSTRATION 7.39

In the given figure, force $F = \alpha t$ is applied on the block of mass m_2. Here α is a constant and t is the time. Find the acceleration of the block. Also draw the acceleration–time graph of both the blocks.

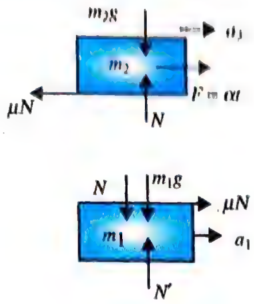


Sol. If $F \le \mu \dfrac{m_2}{m_1}(m_1+m_2)g$, then both blocks will move together. (illustration 7.37)

Here, $t \le \mu \dfrac{m_2}{\alpha m_1}(m_1+m_2)g$

During this time, acceleration of both blocks

$$a_1 = a_2 = \frac{F}{(m_1+m_2)} = \frac{\alpha t}{(m_1+m_2)}$$



If $t \geq \frac{\mu m_2}{\alpha m_1}(m_1+m_2)g$, friction will be of kinetic nature.

Free-body diagrams:

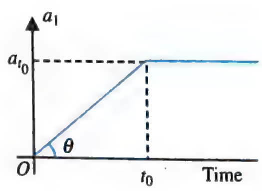
Acceleration of m_1: $a_1 = \frac{\mu N}{m_1} = \frac{\mu m_2 g}{m_1}$

Acceleration of m_2: $a_2 = \frac{\alpha t - \mu m_2 g}{m_2}$

Acceleration–time graph for m_1:

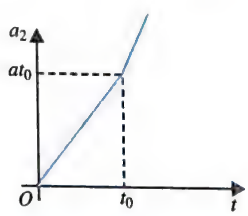
$$t_0 = \frac{\mu m_2}{\alpha m_1}(m_1+m_2)g$$

$$a_{t_0} = \frac{\alpha t_0}{(m_1+m_2)} = \frac{\mu m_2 g}{m_1}$$



Acceleration–time graph for m_2:

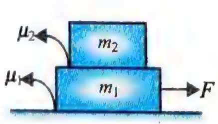
Before time t_0, slope of the graph is $\frac{\alpha}{m_1+m_2}$ and after t_0, slope of the graph becomes α/m_2. Slope increases after t_0, hence the graph.



FRICTION ACTING ON MULTIPLE SURFACES

If Force Applied on Lower Block

The maximum friction acting on m_2 is $f_2 = \mu_2 m_2 g$. If the system moves with the common acceleration, then



$$F - \mu_1(m_1+m_2)g = (m_1+m_2)a$$

and $f_2 = m_2 a$

Maximum acceleration of m_2,

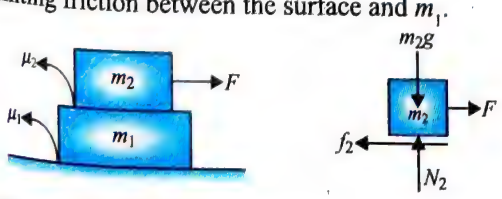
$$\mu_2 m_2 g = m_2 a_{max}$$

$\therefore$ $a_{max} = \mu_2 g$

If Force Applied on Upper Block

f_2 = limiting friction between m_1 and m_2.

f_1 = limiting friction between the surface and m_1.



If $F > f_2$, then both blocks move with different acceleration and the maximum friction acts between the blocks.

$$F - f_2 = m_2 a_2$$

$\Rightarrow$ $F - \mu_2 m_2 g = m_2 a_2$

and $N_2 = m_2 g$

$$N_1 = N_2 + m_1 g = (m_1+m_2)g$$

$\therefore$ $f_1 = \mu_1 N_1 = \mu_1(m_1+m_2)g$

$$f_2 = \mu_2 m_2 g$$

If $f_2 < f_1$, then m_1 remains at rest.

If $f_2 > f_1$, then m_1 moves in the direction of f_2.

$$f_2 - f_1 = m_1 a_1$$

If $F < f_2$, then no slipping is found between m_1 and m_2, i.e., m_1 and m_2 move together.

If $F < f_1$, then the system is in rest.

If $F > f_1$, the system moves with the common acceleration a. In this case,

$$F - f_1 = (m_1+m_2)a$$

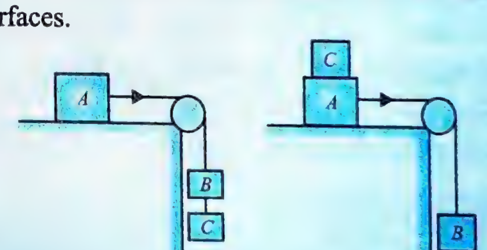
or $$F - \mu_1(m_1+m_2)g = (m_1+m_2)a$$

ILLUSTRATION 7.40

The masses of the blocks A, B, and C shown in figure are 4 kg, 2 kg, and 2 kg, respectively. Block A moves with an acceleration of 2.5 m s^{-2}.

(a) Block C is removed from its position and placed on block A, shown in figure. What is now the acceleration of block C?

(b) The positions of the blocks A and B is subsequently interchanged. Find the new acceleration of C. The coefficient of friction is the same for all the contact surfaces.



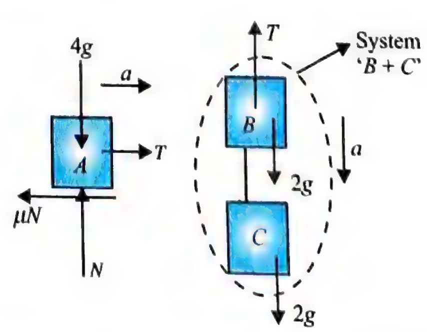
(a) (b)

Sol. Given acceleration of the system, $a = 2.5$ m s^{-2}

For B and C: $2g + 2g - T = (2+2)a$ ...(i)

For A: $T - \mu N = 4a \Rightarrow T - \mu 4g = 4a$ ...(ii)

Solving (i) and (ii), we get $\mu = 0.5$.

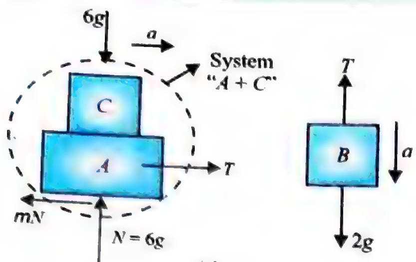


(a) Let A and C move together with common acceleration a. Then the acceleration of B is also a.

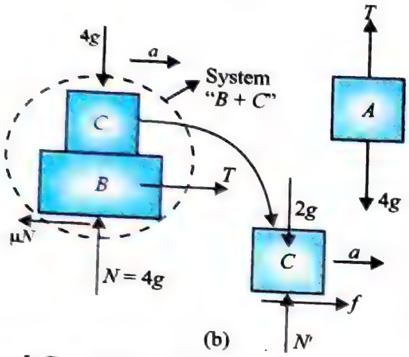
$$2g - T = 2a \quad ...(i)$$

$$T - \mu\, 6g = 6a \quad ...(ii)$$

Solving these equations, a comes out to be negative, which is not possible. It means system will remain at rest and acceleration of C will be zero.



(a)



(b)

(b) Let B and C move together.

$4g - T = 4a$, $T - 4\mu g = 4a$

Solving them: $a = 2.5\ \text{m s}^{-2}$

Let f be the friction between B and C. Then

For C:

$$f = 2a = 2 \times 2.5 = 5\ \text{N}$$

It is because this acceleration to C is given by friction f.

Limiting friction between B and C: $f_l = \mu 2g = (1/2)2g = 10\ \text{N}$.

Since $f < f_l$, so C will not slip on B.

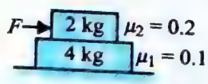
Hence, acceleration of $C = 2.5\ \text{m s}^{-2}$

ILLUSTRATION 7.41

In the situation shown in the figure,

(a) what minimum force F will make any part or whole system move?

(b) find the acceleration of two blocks and value of friction at the two surfaces if $F = 6$ N.



Sol. Between 4 kg and ground:

$$f_{l_1} = f_{k_1} = \mu_1(2+4)g = 6\ \text{N}$$

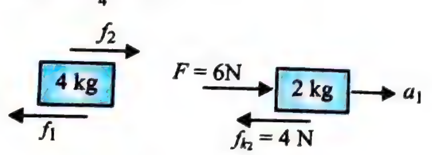
Between 2 kg and 4 kg:

$$f_{l_2} = f_{k_2} = \mu_2 2g = 4\ \text{N}$$

(a) Limiting friction between 2 kg and 4 kg is less than that between 4 kg and ground. So 2 kg will slip over 4 kg if

$$F > f_{l_2} = 4\ \text{N} \Rightarrow F_{min} = 4\ \text{N}$$

(b) Let us consider the FBD of 4 kg: Acceleration to 4 kg block will be provided by f_2. But the maximum value of f_2 is less than the maximum value of f_1. Hence, 4 kg will not accelerate. So $a_4 = 0$.

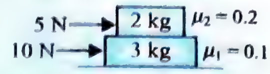


For 2 kg: $a_1 = \dfrac{6-4}{2} = 1\ \text{m s}^{-2}$

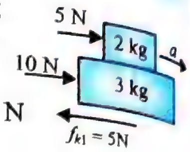
ILLUSTRATION 7.42

(a) Find the acceleration of the two blocks shown in the figure.

(b) Find the friction force between all contact surfaces.



Sol. Let us first calculate the limiting and kinetic friction between various surfaces.



Between 3 kg and ground:

$$f_{l_1} = f_{k_1} = \mu_1(2+3)g = 0.1 \times 5g = 5\ \text{N}$$

Between 2 kg and 3 kg:

$$f_{l_2} = f_{k_2} = \mu_2 2g = 0.2 \times 2g = 4\ \text{N}$$

Let us first assume that both blocks move together with common acceleration a.

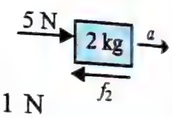
$$a = \frac{5+10-f_{k_1}}{2+3} \Rightarrow a = \frac{15-5}{5} = 2\ \text{m s}^{-2}$$

Now let us see how much frictional force is required between 2 kg and 3 kg for common acceleration a.

$$5 - f_2 = 2a \Rightarrow 5 - f_2 = 2 \times 2 \Rightarrow f_2 = 1\ \text{N}$$

Since $f_2 < f_{l_2}$,

1. Both blocks will move together with common acceleration $a = 2\ \text{m s}^{-2}$

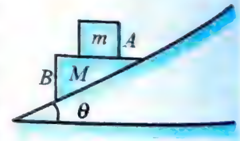


2. Friction between 2 kg and 3 kg $= f_2 = 1$ N

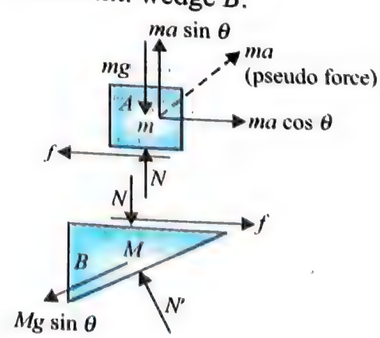
   Friction between 3 kg and ground $= f_{k_1} = 5$ N

ILLUSTRATION 7.43

The coefficient of friction between the block A of mass m and the triangular wedge B of mass M is μ. There is no friction between the wedge and the plane. If the system (block A + wedge B) is released so that there is no sliding between A and B. Find the inclination θ.



Sol. Acceleration of the system $(A + B)$ down the plane $= g\sin\theta$. In both the block and wedge, friction between them will be static. The block A is at rest w.r.t. wedge B.



From FBD of A:

$$N = mg - ma\sin\theta \quad \text{...(i)}$$

$$f = ma\cos\theta \quad \text{...(ii)}$$

For no sliding between A and B,

$$f < \mu N$$

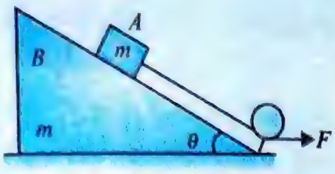
$\sin\theta \cdot \cos\theta < \mu(mg - mg\sin^2\theta)$

$mg\sin\theta\cos\theta < \mu\, mg\,(1 - \sin\theta)$

$\sin\theta\cos\theta < \mu\cos^2\theta$

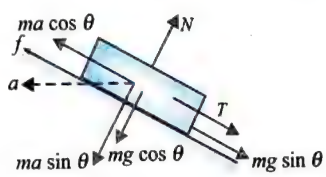
$\tan\theta < \mu$ or $\theta = \tan^{-1}(\mu)$

ILLUSTRATION 7.44



A block *A* of mass *m* which is placed on a rough inclined face of a wedge of same mass being pulled through light string and with force *F* as shown in figure. The coefficient of friction between inclined face and block *A* is μ while there is no friction between the ground and wedge. If the whole system moves with same acceleration, then find the value of *F*.

Sol. If the whole system moves together, then the acceleration of system,



$$a = \frac{F}{2m}$$

$$T + mg\sin\theta = f + ma\cos\theta$$

But $T = F$. Therefore,

$$f = F + mg\sin\theta - ma\cos\theta$$

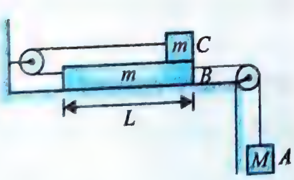
For no relative sliding $f < \mu N$

$$F + mg\sin\theta - ma\cos\theta < \mu(mg\cos\theta + ma\sin\theta)$$

$$F + mg\sin\theta - m\cdot\left(\frac{F}{2m}\right)\cos\theta < \mu\left(mg\cos\theta + m\left(\frac{F}{2m}\right)\sin\theta\right)$$

$$F < \frac{2mg(\mu\cos\theta - \sin\theta)}{(2 - \cos\theta - \mu\sin\theta)}$$

ILLUSTRATION 7.45



Given the setup shown in figure. Block *A*, *B*, and *C* have masses $m_A = M$ and $m_B = m_C = m$. The strings are assumed massless and unstretchable, and the pulleys frictionless. There is no friction between blocks *B* and the support table, but there is friction between blocks *B* and *C*, denoted by a given coefficient μ.

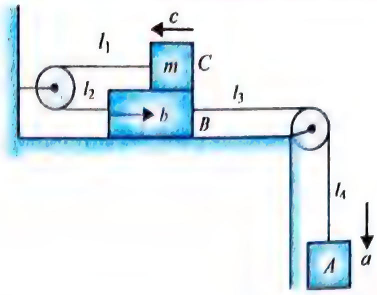
(a) In terms of the given, find (i) the acceleration of block *A*, and (ii) the tension in the string connecting *A* and *B*.

(b) Suppose the system is released from rest with block *C* near the right end of block *B* as shown in the above figure. If the length *L* of block *B* is given, what is the speed of block *C* as it reaches the left end of block *B*? Treat the size of *C* as small.

(c) If the mass of block *A* is less than some critical value, the blocks will not accelerate when released from rest. Write down an expression for that critical mass.

Sol.

(a) Apply constraint equation on strings, the length of strings is constant. Differentiate twice to get relation between acceleration. Let the acceleration of block *A*, *B*, and *C* be *a*, *b*, and *c*, respectively.



$$l_1 + l_2 = \text{constant}$$

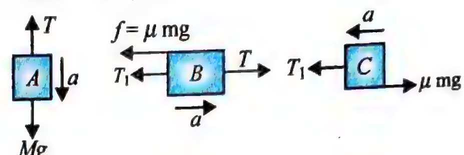
and $$l_3 + l_4 = \text{constant}$$

$$\ddot{l}_1 + \ddot{l}_2 = 0 \Rightarrow |b| = |c|$$

$$\ddot{l}_3 + \ddot{l}_4 = 0 \Rightarrow |a| = |b|$$

From which we get $a = b = c$.

From FBDs of *A*, *B*, and *C*



Writing equations of motion for block *A*:

$$Mg - T = Ma \quad \text{...(i)}$$

For block *B*, $T - T_1 - \mu mg = ma$ ...(ii)

For block *C*, $T_1 - \mu mg = ma$ ...(iii)

Solving equations (i), (ii), and (iii), we get

$$a = \left(\frac{M - 2\mu m}{M + 2m}\right)g \quad \text{...(iv)}$$

Putting *a* in Eq. (i), we get

$$Mg - T = M\left(\frac{M - 2\mu m}{M + 2m}\right)g \Rightarrow T = \frac{2mMg(1+\mu)}{(M+2m)}$$

(b) As there is relative motion between blocks, we apply

$$v_{\text{rel}}^2 = v_{\text{rel}}^2 + 2a_{\text{rel}}S_{\text{rel}}$$

If system is released from rest, $u_{\text{rel}} = 0$

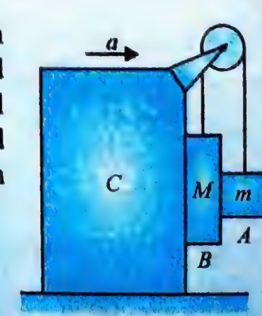
$$v_{\text{rel}}^2 = 2a_{\text{rel}}S_{\text{rel}} \Rightarrow v_{\text{rel}} = \sqrt{2a_{\text{rel}}S_{\text{rel}}}$$

$$a_{\text{rel}} = 2a \text{ and } S_{\text{rel}} = L$$

$$\Rightarrow \quad v = \sqrt{\frac{4gL(M - 2\mu m)}{(M + 2m)}}$$

(c) If blocks will not accelerate, then put $a = 0$ in Eq. (iv) to get $M = 2\mu m$.

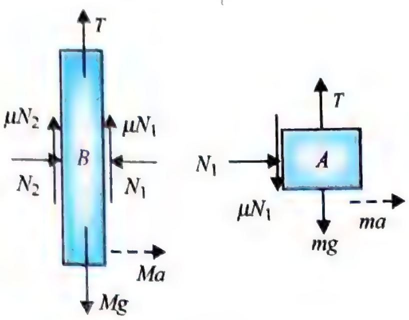
ILLUSTRATION 7.46



The pulley of the system shown in figure is massless and frictionless and thread is inextensible. The horizontal surface over which block *C* is placed is smooth while coefficient of friction for all the remaining surfaces is μ. Calculate minimum acceleration *a* with which the system should be moved to the right so that suspended blocks *A* (mass *m*) and *B* (mass *M* and $M > m$) can remain stationary relative to *C*.

Sol. Since blocks A and B are stationary relative to C. It means their acceleration will also be equal to a (rightward). Since mass m of block A is less than mass M of block B, hence block A has tendency to slip upward while B has tendency to slip downward.

Since A has tendency to slip upward therefore, friction on its left surface will act downward and friction on both the surfaces of block B will act upward because it has tendency to slip downward. Hence, their free body diagram will be as shown in figure.



For horizontal forces on A, $N_1 = ma$

For vertical equilibrium of A,

$$T = (\mu N_1 + mg) = (\mu ma + mg) \quad \text{...(i)}$$

For horizontal forces on B,

$$N_2 - N_1 = Ma$$

or $\quad N_2 = (m + M)a$

For vertical equilibrium of B,

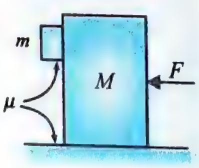
$$T + \mu N_1 + \mu N_2 = Mg$$

$$(\mu ma + mg) + \mu ma + \mu(m + M)a = Mg$$

$$a = \frac{M - m}{\mu(3m + M)} \cdot g$$

ILLUSTRATION 7.47

In the given figure, the co-efficient of friction between the walls of block of mass m and the plank of mass M is μ. The same co-efficient of friction is there between the plank and the horizontal floor. The force F is of 100 N and the masses m and M are of 1 kg and 3 kg, respectively. Find the value of μ, if the block does not slip along the wall of the plank.



Sol. Equation of motion for m:

$$\Sigma F_x = ma \Rightarrow N = ma$$

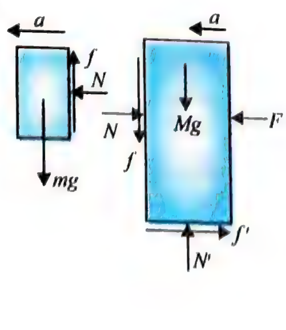
$$\Sigma F_y = 0$$

FBD of m and M,

Since m is at rest relation to M at the verge of slipping.

$\Rightarrow \quad f = mg$

Taking "m and M" together as system,



$$f' = \mu N' = \mu(m + M)g$$

$$F - f' = (M + m)a$$

$$\Rightarrow \quad a = \frac{F - \mu g(M + m)}{M + m}$$

$$f \le f_{max}$$

$$\Rightarrow \quad mg \le \mu N$$

$$\Rightarrow \quad mg \le \mu ma$$

$$\Rightarrow \quad g \le \mu \cdot \frac{F - \mu g(M + m)}{M + m}$$

$$\Rightarrow \quad (M + m)g \le \mu F - \mu^2 g(M + m)$$

$$\Rightarrow \quad \mu^2 g(M + m) - \mu F + (M + m)g \le 0$$

$$\Rightarrow \quad 2\mu^2 - 5\mu + 2 \le 0$$

For critical, $2\mu^2 - 5\mu + 2 = 0$

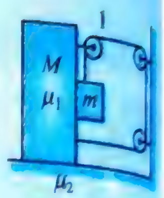
$$(\mu - 2)(2\mu - 1) = 0$$

$\therefore \quad \mu = 2$ or $1/2$

Practically μ does not become 2. Therefore, $\mu = 0.5$.

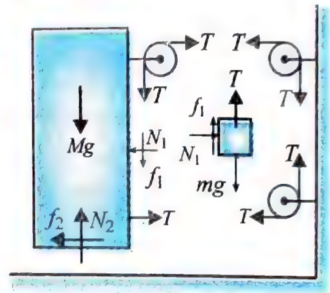
ILLUSTRATION 7.48

The blocks of mass M and m are arranged as the situation in figure is shown. The coefficient of friction between two blocks is μ_1 and that between the bigger block and the ground is μ_2. Find the accelerations of the blocks.



Sol.

For the motion of block m:



Along horizontal direction, $N_1 = ma$

Along vertical direction, $mg - (\mu_1 N_1 + T) = m(2a)$ ...(i)

After substituting value of N_1 in Eq. (i), we have

$$Mg - (\mu_1 ma + T) = m(2a) \quad \text{...(ii)}$$

For the motion of block M:

Along vertical direction $\Sigma F_V = 0$,

or $\quad N_2 = T + \mu_1 N_1 + Mg$ ...(iii)

Along horizontal direction,

$$2T - (N_1 + \mu_2 N_2) = Ma \quad \text{...(iv)}$$

From equation (i) and (iii), we get

$$2T - [N_1 + \mu_2(T + \mu_1 N_1 + Mg)] = Ma$$

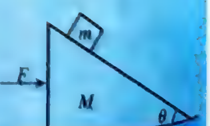
or. $\quad 2T - [ma + \mu_2 T + \mu_1\mu_2(ma) + \mu_2 Mg] = Ma$ ...(v)

Now solving equations (ii) and (iv), we get

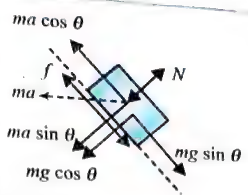
$$a = \frac{2m - \mu_2(M + m)g}{M + m[5 + 2(\mu_1 - \mu_2)]}$$

ILLUSTRATION 7.49

A bar of mass m is placed on a triangular block of mass M as shown in figure. The friction coefficient between the two surface is μ and ground is smooth. Find the minimum and maximum horizontal force F required to be applied on block so that the bar will not slip on the inclined surface of block.



Sol. Here if both the masses are moving together, acceleration of the system will be $a = F/(M + m)$. If we observe the mass m relative to M, it experiences a pseudo force ma towards left. Along the incline it experiences two forces, $mg \sin \theta$ downwards and $ma \cos \theta$ upwards. If $mg \sin \theta$ is more than $ma \cos \theta$, it has a tendency of slipping downwards, so friction on it will act in upward direction. Here if block m is in equilibrium on inclined surface, we must have



$f = \mu N$,

Here $f = (mg \sin \theta - ma \cos \theta)$

and $N = (ma \sin \theta + mg \cos \theta)$.

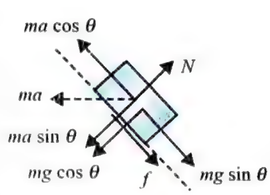
$$f = mg\sin\theta - ma\cos\theta \le \mu(mg\cos\theta + ma\sin\theta)$$

or $$a \ge \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} g$$

or $$(M+m)a \ge \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}(M+m)g$$

or $$F \ge \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}(M+m)g \quad \text{...(i)}$$

If force is more than the value obtained in Eq. (i), $ma \cos \theta$ will increase on m and the static friction on it will decrease. At $a = g \tan \theta$ (when $F = (M + m) g \tan \theta$), we know that the force $mg \sin \theta$ will be balanced by $ma \cos \theta$ at this acceleration no friction will act on it. If applied force will increase beyond this valued, $ma \cos \theta$ will exceed $mg \sin \theta$ and friction starts acting in downward direction. Here if block m is in equilibrium, we must have $f = (ma \cos \theta - mg \sin \theta)$, $N = (mg \cos \theta + ma \sin \theta)$, and $f \le \mu N$.



$$ma \cos \theta - mg \sin \theta \le \mu(mg \cos \theta + ma \sin \theta)$$

$$\Rightarrow \quad a \le \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} g$$

$$\Rightarrow \quad (M+m)a \le \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}(M+m)g$$

$$\Rightarrow \quad F \le \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}(M+m)g$$

Hence, $$F_{\min} \le \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}(M+m)g$$

and $$F_{\max} \le \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}(M+m)g$$

Analyzing Frictional Force While Walking

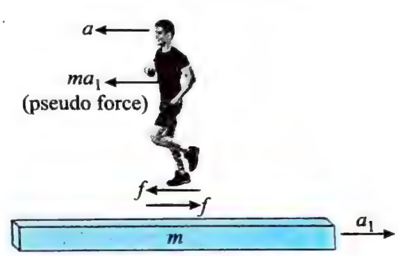
When we walk or run on ground, we get the impression of shoes on ground. It means there is no relative motion between the shoe and the ground. Hence, the frictional force is static friction. To start walking, you push back with your foot on the floor. Without friction, your foot would slide back, moving back relative to the floor, as shown in the figure below. Static friction opposes this motion, the motion that would occur if there were no friction, and thus static friction is directed forward.

Illustration 7.50

A man of mass m is moving with a constant acceleration a w.r.t. plank. The plank lies on a smooth horizontal flo;or. If the mass of plank is also m, calculate the acceleration of plank and man w.r.t. ground, and frictional force extended by plank on man.

Sol. Let us consider the acceleration of the plank be a_1 (rightwards). Now considering the motion of the man w.r.t. plank.

Equation of motion for man: $ma_1 + f = ma$ ...(i)



Equation of motion for plank: $f = ma_1$ ...(ii)

From (i) and (ii): $ma_1 + ma_1 = ma \Rightarrow 2ma_1 = ma \Rightarrow a_1 = \frac{a}{2}$

Frictional force between man and plank

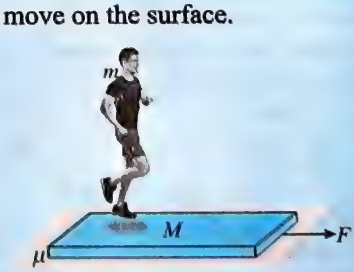
From (i) $f = ma_1 = \frac{ma}{2}$

Acceleration of man $\vec{a}_{\text{man}} = \vec{a}_{\text{man,plank}} + \vec{a}_{\text{plank}} = a + \left(-\frac{a}{2}\right) = \frac{a}{2}$

(Towards left)

Illustration 7.51

A plank of mass M is placed on a rough horizontal surface and a constant horizontal force F is applied on it. A man of mass m runs on the plank. The coefficient of friction between the plank and the surface is μ. Assume that the man does not slip on the plank. Find the acceleration of the man so that the plank does not move on the surface.

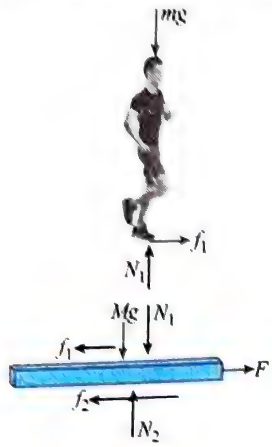


Sol. Let f_1 be the force between the man and plank and f_2 be the force of friction between the plank and surface. Here two cases are possible.

Case-I: Greater the acceleration of man, greater the value of f_1. If f_1 is greater than F, the plank has tendency to slid in backward direction. The direction of f_2 is in forward direction.

Case-II: For smaller value of acceleration of man, lesser the value of f_1. If f_1 is less than F, the plank has tendency to slide in forward direction. The direction of f_2 is in backward direction.

Case-I



Case-II

F.B.D. of man

$f_1 = ma$ ...(i)

and $N_1 = mg$ ...(ii)

Case-I:

From F.B.D. of plank: $f_2 = F - f_1 = ma - F$

But $f_2 \le \mu N_2$

Here, $N_2 = Mg + N_1 = Mg + mg$

$$ma - F \le \mu(M+m)g \Rightarrow a \le \frac{F + \mu(M+m)g}{m}$$

Case-II:

From F.B.D. of plank: $f_2 = F - f_1 = F - ma$

But $f_2 \le \mu N_2$

$\Rightarrow \quad F - ma \le \mu(M+m)g$

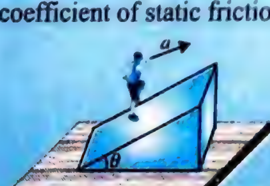
$\Rightarrow \quad a \ge \dfrac{F - \mu(M+m)g}{m}$

Hence the accelerations of the man so that the plank does not move on the surface.

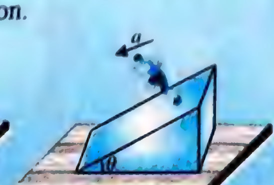
$$\frac{F}{m} - \frac{\mu(m+M)g}{m} \le a \le \frac{F}{m} + \frac{\mu(m+M)g}{M}$$

ILLUSTRATION 7.52

With what maximum acceleration can a person run up and down on an inclined plane of angle of inclination θ? Assume μ = coefficient of static friction.

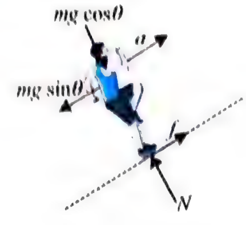


Running up



Running down

Sol. For maximum acceleration, friction should act in the direction of motion.



While running up

While running down

While running up: $f - mg\sin\theta = ma \Rightarrow f = ma + mg\sin\theta$

If friction is static: $f \le \mu N$

$$ma + mg\sin\theta = \mu(mg\cos\theta) \Rightarrow a = (\mu\cos\theta - \sin\theta)g$$

While running down: friction will act in downward direction

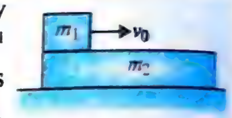
$$f + mg\sin\theta = ma \Rightarrow f = ma + mg\sin\theta$$

If friction is static: $f \le \mu N$

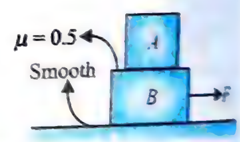
$$ma + mg\sin\theta = \mu(mg\cos\theta) \Rightarrow a = (\mu\cos\theta + \sin\theta)g$$

CONCEPT APPLICATION EXERCISE 7.3

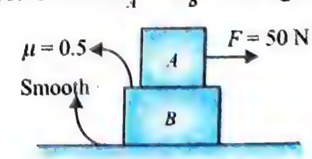
1. A block of mass 1 kg is horizontally thrown with a velocity of 10 m s^{-1} on a stationary long plank of mass 2 kg whose surface has a $\mu = 0.5$. Plank rests on frictionless surface. Find the time when m_1 comes to rest w.r.t. plank.



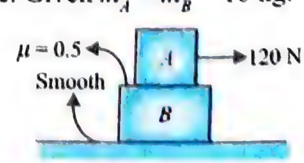
2. In the figure, initially the system is at rest. Find out minimum value of F for which sliding starts between the two blocks. Given $m_A = 10$ kg and $m_B = 20$ kg.



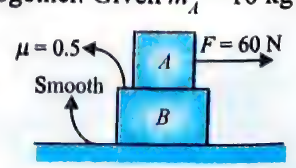
3 Find the acceleration of the two blocks. The system is initially at rest and the friction coefficient are as shown in the figure. Given $m_A = m_B = 10$ kg.



4. Find the acceleration of the two blocks. The system is initially at rest and the friction coefficient are as shown in the figure. Given $m_A = m_B = 10$ kg.



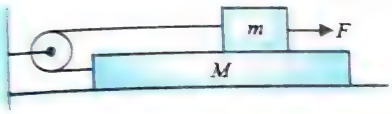
5. Find the acceleration of the two blocks. The system is initially at rest and the friction coefficient are as shown in the figure? Also find maximum F for which two blocks will move together. Given $m_A = 10$ kg and $m_B = 20$ kg.



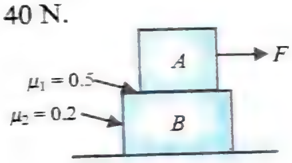
The block A is kept over a plank B. The maximum horizontal acceleration of the system in order to prevent slipping of A over B is $a = 2\ ms^{-2}$. Find the coefficient of friction between A and B.

A small block of mass m kept at the left end of a larger block of mass M and length l. The system can slide on a horizontal road. The system is started towards right with an initial velocity v. The friction coefficient between the road and the bigger block is μ and that between the blocks is $\mu/2$. Find the time elapsed before the smaller block separates from the bigger block.

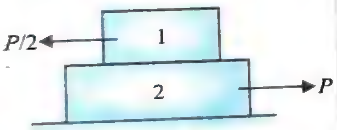
A small block of mass m is placed on a plank of mass M. The block is connected to plank with the help of a light string passing over a light smooth pulley, shown inthe figure. The co-efficient of static friction between the block and plank is μ. The co-efficient of friction between the plank and the horizontal surface is zero. What maximum horizontal force F applied on the block of mass m can make the block and plank not to slide relatively?



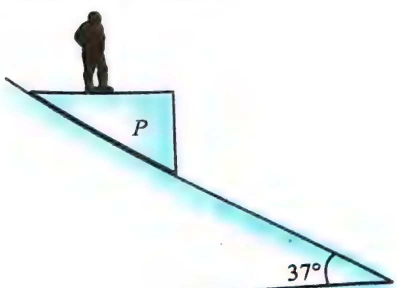
9. Two blocks A and B are arranged as shown in the figure ($m_A = 5$ kg and $m_B = 10$ kg). Find the acceleration of blocks if $F = 40$ N.



10. In the figure, block 1 is placed on top of block 2. Both of them have a mass of 1 kg. The coefficient of friction between blocks 1 and 2 are $\mu_s = 0.75$ and $\mu_k = 0.60$. The table is frictionless. A force $P/2$ is applied on block 1 to the left, and force P on block 2 to the right. Find the minimum value of P such that sliding occurs between the two blocks.

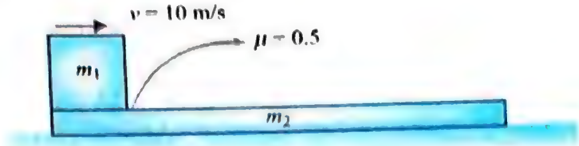


11. A man of mass 80 kg stands on a horizontal weighing machine of negligible mass, attached to a massless platform P that slides down at 37° incline. The weighing machine reads 72 kg. The man is always at rest w.r.t. weighing machine. Calculate:

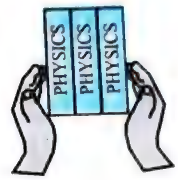


(a) the vertical acceleration of the man.
(b) the coefficient of kinetic friction μ between the platform and incline.

12. A block of mass $m_1 = 1$ kg is horizontally thrown with a velocity of $v = 10$ m/s on a stationary long plank of mass $m_2 = 2$ kg whose surface has $\mu = 0.5$. Plank rests on frictionless surface. Find the time when the block comes to rest w.r.t. plank.



13. Find minimum normal force to be applied by each hand to hold three identical books in vertical position. Each book has mass m and the value of coefficient of friction between the books as well as between hand and the book is μ.



14. A man of mass m is moving with a constant acceleration a w.r.t. plank. The plank lies on a smooth horizontal floor. If the mass of plank is also m, then calculate the acceleration of plank and man w.r.t. ground, and frictional force extended by plank on man.

ANSWERS

1. $\frac{4}{3}$ s   2. 150 N   3. 2.5 ms^{-2}

4. $a_A = 7.0\ ms^{-2}, a_B = 5\ ms^{-2}$

5. 75 N

6. 0.2   7. $\sqrt{\dfrac{4Ml}{(M+m)\mu g}}$

8. $2\mu\, mg$

9. $a_A = 3ms^{-2}, a_B = 0$ 10. $P = 10$ N

11. $\frac{13}{24}$

12. $\frac{4}{3}$ sec   13. $\frac{3mg}{2\mu}$

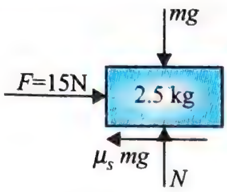
14. $\frac{a}{2}, f = \frac{ma}{2}$

Solved Examples

EXAMPLE 7.1

A block of mass 2.5 kg is kept on a rough horizontal surface. It is found that the block start sliding if a horizontal force of 15 N is applied to it. Also it is found that it takes 5 seconds to slide 10 m if the same horizontal force of 15 N is applied and the block is gently pushed to start the motion with initial velocity 1 ms^{-1}. Taking $g = 10$ m/s^2, calculate the coefficients of static and kinetic friction between the block and the surface.

Sol. The block starts sliding if a horizontal force of 15 N is applied to it.



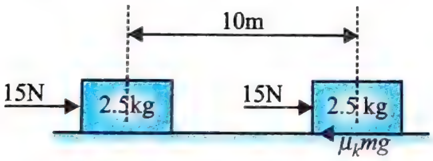
It means friction reaches to its limiting value

$$f_{\text{lim}} = \mu_s N = F \Rightarrow \mu_s = \frac{F}{N} = \frac{F}{mg} = \frac{15}{2.5\times 10} = \frac{3}{5}$$

From F.B.D of the block, $F - \mu_k mg = ma$

Hence acceleration of the block

$$a = \frac{F-\mu_k mg}{m} = \frac{15}{2.5} - \mu_k g = 6 - 10\mu_k \quad \text{...(i)}$$



The block takes 5 seconds to slide 10 m.

Using $s = ut + \dfrac{1}{2}at^2$

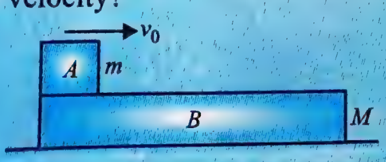
$$10 = 1\times 5 + \frac{1}{2}(6-10\mu_k)(5)^2 \Rightarrow 5 = \frac{1}{2}(6-10\mu_k)(5)^2$$

$$\Rightarrow \quad (6-10\mu_k) = \frac{10}{(5)^2} = \frac{2}{5}$$

$$\Rightarrow \quad 10\mu_k = \frac{28}{5} \Rightarrow \mu_k = \frac{14}{25}$$

EXAMPLE 7.2

The masses of the blocks A and B are m and M, respectively. Between A and B there is a constant frictional force F, but B can slide frictionlessly on the horizontal surface (figure). A is set in motion with velocity v_0 while B is at rest. What is the distance moved by A relative to B before they move with the same velocity?



Sol. The same frictional force is effective on A and B. This force produces retardation on A and acceleration on B till they acquire a common velocity.

$$F = ma = Ma'$$

where a = absolute retardation of m

a' = absolute acceleration of M

Relative retardation of $m = a-(-d) = a+d$

Initial relative velocity $= v_0$

Final relative velocity $= 0$

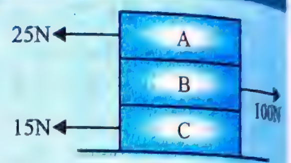
$$\therefore \quad v_0^2 = 2(a+a')s$$

where s = distance covered by m relative to M

$$\text{or} \quad v_0^2 = 2\left(\frac{F}{m}+\frac{F}{M}\right)s = \frac{2F(m+M)}{mM}s \quad \text{or} \quad s = \frac{mMv_0^2}{2F(m+M)}$$

EXAMPLE 7.3

Each of the three blocks in the figure has a mass of 10 kg. The coefficient of static and kinetic friction at each surface of contact between the blocks are $\mu_s = 0.3$ and $\mu_s = 0.3$, respectively. The ground is smooth. Determine the acceleration of each block when the horizontal forces as shown are applied.

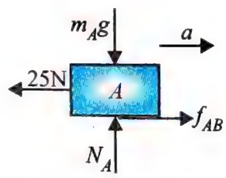


Sol. Let us assume that all the three blocks A, B and C move together.

The acceleration of the system, $a = \dfrac{100-(25+15)}{(10+10+10)} = 2\,\text{m/s}^2$

Maximum possible resistance force between A and B

$$= (f_{\text{lim}})_{A,B} = \mu_s N_A = 0.3\times 100 = 30\,\text{N}$$



Let us calculate actual frictional force between the blocks A and B.

Equation of motion of A;

$$f_{AB} - 25 = 10\times a = 10\times 2 \Rightarrow f_{AB} = 45\,\text{N}$$

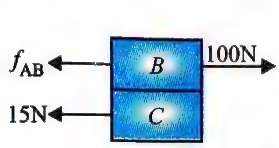
But the friction between the blocks cannot be greater than 30 N. It means the block A will slip on the block B and friction between the blocks A and B will be of kinetic nature.

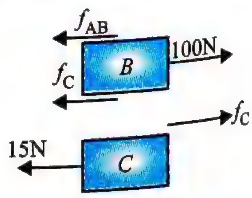
Hence $f_{AB} = \mu_k N_A = 0.2\times 100 = 20\,\text{N}$

Hence the acceleration of the block A should be

$$a_A = \frac{25-20}{10} = 0.5\,\text{m/s}^2 \text{ (towards left)}$$

Now let us consider the blocks B and C move together.



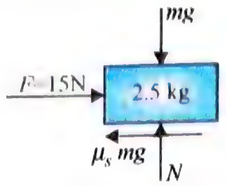


Solved Examples

EXAMPLE 7.1

A block of mass 2.5 kg is kept on a rough horizontal surface. It is found that the block start sliding if a horizontal force of 15 N is applied to it. Also it is found that it takes 5 seconds to slide 10 m if the same horizontal force of 15 N is applied and the block is gently pushed to start the motion with initial velocity 1 ms^{-1}. Taking g = 10 m/s^2, calculate the coefficients of static and kinetic friction between the block and the surface.

Sol. The block starts sliding if a horizontal force of 15 N is applied to it.



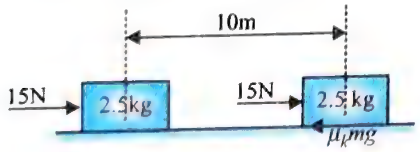
It means friction reaches to its limiting value

$$f_{\lim} = \mu_s N = F \Rightarrow \mu_s = \frac{F}{N} = \frac{F}{mg} = \frac{15}{2.5\times 10} = \frac{3}{5}$$

From F.B.D of the block, $F - \mu_k mg = ma$

Hence acceleration of the block

$$a = \frac{F-\mu_k mg}{m} = \frac{15}{2.5} - \mu_k g = 6 - 10\mu_k \qquad ...(i)$$



The block takes 5 seconds to slide 10 m.

Using $s = ut + \frac{1}{2}at^2$

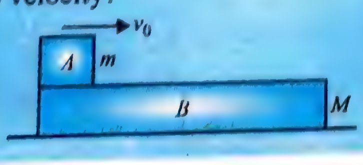
$$10 = 1\times 5 + \frac{1}{2}(6-10\mu_k)(5)^2 \Rightarrow 5 = \frac{1}{2}(6-10\mu_k)(5)^2$$

$$\Rightarrow \quad (6-10\mu_k) = \frac{10}{(5)^2} = \frac{2}{5}$$

$$\Rightarrow \quad 10\mu_k = \frac{28}{5} \Rightarrow \mu_k = \frac{14}{25}$$

EXAMPLE 7.2

The masses of the blocks A and B are m and M, respectively. Between A and B there is a constant frictional force F, but B can slide frictionlessly on the horizontal surface (figure). A is set in motion with velocity v_0 while B is at rest. What is the distance moved by A relative to B before they move with the same velocity?



Sol. The same frictional force is effective on A and B. This force produces retardation on A and acceleration on B till they acquire a common velocity.

$$F = ma = Ma'$$

where a = absolute retardation of m

a' = absolute acceleration of M

Relative retardation of $m = a-(-a') = a + a'$

Initial relative velocity = v_0

Final relative velocity = 0

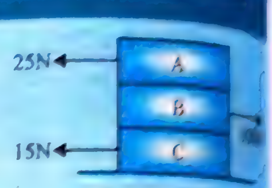
$\therefore \quad v_0^2 = 2(a+a')s$

where s = distance covered by m relative to M

or $\quad v_0^2 = 2\left(\frac{F}{m}+\frac{F}{M}\right)s = \frac{2F(m+M)}{mM}s \quad$ or $\quad s = \frac{mMv_0^2}{2F(m+M)}$

EXAMPLE 7.3

Each of the three blocks in the figure has a mass of 10 kg. The coefficient of static and kinetic friction at each surface of contact between the blocks are $\mu_s = 0.3$ and $\mu_k = 0.3$, respectively. The ground is smooth. Determine the acceleration of each block when the horizontal forces as shown are applied.

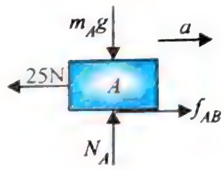


Sol. Let us assume that all the three blocks A, B and C move together.

The acceleration of the system, $a = \frac{100-(25+15)}{(10+10+10)} = 2\,\text{m/s}^2$

Maximum possible resistance force between A and B

$$= (f_{\lim})_{A,B} = \mu_s N_A = 0.3\times 100 = 30\,\text{N}$$



Let us calculate actual frictional force between the blocks A and B.

Equation of motion of A;

$$f_{AB} - 25 = 10\times a = 10\times 2 \Rightarrow f_{AB} = 45\,\text{N}$$

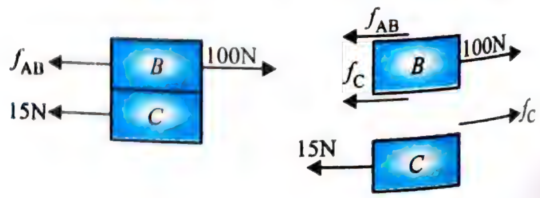
But the friction between the blocks cannot be greater than 30 N. It means the block A will slip on the block B and friction between the blocks A and B will be of kinetic nature.

Hence $f_{AB} = \mu_k N_A = 0.2\times 100 = 20\,\text{N}$

Hence the acceleration of the block A should be

$$a_A = \frac{25-20}{10} = 0.5\,\text{m/s}^2 \text{ (towards left)}$$

Now let us consider the blocks B and C move together.



The acceleration of the system (B and C)

$$a=\frac{100-(15+20)}{(10+10)}=3.25\ \text{m/s}^2$$

From F.B.D of the block C

$$f_C-15=10\times 3.25 \Rightarrow \qquad f_C=47.5\,\text{N}$$

Maximum possible friction between the blocks B and C

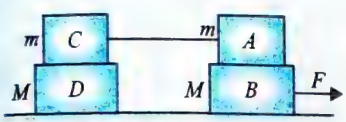
$$(f_{\min})_{BC}=(f_C)_{\max}=\mu_s(m_A+m_B)g$$

$$\Rightarrow\ (f_C)_{\max}=0.3\times 20\times 10=60\,\text{N}$$

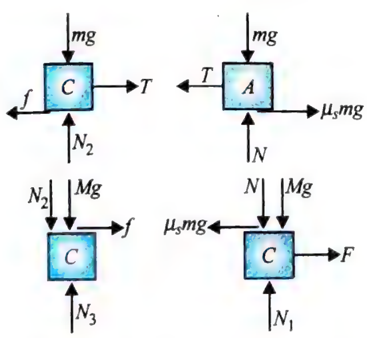
As $(f_C)_{\max}<f_C$, hence blocks B and C should move together with acceleration 3.25 m/s².

EXAMPLE 7.4

Four blocks are arranged on a smooth horizontal surface as shown in the figure. The masses of the blocks are given (see the figure). The coefficient of static friction between the top and the bottom blocks is μ_s. What is the maximum value of the horizontal force F, applied to one of the bottom blocks as shown, that makes all four blocks move with the same acceleration?



Sol. Let a be the acceleration of the system of the blocks. Drawing the free body diagram of all the blocks.



Let us identify the cause of motion of blocks A, C and D.

Block A moves due to static friction. When slipping starts it is $f_{\max}=\mu_s mg$. This force must be greater than tension T, only then it will accelerate forward. Block C moves due to tension T, which must be greater than f, the static friction between C and D. Block D moves due to the static friction between the blocks C and D.

Writing equation of motions of the blocks,

For block B: $F-\mu_s mg=Ma$ ...(i)

For block A: $\mu_k mg-T=ma$ ...(ii)

For block C: $T-f=ma$ ...(iii)

For block D: $f=Ma$ ...(iv)

From equations (iii) and (iv), $T=(m+M)a$ ...(v)

$$\Rightarrow\ a=\left(\frac{T}{m+M}\right)$$

Putting T in equations (ii) from (v)

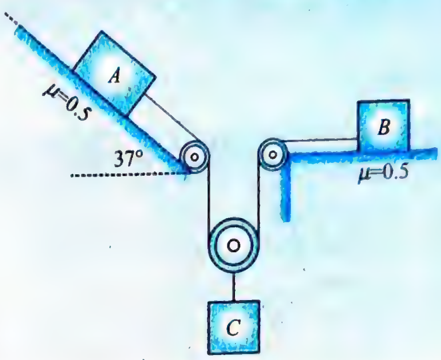
$$\mu_s mg-(m+M)a=ma \text{ or } a=\frac{\mu_s mg}{(M+2m)}$$

Putting a in (i), we get $F-\mu mg=\dfrac{\mu_s Mmg}{(M+2m)}$

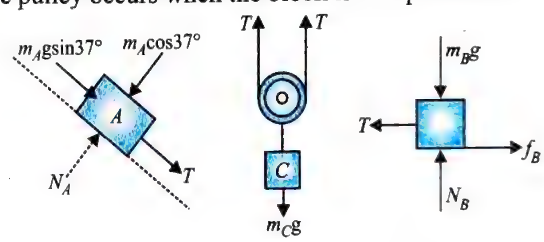
$$\Rightarrow\ F_{\max}=2\mu_s mg\left(\frac{m+M}{2m+M}\right)$$

EXAMPLE 7.5

In the arrangement shown in figure, pulleys are small, light and frictionless; threads are inextensible and mass of blocks A, B and C is $m_1=5$ kg, $m_2=4$ kg and $m_3=2.5$kg, respectively. Co-efficient of friction for both the planes is $\mu=0.50$. Calculate acceleration of each block when system is released from rest.



Sol. Let us investigate if the block B is moving or not. The maximum possible tension in the string connecting the block C and the pulley occurs when the block is in equilibrium.



$$2T_{\max}=m_cg \Rightarrow\ T_{\max}=\frac{m_cg}{2}=\frac{2.5\times 10}{2}=12.5\,N$$

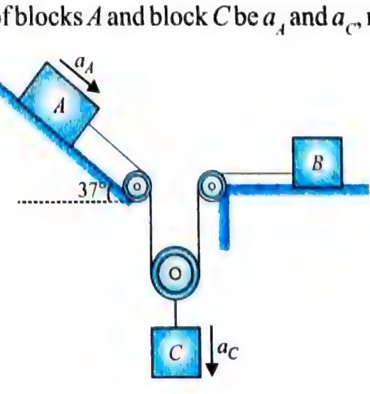
Hence, maximum possible driving force for block B should be 12.5 N.

Maximum possible resisting force on the block B,

$$(f_B)_{\max}=\mu N_B=0.5\times 4\times 10=20\,\text{N}$$

As maximum possible resisting force acting on block B is greater than the maximum possible driving force acting on it, hence, the block B will not move. It means the acceleration of the block B will be zero.

Let acceleration of blocks A and block C be a_A and a_C, respectively.



Here, $a_C = \dfrac{a_A + 0}{2} \Rightarrow a_A = 2a_C$ ...(i)

Now writing equations of motion:

For block A: $T + m_A g \sin 37° - f_A - m_A a_A$

As block A is sliding, it means $f_A = \mu m_A g \cos 37°$

or $\quad f_A = 0.5 \times 5 \times 10 \times \dfrac{4}{5} = 20\,\text{N}$

Hence $T + 5 \times 10 \times \dfrac{3}{5} - 20 = 5a_A$ or $T + 10 = 5a_A$ ...(ii)

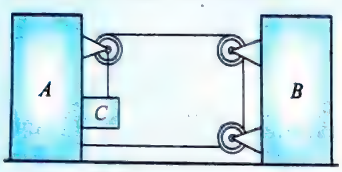
For block C: $m_c g - 2T = m_c a_c$

$$2.5 \times 10 - 2T = 2.5 \times \frac{a_A}{2} \text{ or } 2.5 - 2T = \frac{5}{4}a_A \quad \text{...(iii)}$$

From (ii) and (iii), $a_A = 4\,\text{m/s}^2$ and $a_C = 2\,\text{m/s}^2$

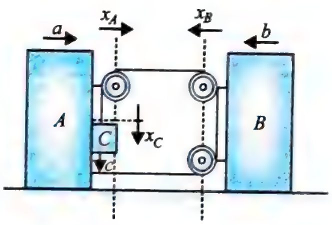
EXAMPLE 7.6

In the arrangement shown in figure, mass of blocks A, B and C is 18.5 kg, 8 kg and 1.5 kg, respectively.



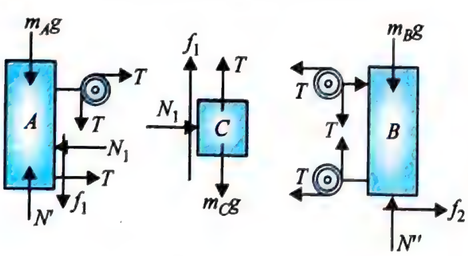
The bottom surface of A is smooth, while co-efficient of friction between block A and C is $\mu_1 = 1/3$, that between B and floor is $\mu_2 = 1/5$. System is released from rest at $t = 0$ and pulleys are light and frictionless. Calculate accelerations of blocks A, B and C.

Sol. Let the acceleration of A, B and C are a, b and c respectively.



Net sum of change of segment length in string should be zero.

$$2(-x_A) + 2(-x_B) + x_C = 0 \Rightarrow c = 2(a + b)$$



For A: In vertical direction,

$$N' = m_A g + T + f_1 \quad \text{...(i)}$$

where $f_1 = \mu_1 N_1 = \dfrac{N_1}{3}$

In horizontal direction,

$$2T - N_1 = m_A a \quad \text{...(ii)}$$

For C: horizontal direction

$$N_1 = m_c a \quad \text{...(iii)}$$

In vertical direction, $m_c g - f_1 - T = m_c 2(a + b)$ ...(iv)

$$m_c g - \frac{N_1}{3} - T = m_c 2(a + b) \quad \text{...(v)}$$

For B: In vertical direction $N'' = mB\,g$ ...(vi)

In horizontal direction

$$2T - f_2 = m_B b$$

where $f_2 = \mu_2 N'' = \dfrac{N''}{5}$ ...(vii)

After solving above equations, we get

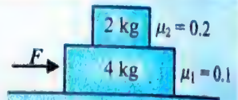
$a = 1\,\text{m s}^{-2}$, $b = 0.5\,\text{m s}^{-2}$

$c = 2(a + b) = 2(1 + 0.5) = 3\,\text{m/s}^2$

Net acceleration of $c = \sqrt{a^2 + c^2} = \sqrt{1^2 + 3^2} = \sqrt{10}\,\text{m/s}^2$

EXAMPLE 7.7

Consider the two blocks as shown in figure. A force is applied on the lower block.



(a) Where does the sliding begins first?

(b) What is the minimum force at which any part of system starts sliding?

(c) At what value of force F will the sliding start at the other surfaces?

(d) For the following values of F, find the acceleration of the two blocks, nature, and value of friction at both rough surfaces.

(i) 3 N, (ii) 12 N (iii) 24 N

Sol. Between 4 kg and ground:

$$f_{l_1} = f_{k_1} = \mu_1 (2 + 4) g = 6\,\text{N}$$

Between 2 kg and 4 kg: $f_{l_2} = f_{k_2} = \mu_2 2g = 4\,\text{N}$

(a) Obviously sliding will occur first between 4 kg and ground. Because 2 kg can slide over 4 kg only after 4 kg gains some acceleration.

(b) 4 kg will start sliding on ground if

$$F > f_{l_1} = 6\,\text{N or } F_{\min} = 6\,\text{N}$$

(c) Maximum acceleration of 2 kg block can be

$$\mu_2 g = 0.2 \times 10 = 2\,\text{m s}^{-2}$$

Both blocks can move together up to the combined acceleration of $2\,\text{m s}^{-2}$. To produce the acceleration of $2\,\text{m s}^{-2}$,

$$F - f_{k_1} = (2 + 4) \times 2 \Rightarrow F = 18\,\text{N}$$

When $F > 18$ N, the acceleration of 4 kg will become more than $2\,\text{m s}^{-2}$ but that of 2 kg cannot become more than $2\,\text{m s}^{-2}$. Hence, sliding will occur between 2 kg and 4 kg for $F > 18$ N

(d) (i) $F = 3$ N, system cannot move because $F < f_{l_1}$, so acceleration of both blocks is zero.

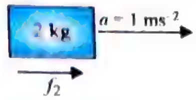
Friction between 4 kg and ground $= f_1 = F = 3$ N

As 2-kg block has no tendency to slide over 4 kg, friction between 2 kg and 4 kg blocks is zero.

(ii) $F = 12\text{ N} < 18\text{ N}$, but $F < f_{l_1}$. So both blocks will move with common acceleration.

$$a = \frac{F - f_{k_1}}{2+4} = \frac{12-6}{6} = 1\text{ ms}^{-2}$$

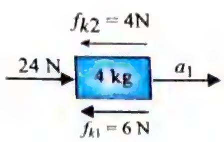
Friction between 4 kg and ground is $f_{k_1} = 6$ N



Friction between 2 kg and 4 kg

$f_2 = 2a = 2 \times 1 = 2$ N

(iii) $F = 24\text{ N} > 18\text{ N}$, here slipping will occur at both the contact surfaces. Hence friction between 4 kg and ground will be $f_{k_1} = 6$ N and between 2 kg and 4 kg will be $f_{k_2} = 4$ N. Acceleration of 4 kg:

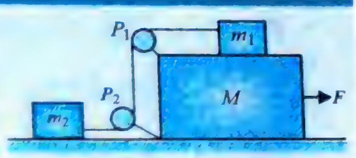


$$a_1 = \frac{24-6-4}{4} = 3.5\text{ m s}^{-2}$$

Acceleration of 2 kg block: $a_2 = \frac{f_{k_2}}{2} = \frac{4}{2} = 2\text{ m s}^{-2}$

EXAMPLE 7.8

In the given figure, m_1, m_2, and M are 20 kg, 5 kg, and 50 kg, respectively. The coefficient of friction between and ground is zero. The coefficient of friction between m_1 and M and that between m_2 and ground is 0.3. The pulleys and the spring are massless.



The string is perfectly horizontal between P_1 and m_1 and also between P_2 and m_2.

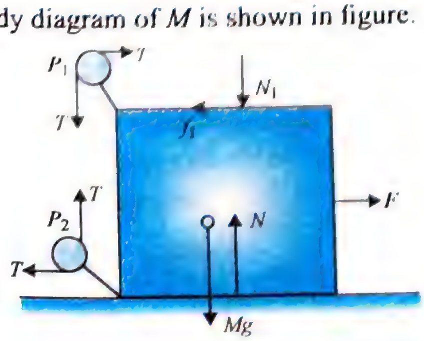
The string is perfectly vertical between P_1 and P_2. An external horizontal force F is applied to mass M. Take $g = 10\text{ m s}^{-2}$.

(a) Draw a free-body diagram of mass M, clearly showing all the forces.

(b) Let the magnitude of the force of the friction between m_1 and M be f_1 and that between m_2 and ground be f_2. For a particular F it is found that $f_1 = 2f_2$. Find f_1 and f_2. Write down equations of motion of all the masses. Find F, tension in the string, and the acceleration of the masses.

Sol.

(a) Free-body diagram of M is shown in figure.



(b) Supposing all the blocks are in motion

(i) $f_{1_{max}} = \mu_1 N_1 = \mu_1 m_1 g = 0.3 \times 20 \times 10 = 60$ N

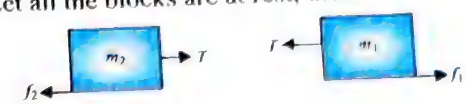
and $f_{2_{max}} = \mu_2 N_2 = \mu_2 m_2 g = 0.3 \times 5 \times 10 = 15$ N

Friction between m_2 and ground will be maximum, which is 15 N. Given that $f_1 = 2f_2$, so

$f_1 = 2 \times 15 = 30\text{ N} < f_{1,max}$

The block m_1 cannot move on M.

(ii) Let all the blocks are at rest, then



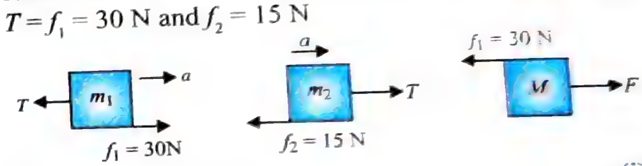
For M: $F - f_1 = 0$,

for m_1: $T - f_1 = 0$, and

for m_2: $T - f_2 = 0$

which gives $f_1 = f_2$, which does not satisfy the given condition.

(iii) Since m_1 cannot move over the block M, m_2 cannot move relative to M; therefore, all the blocks move together and $T = f_1 = 30$ N and $f_2 = 15$ N



For m_1: $30 - T = 20a$ ...(i)

For M: $F - 30 = 50a$ ...(ii)

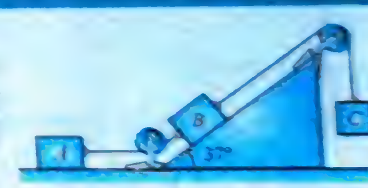
For m_2: $T - 15 = 5a$ ...(iii)

After solving these equations, we get

$$a = \frac{3}{5}\text{ m s}^{-2}, T = 18\text{ N}, F = 60\text{ N}$$

EXAMPLE 7.9

Blocks A, B, and C are placed as shown in figure and connected by the ropes of negligible mass. Both A and B weigh 25.0 N each, and the coefficient of kinetic friction between each block and the surface is 0.35. Block C descends with constant velocity.



(a) Draw two separate free-body diagrams showing the forces acting on A and B.

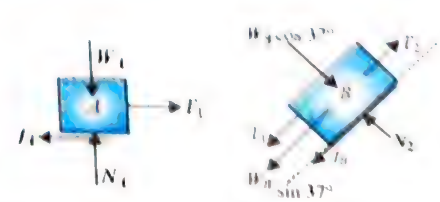
(b) Find the tension in the rope connecting blocks A and B.

(c) What is the weight of block C?

(d) If the rope connecting A and B were cut, what would be the acceleration of C?

Sol.

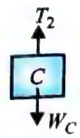
(a)



(b) The blocks move with constant speed, so there is no net force on block A; the tension in the rope connecting A and B must be equal to the frictional force on block A.

$$T_1 = \mu_k N_A = (0.35)(25.0) = 8.75\text{ N}$$

(c) The weight of block C will be the tension in the rope connecting B and C



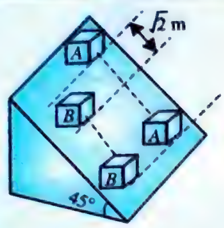
$$\begin{aligned} W_C &= T_2 \\ &= 9\text{ N} + W_B(\sin 37° + \mu_k \cos 37°) \\ &= 9\text{ N} + (25.0\text{ N})\,[\sin 37° + (0.35)\cos 37°] \\ &= 31.0\text{ N} \end{aligned}$$

(d) Applying Newton's second law to the remaining masses (B and C) gives

$$a = \frac{g(W_c - \mu_k W_B \cos 37° - W_B \sin 37°)}{(W_B + W_c)} = 1.6\text{ ms}^{-2}$$

EXAMPLE 7.10

Two blocks A and B of equal masses are placed on rough inclined plane as shown in figure. When and where will the two blocks come on the same line on the inclined plane if they are released simultaneously? Initially the block A is $\sqrt{2}$ m behind the block B. Coefficients of kinetic friction for the blocks A and B are 0.2 and 0.3, respectively ($g = 10\text{ ms}^{-2}$).



Sol. $a = \dfrac{mg\sin\theta - \mu_k mg\cos\theta}{m}$

$\therefore \quad a_A = g\sin\theta - \mu_{k,A} g\cos\theta = 4\sqrt{2}\text{ ms}^{-2}$ ...(i)

and $\quad a_B = g\sin\theta - \mu_{k,B} g\cos\theta = 3.5\sqrt{2}\text{ ms}^{-2}$ ...(ii)

Putting values, we get

a_{AB} is relative acceleration of A w.r.t.

$$B = a_A - a_B = \frac{1}{\sqrt{2}}\text{ ms}^{-2}$$

$$L = \sqrt{2}\text{ m},\ L = \frac{1}{2} a_{A/B} t^2 \Rightarrow t = 2\text{ s}$$

Distance moved by B during that time is given by

$$S_B = \frac{1}{2} a_B t^2 = \frac{1}{2} 3.5\sqrt{2} \times 4 = \frac{2 \times 0.7}{\sqrt{2}} \times 10 = 7\sqrt{2}\text{ m}$$

Similarly for A, $S_A = 8\sqrt{2}$ m

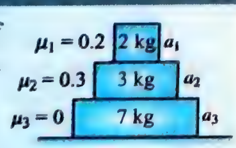
EXAMPLE 7.11

Find the accelerations a_1, a_2, and a_3 of the three blocks shown in figure, if a horizontal force of 10 N is applied on

(a) 2 kg block,

(b) 3 kg block, and

(c) 7 kg block (Take $g = 10\text{ ms}^{-2}$)

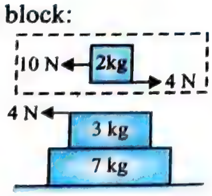


Sol.

(a) When force of 10 N is applied on 2 kg block:

The limiting frictional force between 2 kg and 3 kg blocks, $f_{l_1} = 0.2 \times 2g = 0.2 \times 2 \times 10 = 4$ N

The limiting frictional force between 3 kg and 7 kg blocks



$$\begin{aligned} f_{l_2} &= 0.3 \times 5g \\ &= 0.3 \times 5 \times 10 = 15\text{ N} \end{aligned}$$

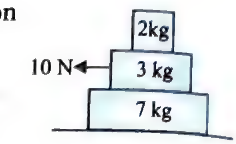
After proper analysis, we find that 2 kg will slip over 3 kg, but 3 kg will not slip over 7 kg.

Thus we have: $a_1 = \dfrac{10 - 4}{2} = 3\text{ ms}^{-2}$

$$a_2 = a_3 = \frac{4}{3+7} = 0.4\text{ ms}^{-2}$$

Since a_1 is coming out to be greater than a_2, 2 kg will slip over 3 kg.

(b) When force of 10 N is applied on 3 kg block:



Let us assume that all these blocks move together with common acceleration a, then

$$a = \frac{10}{2+3+7} = \frac{5}{6}\text{ms}^{-2}$$

Under this assumption:

Friction between 2 kg and 3 kg:

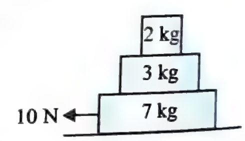
$$f_1 = 2a = \frac{2 \times 5}{6} = \frac{5}{3}\text{ N} < f_{l_1}$$

Friction between 3 kg and 7 kg:

$$f_2 = 7a = \frac{7 \times 5}{6} = \frac{35}{6}\text{ N} < f_{l_2}$$

Hence, no slipping occurs anywhere.

(c) When force of 10 N is applied on 7 kg block:



Suppose 3 kg and 2 kg blocks move together with the 7 kg block. The acceleration of the whole system,

$$a = \frac{10}{2+3+7} = \frac{5}{6}\text{ms}^{-2}$$

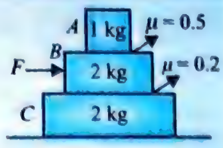
The pseudo force on 2 kg block $= \dfrac{2 \times 5}{6} = \dfrac{5}{3}$ N, which is less than the frictional force between 2 kg and 3 kg blocks, so they move together.

Now check whether 2 kg and 3 kg move together over 7 kg block. The pseudo force on (2 + 3) kg is $5 \times \dfrac{5}{6} = \dfrac{25}{6}$ N, which is also less than frictional force between 3 kg and 7 kg blocks, so all the blocks move together with a common acceleration of 5/6 ms^{-2}. Therefore,

$$a = a_1 = a_2 = a_3\ \frac{10}{2+3+7} = \frac{5}{6}\text{ms}^{-2}$$

EXAMPLE 7.12

In the situation shown in figure, there is no friction between 2 kg and ground.



(a) For what maximum value of force F can all three blocks move together?

(b) Find the value of force F at which sliding starts at other rough surfaces.

(c) Find acceleration of all blocks, nature, and value of friction force for the following values of force F: (i) 10 N, (ii) 18 N, and (iii) 25 N.

Sol. Limiting friction between blocks A and B:

$$f_{l_1} = f_{k_1} = 0.5 \times 1 \times g = 5\text{ N}$$

Limiting frictional force between block B and block C

$$f_{l_2} = f_{k_2} = 0.2 \times (1+2) \times g = 6\text{ N}$$

Maximum possible acceleration of block A can be

$$a_{A,\max} = 0.5\text{ g} = 5\text{ m s}^{-2}$$

Maximum possible acceleration of block C:

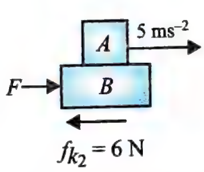
$$a_{C\max} = \frac{f_{l2}}{2} = \frac{6}{2} = 3\text{ m s}^{-2}$$

For the entire system to move together, we have to take lesser acceleration.

(a) Maximum force F that can be applied for the entire system to move together : $F_{\max} = m_{\text{total}}\,a$

$$\Rightarrow \quad F_{\max} = (1+2+2) \times 3 = 15\text{ N}$$

(b) Sliding will start between block B and block C if $F > 15$ N

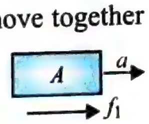


For sliding to start between block A and block B: This will occur when acceleration of combined block A and block B will exceed 5 m s^{-2}

$$F - f_{k_2} = (m_A + m_B)a$$

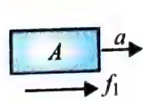
$$F - 6 = (1+2) \times 5 \Rightarrow F = 21\text{ N}$$

(c) (i) $F = 10\text{ N} < 15\text{ N}$, so entire system will move together with common acceleration.



$$a = \frac{10}{1+2+2} = 2\text{ m s}^{-2}$$

Friction between block B and block C:

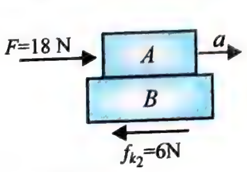


$$f_2 = 2a = 2 \times 2 = 4\text{ N}$$

Friction between blocks A and B:

$$f_1 = 1 \times a = 1 \times 2 = 2\text{ N}$$

(ii) $F = 18$ N, here $15\text{ N} < F < 21\text{ N}$, so sliding will start between block B and block C and not between block A and B.



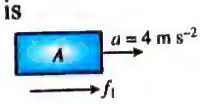
Acceleration of block C:

$$a = -f_{k_2}/m = 6/2 = 3\text{ m s}^{-2}$$

Acceleration of blocks A and B:

$$a = \frac{18-6}{1+2} = 4\text{ m s}^{-2}$$

Friction between block C and ground is

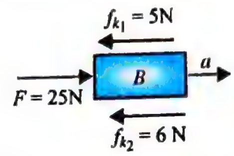


$$f_{k_2} = 6\text{ N}$$

Friction between blocks A and B:

$$f_1 = 1 \times 4 = 4\text{ N}$$

(iii) $F = 25\text{ N} > 21$ N, so sliding occurs at all the contact surfaces. Friction between block A and B is $f_{k_1} = 5$ N



Friction between blocks B and C is $f_{k_2} = 6$ N

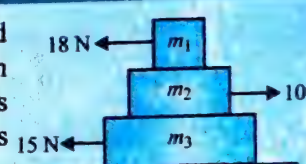
Accelerations of blocks A and C are their maximum possible acceleration.

Acceleration of block B:

$$a = \frac{25-5-6}{2} = 7\text{ m s}^{-2}$$

EXAMPLE 7.13

Consider three blocks placed one over the other as shown in figure. Let us now pull the blocks with the forces of magnitudes 18 N, 100 N, and 15 N. Take $m_1 = m_2 = m_3 = 10$ kg. If the coefficients of static and kinetic friction between all contacting surfaces are $\mu_s = 0.3$ and $\mu_k = 0.2$, respectively, find the



(a) Acceleration of the blocks

(b) Friction at each surface

Sol. Let us first calculate the limiting and kinetic frictional force between various contact surfaces:

Between m_1 and m_2:

$$f_{l_1} = 0.3 \times 10\text{ g} = 30\text{ N}$$

$$f_{k_1} = 0.2 \times 10\text{ g} = 20\text{ N}$$

Between m_2 and m_3:

$$f_{l_2} = 0.3 \times 20g = 60\text{ N}$$

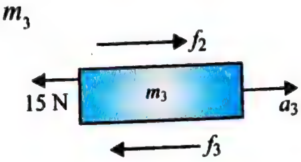
$$f_{k_2} = 0.2 \times 20\text{ g} = 40\text{ N}$$

Between m_3 and ground:

$$f_{l_3} = 0.3 \times 30\text{ g} = 90\text{ N}$$

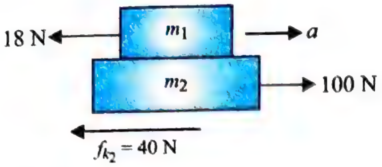
$$f_{k_3} = 0.2 \times 30\text{ g} = 60\text{ N}$$

Consider FBD of m_3



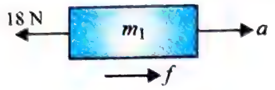
Maximum value of f_2 is $f_{l_2} = 60$ N, which is less than the maximum value of $15 + f_3$ which is $15 + 90 = 105$ N. Hence, m_3 will not accelerate, so $a_3 = 0$.

Now assume that m_1 and m_2 move together with common acceleration a.



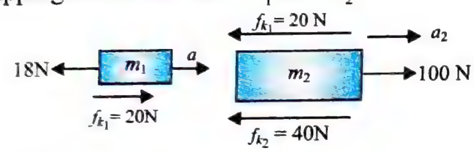
$$a = \frac{100-18-40}{20} = 2.1\ \mathrm{m\,s^{-2}}$$

Let us calculate how much friction is required between m_1 and m_2 for common acceleration a. For this, consider FBD of m_1.



$$f_1 - 18 = 10a \Rightarrow f_1 - 18 = 10 \times 2.1 \Rightarrow f_1 = 39\ \mathrm{N}$$

This is not possible because the maximum value of f_1 is $f_{l_1} = 30$ N. Hence, slipping occurs between m_1 and m_2.

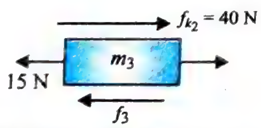


$$a_1 = \frac{f_{k_1} - 18}{10} = \frac{20-18}{10} = 0.2\ \mathrm{m\,s^{-2}}$$

$$a_2 = \frac{100 - f_{k_1} - f_{k_2}}{10} = \frac{100-20-40}{10} = 4\ \mathrm{m\,s^{-2}}$$

(a) So acceleration of m_1 is $a_1 = 0.2\ \mathrm{m\,s^{-2}}$
Acceleration of m_2 is $a_2 = 4\ \mathrm{m\,s^{-2}}$
Acceleration of m_3 is $a_3 = 0$

(b) Since slipping occurs between m_1 and m_2, friction between m_1 and m_2 is $f_{k_1} = 20$ N. Similarly, slipping occurs between m_2 and m_3, so friction between m_2 and m_3 is $f_{k_2} = 40$ N

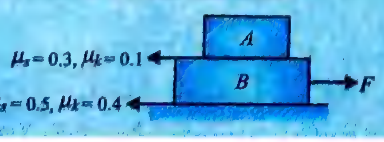


To find friction between m_3 and ground:

$$f_{k_2} = 15 + f_3 \Rightarrow 40 = 15 + f_3 \Rightarrow f_3 = 25\ \mathrm{N}$$

EXAMPLE 7.14

Two blocks $A(m_A = 5$ kg) and $B(m_B = 15$ kg) are placed as shown in figure. A variable force $F = 20t$ starts acting from time $t = 0$ on lower block B, just large enough to. Determine the force F to make block B sliding out from between the block A and the ground at this instant. Plot a graph between acceleration of both the blocks and time.

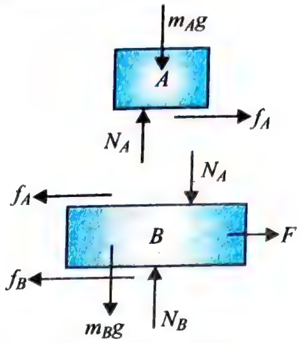


Sol. If upper block starts sliding w.r.t. lower block the friction between the surfaces of A and B should be limiting. Let f_A and f_B are friction between A and B, respectively.

Maximum friction between A and B
= Limiting friction between A and B

$$\Rightarrow \quad (f_A)_{\max} = (f_A)_{\lim}$$
$$= \mu_s N_A = \mu_s m_A g$$
$$= 03 \times 5 \times 10 = 15\ \mathrm{N}$$

and kinetic friction between A and B



$$(f_A)_{\text{kin}} = \mu_k N_A = 0.1 \times 5 \times 10 = 5\ \mathrm{N}$$

Similarly for block B and ground

$$(f_B)_{\max} = \mu_k N_B = \mu_s(N_A + m_B g) = \mu_s(m_A g + m_B g)$$

or $\quad (f_B)_{\max} = \mu_s(m_A + m_B)g = 0.5(5 + 15) \times 10 = 100\ \mathrm{N}$

$$(f_B)_{\text{kin}} = \mu_k(m_A + m_B)g = 0.4(5 + 15) \times 10 = 80\ \mathrm{N}$$

If $F \le 100$ N, upto $t = 5$ s the system of block A and B will not move.

or $\quad a_A = a_B = 0$

In this case, the friction on upper block A (f_A) will be zero.

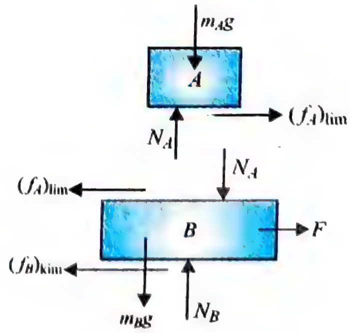
For $F > 100$ N (or after 5 s) the block B will starts sliding with respect to ground and friction appear between the surfaces of A and B.

Upto the time when friction between the blocks A and B reaches to its limiting value, both the blocks move with common acceleration. After this, both the blocks move with different accelerations.

Acceleration of block A when slipping starts between

$$\frac{(f_A)_{\max}}{m_A} = \frac{15}{5} = 3\ \mathrm{m\,s^{-2}}$$

At this time, block B is sliding on the ground and friction between block B and ground is kinetic nature.



From FBD, equation of motion of block B,

$$f - [(f_A)_{\lim} + (f_B)_{\text{kin}}] = m_B a$$
$$F - [15 + 80] = 15 \times 3$$
$$\Rightarrow \quad F = 140\ \mathrm{N}$$

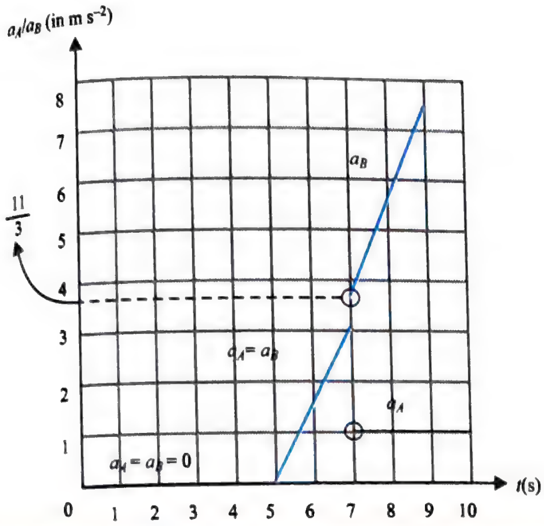
It occurs at time $t = \frac{140}{20} = 7\,\text{s}$ (time $t > 7$ s)

For force range $100\text{ N} < F \le 140\text{ N}$

or $100\text{ N} < 20t \le 140\text{ N}$

i.e., for time range $5\text{ s} < t \le 7\text{ s}$ both blocks move with common acceleration.

$$a_A = a_B = \frac{F - (f_B)_{\text{kin}}}{(m_A + m_B)} = \frac{20t - 80}{20} = (t-4)\,\text{m s}^{-2}$$

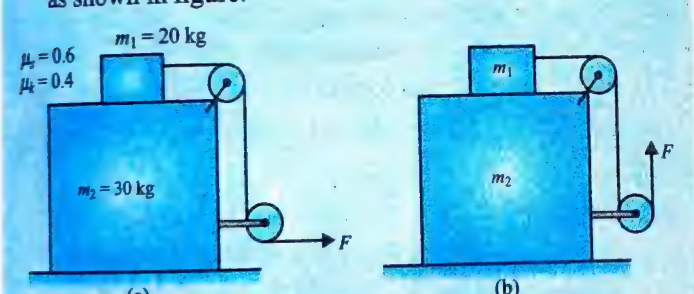


EXAMPLE 7.15

A block of mass $m_1 = 20$ kg is placed on a wedge of mass $m_2 = 30$ kg rests on a smooth floor as shown in figure. Friction exists between block and wedge.

(a) Find the acceleration of each block if (i) $F = 180$ N and (ii) $F = 400$ N.

(b) Solve the problem if force $F = 180$ N is directed upwards as shown in figure.



Sol.

(a) Assume that the system moves together and friction between m_1 and m_2 is static nature.

Common acceleration,

$$a = \frac{F}{(m_1 + m_2)} = \frac{F}{(20+30)} = \frac{F}{50}\,\text{m s}^{-2} \quad \text{...(i)}$$

For block: $F - f = 20a$ ...(ii)

For wedge: $f + F - F = 30a$

or $f = 30a$ ...(iii)

From (i) and (iii), $f = 30\left(\frac{F}{50}\right) \Rightarrow f = \frac{3}{5}F$ ...(iv)

If friction is of static nature, $f \le \mu_s N$

$$\frac{3}{5}F \le 0.6 \times 20 \times 10 \Rightarrow F \le 200\text{ N}$$

or $F_{\max} = 200$ N

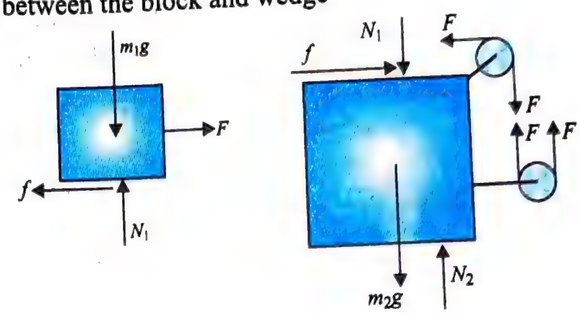
For case (i), $F = 180\text{ N} < F_{\max}$

Hence, friction will be of static nature and both will move with common acceleration.

$$a_1 = a_2 = \frac{180}{(20+30)} = \frac{18}{5}\,\text{m s}^{-2}$$

For case (ii), $F = 400\text{ N} > F_{\max}$

The friction will be of kinetic nature, both the block and the wedge will move with different accelerations. The friction between the block and wedge



$$f_k = \mu N = 0.4 \times 20 \times 10 = 80\text{ N}$$

Acceleration of block, $a_1 = \frac{400 - 80}{20} = 16\,\text{m s}^{-2}$

Acceleration of wedge, $a_2 = \frac{80}{30} = \frac{8}{3}\,\text{m s}^{-2}$

(b) First we will determine the force for which block and wedge move in combination. From Newton's second law, the equations are:

Block: $F - f = m_1 a$

$f - F = m_2 a$

On solving the above equations simultaneously, we obtain

$a = 0 \quad F = f$

For no sliding $f \le \mu_s N$

$f \le 0.6 \times 20 \times 10$

or $F_{\max} = 120$ N

Here F is greater than 120 N; therefore slipping occurs. Now equations are:

Block: $-\mu_k m_1 g = m_1 a_1$

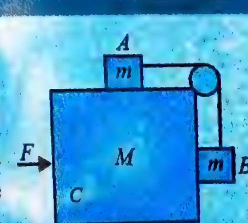
$a_1 = 5\text{ m s}^{-2}$

Wedge: $\mu_k m_1 g - F = m_2 a_2$

$a_2 = -3.33\text{ m s}^{-2}$

EXAMPLE 7.16

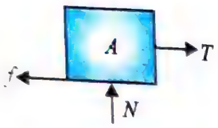
Consider the situation shown in figure. The horizontal surface below the bigger block is smooth the coefficient of friction between the blocks is μ find the minimum and the maximum force F that can be applied in order to keep the smaller blocks at rest with respect to the bigger block.



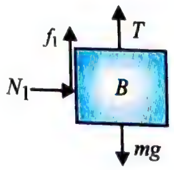
Sol. If no force is applied block A will slip on C towards right and block B will move downward. Suppose the minimum force needed to prevent slipping is F. Taking $A + B + C$ as the system is F. Hence, the acceleration of the system is

$$a = \frac{F}{M+2m} \qquad \text{...(i)}$$

Now take the block A as the system. The forces on A are shown in figure (a).



(a)



(b)

(a) Tension T by the string towards right
(b) Friction f by the block C towards left
(c) Weight mg downward and
(d) Normal force N upward

For vertical equilibrium, $N = mg$.

As the minimum force needed to prevent all slipping is applied the friction is limiting. Thus,

$$f = \mu N = \mu mg$$

As the block moves toward right with an acceleration a,

$$T - f = ma$$

or $\quad T - \mu\, mg = ma \qquad \text{...(ii)}$

Now take block B as the system. The forces are [figure],

(a) Tension T upward
(b) weight mg downward,
(c) normal force N 'towards right', and
(d) friction f upward.

As the block moves towards right with an acceleration α,

$$N' = ma$$

As the friction is limiting, $f' = \mu N' = \mu ma$

For vertical equilibrium, $T + f' = mg$

or $\quad T + \mu\, ma = mg$

Eliminating T form (ii) and (iii), we get

$$a_{\text{min}} = \frac{1-\mu}{1+\mu} g.$$

When a large force is applied, block A slips on C toward left and block B slips on C in the upward direction. The friction on A is towards right and that on B is downwards. Solving as above the acceleration in this case is

$$a_{\text{max}} = \frac{1+\mu}{1-\mu} g$$

Thus lies between $\dfrac{1-\mu}{1+\mu} g$ and $\dfrac{1+\mu}{1-\mu} g$.

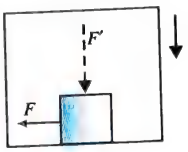
From (i), the force F should be between $\dfrac{1-\mu}{1+\mu}(M+2m)g$ and $\dfrac{1+\mu}{1-\mu}(M+2m)g$.

Exercises

Single Correct Answer Type

1. A rectangular wooden box 10 cm × 20 cm × 40 cm in size is kept on a horizontal surface with its face of largest area on the surface. A minimum force of 10 N applied parallel to the surface sets the box in sliding motion along the surface. If the box is now kept with its face of smaller area in contact with the surface, the minimum force applied parallel to the surface, to set the box in motion, is
(1) less than 10 N
(2) may be greater or less than 10 N
(3) greater than 10 N
(4) equal to 10 N

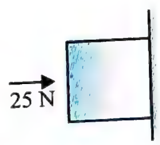
2. A block of mass m is kept on the floor of a freely falling lift. During the free fall of the lift, the block is pulled horizontally with a force of $F = 5$ N. $\mu_s = 0.1$. The frictional force on the block will be



(1) 5 N (2) 2 N
(3) zero (4) 10 N

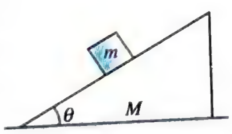
3. If in previous problem, an additional force $F' = 100$ N is applied in vertical direction as shown in figure. The friction force acting on the block is
(1) zero (2) 10 N
(3) 20 N (4) 5 N

4. A horizontal force of 25 N is necessary to just hold a block stationary against a wall. The coefficient of friction between the block and the wall is 0.4. The weight of the block is



(1) 2.5 N (2) 20 N
(3) 10 N (4) 5 N

5. In the figure shown, the block of mass m is at rest relative to the wedge of mass M and the wedge is at rest with respect to ground. This implies that



(1) Net force applied by m on M is mg.
(2) Normal force applied by m on M is mg.
(3) Force of friction applied by m on M is mg.
(4) None of the above.

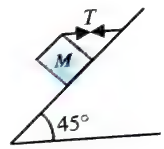
6. The given figure shows a wooden block at rest in equilibrium on a rough horizontal plane being acted upon by forces $F_1 = 10$ N, $F_2 = 2$ N as shown. If F_1 is removed, the resultant force acting on the block will be

(1) 2 N towards left (2) 2 N towards right
(3) 0 N (4) cannot be determined

7. A lift is moving upwards with a uniform velocity v in which a block of mass m is lying. The frictional force offered by the block, when coefficient of friction is $\mu = 0.5$, will be
(1) zero (2) $mg/2$
(3) mg (4) $2mg$

8. A box of mass 8 kg is placed on a rough inclined plane of inclination θ. Its downward motion can be prevented by applying an upward pull F and it can be made to slide upwards by applying a force $2F$. The coefficient of friction between the box and the inclined plane is
(1) $(\tan\theta)/3$ (2) $3\tan\theta$
(3) $(\tan\theta)/2$ (4) $2\tan\theta$

9. A block of mass 15 kg is resting on a rough inclined plane as shown in figure. The block is tied by a horizontal string which has a tension of 50 N. The coefficient of friction between the surfaces of contact is



(1) 1/2 (2) 2/3
(3) 3/4 (4) 1/4

10. A horizontal force just sufficient to move a body of mass 4 kg lying on a rough horizontal surface is applied on it. The coefficient of static and kinetic friction between the body and the surface are 0.8 and 0.6, respectively. If the force continues to act even after the block has started moving, the acceleration of the block in m s^{-2} is ($g = 10$ m s^{-2})
(1) 1/4 (2) 1/2
(3) 2 (4) 4

11. The upper half of an inclined plane with inclination ϕ is perfectly smooth while the lower half is rough. A body starting from rest at the top will again come to rest at the bottom if the coefficient of friction for the lower half is given by
(1) $2\tan\phi$ (2) $\tan\phi$
(3) $2\sin\phi$ (4) $2\cos\phi$

12. A body of mass m is launched up on a rough inclined plane making an angle 45° with horizontal. If the time of ascent is half of the time of descent, the frictional coefficient between plane and body is
(1) $\frac{2}{5}$ (2) $\frac{3}{5}$
(3) $\frac{3}{4}$ (4) $\frac{4}{5}$

13. A wooden block of mass M resting on a rough horizontal floor is pulled with a force F at an angle ϕ with the horizontal. If μ is the coefficient of kinetic friction between the block and the surface, then the acceleration of the block is
(1) $\frac{F}{M}(\cos\phi - \mu\sin\phi) - \mu g$
(2) $\frac{\mu F}{M}\cos\phi$

(3) $\frac{F}{M}(\cos\phi + \mu\sin\phi) - \mu g$

(4) $\frac{F}{M}\sin\phi$

14. A given object takes n times more time to slide down $45°$ rough inclined plane as it takes to slide down a perfectly smooth $45°$ incline. The coefficient of kinetic friction between the object and the incline is

(1) $\sqrt{\frac{1}{1-n^2}}$ (2) $\sqrt{1-\frac{1}{n^2}}$

(3) $1-\frac{1}{n^2}$ (4) $\frac{1}{2-n^2}$

15. Two blocks of masses 0.2 kg and 0.5 kg, which are placed 22 m apart on a rough horizontal surface ($\mu = 0.5$), are acted upon by two forces of magnitude 3 N each as shown in figure at time $t = 0$. Then, the time t at which they collide with each other is

(1) 1 s (2) $\sqrt{2}$ s

(3) 2 s (4) None

16. A block of mass M is being pulled along rough horizontal surface. The coefficient of friction between the block and the surface is μ. If another block of mass $M/2$ is placed on the block and it is again pulled on the surface, the coefficient of friction between the block and the surface will be

(1) μ (2) $\frac{3\mu}{2}$

(3) 2μ (4) $\frac{5\mu}{2}$

17. A body of mass M is resting on a rough horizontal plane surface, the coefficient of friction being equal to μ. At $t = 0$, a horizontal force $F = F_0 t$ starts acting on it, where F_0 is a constant. Find the time T at which the motion starts?

(1) $\mu Mg/F_0$ (2) $Mg/\mu F_0$

(3) $\mu F_0/Mg$ (4) None of these

18. A block A of mass 2 kg is placed over another block B of mass 4 kg, which is placed over a smooth horizontal floor.

The coefficient of friction between A and B is 0.4. When a horizontal force of magnitude 10 N is applied on A, the acceleration of blocks A and B are

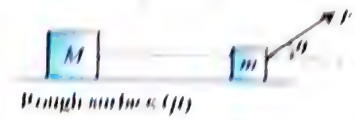
(1) 1 m s^{-2} and 2 m s^{-2}, respectively

(2) 5 m s^{-2} and 2.5 m s^{-2}, respectively

(3) Both the blocks will moves together with acceleration 1/3 m s^{-2}.

(4) Both the blocks will move together with acceleration, 5/3 m s^{-2}

19. Two blocks m and M tied together with an inextensible string are placed at rest on a rough horizontal surface with coefficient of friction μ. The block m is pulled by a variable force F at a varying angle θ with the horizontal. The value of θ at which the least value of F is required to move the blocks is given by



(1) $\theta = \tan^{-1}\mu$ (2) $\theta > \tan^{-1}\mu$

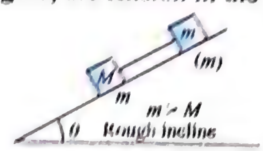
(3) $\theta < \tan^{-1}\mu$ (4) Insufficient data

20. A passenger is travelling a train moving at 40 m s^{-1}. His suitcase is kept on the berth. The driver of train applies breaks such that the speed of the train decreases at a constant rate to 20 m s^{-1} in 4 s. What should be the minimum coefficient of friction between the suitcase and the berth if the suitcase is not to slide during retardation of the train?

(1) 0.3 (2) 0.5

(3) 0.1 (4) 0.2

21. In the given figure, the tension in the rope (rope is light) is



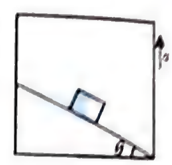
(1) $(M+m)g\sin\theta$

(2) $(M+m)g\sin\theta - \mu mg\cos\theta$

(3) Zero

(4) $(M+m)g\cos\theta$

22. A block of mass m is at rest with respect to a rough incline kept in elevator moving up with acceleration a. Which of following statements is correct?



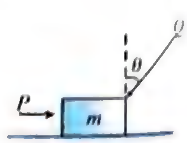
(1) The contact force between block and incline is parallel to the incline.

(2) The contact force between block and incline is of the magnitude $m(g+a)$.

(3) The contact force between block and incline is perpendicular to the incline.

(4) The contact force is of the magnitude $mg\cos\theta$

23. A block of mass m, lying on a horizontal plane, is acted upon by a horizontal force P and another force Q, inclined at an angle θ to the vertical. The block will remain in equilibrium if the coefficient of friction between it and the surface is (assume $P > Q$)



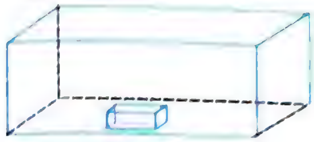
(1) $(P\sin\theta - Q)/(mg - \cos\theta)$

(2) $(P - Q\sin\theta)/(mg + Q\cos\theta)$

(3) $(P\cos\theta + Q)/(mg - Q\cos\theta)$

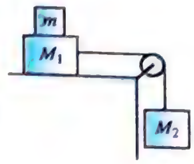
(4) $(P + Q\sin\theta)/(mg + Q\cos\theta)$

24. A solid block of mass 2 kg is resting inside a cube as shown in figure. The cube is moving with a velocity $\vec{v} = 5\hat{i} + 2\hat{j}$ m s^{-1}. If the coefficient of friction between the surface of cube and block is 0.2, then the force of friction between the block and cube is



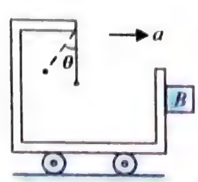
(1) 10 N (2) 4 N
(3) 14 N (4) Zero

25. Two blocks of masses M_1 and M_2 are connected with a string passing over a pulley as shown in figure. The block M_1 lies on a horizontal surface. The coefficient of friction between the block M_1 and the horizontal surface is μ. The system accelerates. What additional mass m should be placed on the block M_1 so that the system does not accelerate?



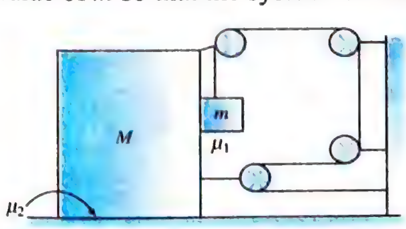
(1) $\dfrac{M_2 - M_1}{\mu}$ (2) $\dfrac{M_2}{\mu} - M_1$
(3) $M_2 - \dfrac{M_1}{\mu}$ (4) $(M_2 - M_1)\mu$

26. A trolley A has a simple pendulum suspended from a frame fixed to its desk. A block B is in contact on its vertical slide. The trolley is on horizontal rails and accelerates towards the right such that the block is just prevented from falling. The value of coefficient of friction between A and B is 0.5. The inclination of the pendulum to the vertical is



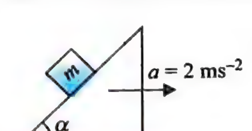
(1) $\tan^{-1}\left(\dfrac{1}{2}\right)$ (2) $\tan^{-1}(3)$
(3) $\tan^{-1}(\sqrt{2})$ (4) $\tan^{-1}(2)$

27. Two blocks M and m are arranged as shown in the diagram. The coefficient of friction between the blocks is $\mu_1 = 0.25$ and between the ground and M is $\mu_2 = \dfrac{1}{3}$. If $M = 8$ kg, then find the value of m so that the system will remain at rest.



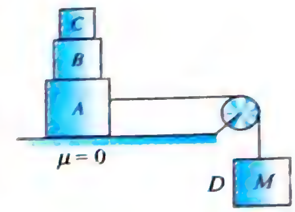
(1) 4/3 kg (2) 8/9 kg
(3) 1 kg (4) 8/5 kg

28. A block of mass m is lying on a wedge having inclination angle $\alpha = \tan^{-1}\left(\dfrac{1}{5}\right)$. Wedge is moving with a constant acceleration $a = 2$ m s^{-2}. The minimum value of coefficient of friction μ so that m remains stationary w.r.t. wedge is



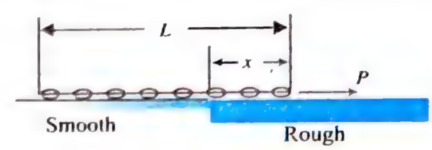
(1) 2/9 (2) 5/12
(3) 1/5 (4) 2/5

29. Three blocks A, B, and C of equal mass m are placed one over the other on a frictionless surface (table) as shown in figure. The coefficient of friction between any blocks A, B, and C is μ. The maximum value of mass of block D so that the blocks A, B, and C move without slipping over each other is



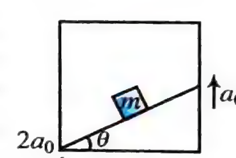
(1) $\dfrac{3m\mu}{\mu+1}$ (2) $\dfrac{3m(1-\mu)}{\mu}$
(3) $\dfrac{3m(1+\mu)}{\mu}$ (4) $\dfrac{3m\mu}{(1-\mu)}$

30. A chain of length L is placed on a horizontal surface as shown in figure. At any instant x is the length of chain on rough surface and the remaining portion lies on smooth surface. Initially $x = 0$. A horizontal force P is applied to the chain (as shown in figure). In the duration, x changes from $x = 0$ to $x = L$. For chain to move with constant speed,



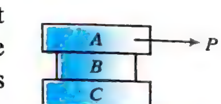
(1) The magnitude of P should increase with time.
(2) The magnitude of P should decrease with time.
(3) The magnitude of P should increase first and then decrease with time.
(4) The magnitude of P should decrease first and then increase with time.

31. For the situation shown in figure, the block is stationary w.r.t. incline fixed in an elevator. The elevator is having an acceleration of $\sqrt{5}\, a_0$ whose components are shown in the figure.
The surface is rough and coefficient of static friction between the incline and block is μ_s. Determine the magnitude of force exerted by incline on the block. (Take $a_0 = g/2$ and $\theta = 37°$, $\mu_s = 0.2$.)



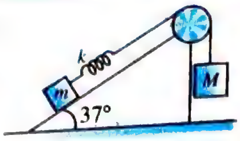
(1) $\dfrac{mg}{10}$ (2) $\dfrac{9mg}{25}$
(3) $\dfrac{3mg}{25} \times \sqrt{41}$ (4) $\dfrac{\sqrt{13}\,mg}{2}$

32. Find the least horizontal force P to start motion of any part of the system of the three blocks resting upon one another as shown in figure. The weights of blocks are $A = 300$ N, $B = 100$ N, and $C = 200$ N. Between A and B, the coefficient of friction is 0.3, between B and C is 0.2, and between C and the ground is 0.1.



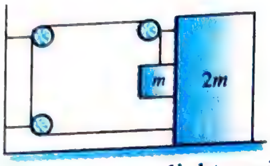
(1) 60 N (2) 90 N
(3) 80 N (4) 70 N

33. A block of mass m is attached with a massless spring of force constant k. The block is placed over a rough inclined surface for which the coefficient of friction is 0.5. M is released from rest when the spring was unstretched. The minimum value of M required to move the block m up the plane is (neglect mass of string and pulley and friction in pulley)



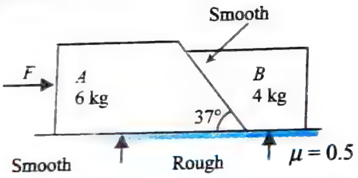
(1) $m/2$
(2) $m/3$
(3) $m/4$
(4) None of these

34. In the system shown in figure, the friction coefficient between ground and bigger block is μ. There is no friction between both the blocks. The string connecting both the block is light; all three pulleys are light and frictionless. Then the minimum limiting value of μ, so that the system remains in equilibrium, is



(1) $\frac{1}{2}$
(2) $\frac{1}{3}$
(3) $\frac{2}{3}$
(4) $\frac{3}{2}$

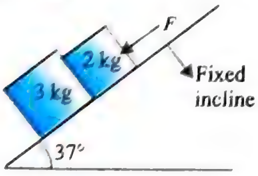
35. Two blocks A of 6 kg and B of 4 kg are placed in contact with each other as shown in figure.



There is no friction between A and ground and between both the blocks. The coefficient of friction between B and ground is 0.5. A horizontal force F is applied on A. Find the minimum and maximum values of F, which can be applied so that both blocks can move combinely without any relative motion between them.

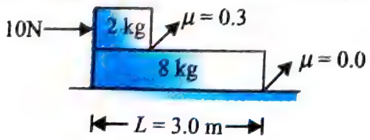
(1) 10 N, 50 N
(2) 12 N, 50 N
(3) 12 N, 75 N
(4) None of these

36. Two blocks of masses 3 kg and 2 kg are placed side by side on an incline as shown in figure. A force F = 20 N is acting on 2 kg block along the incline. The coefficient of friction between the block and the incline is same and equal to 0.1. Find the normal contact force exerted by 2 kg block on 3 kg block.



(1) 18 N
(2) 30 N
(3) 12 N
(4) 27.6 N

37. Determine the time in which the smaller block reaches other end of bigger block as shown in figure.

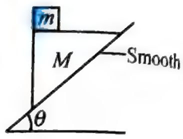


(1) 4 s
(2) 8 s
(3) 2.19 s
(4) 2.13 s

38. A uniform chain is placed at rest on a rough surface of base length l and height h on an irregular surface as shown in figure. Then, the minimum coefficient of friction between the chain and the surface must be equal to

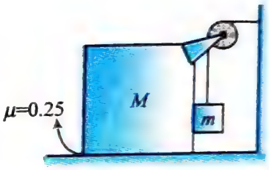
(1) $\mu = \frac{h}{2l}$
(2) $\mu = \frac{h}{l}$
(3) $\mu = \frac{3h}{2l}$
(4) $\mu = \frac{2h}{3l}$

39. A triangular prism of mass M with a block of mass m placed on it is released from rest on a smooth inclined plane of inclination θ. The block does not slip on the prism. Then



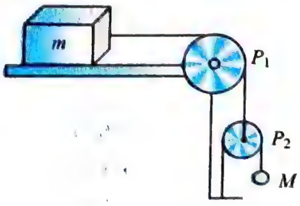
(1) The acceleration of the prism is $g \cos \theta$.
(2) The acceleration of the prism is $g \tan \theta$.
(3) The minimum coefficient of friction between the block and the prism is $\mu_{min} = \cot \theta$.
(4) The minimum coefficient of friction between the block and the prism is $\mu_{min} = \tan \theta$.

40. Two blocks (m and M) are arranged as in figure. There is friction between ground and M only and other surfaces are frictionless. The coefficient of friction between ground and M is $\mu = 0.25$. The maximum ratio of m and M (m/M) so that the system remains is at rest



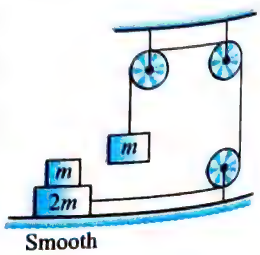
(1) $\frac{1}{3}$
(2) $\frac{1}{4}$
(3) 3
(4) none of these

41. In the pulley arrangement shown in figure, the pulley P_2 is movable. Assuming the coefficient of friction between m and surface to be μ, the minimum value of M for which m is at rest is



(1) $M = \frac{\mu m}{2}$
(2) $m = \frac{\mu M}{2}$
(3) $M = \frac{m}{2\mu}$
(4) $m = \frac{M}{2\mu}$

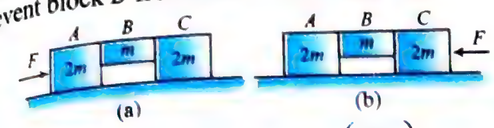
42. In the arrangement shown in figure, there is no friction between the block of mass $2m$ and ground, but there is friction between the blocks of masses m and $2m$. The block of mass m is stationary with respect to block of mass $2m$. The value of coefficient of friction between m and $2m$ is



(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{2}}$

(3) $\frac{1}{4}$ (4) $\frac{1}{3}$

43. The system is pushed by a force F as shown in figure. All surfaces are smooth except between B and C. Friction coefficient between B and C is. Minimum value of F to prevent block B from down ward slipping is



(1) $\left(\frac{3}{2\mu}\right)mg$ (2) $\left(\frac{5}{2\mu}\right)mg$

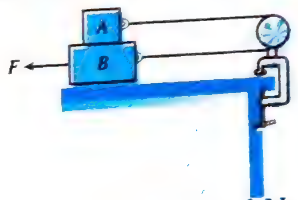
(3) $\left(\frac{5}{2}\right)\mu mg$ (4) $\left(\frac{3}{2}\right)\mu mg$

44. If in the previous problem, all situations are same expect force is applied on block C as shown in case (b), the minimum value of F to prevent block B from downward slipping will be

(1) $\left(\frac{5}{3\mu}\right)mg$ (2) $\left(\frac{5}{2\mu}\right)mg$

(3) $\left(\frac{5}{2}\right)\mu mg$ (4) None of these

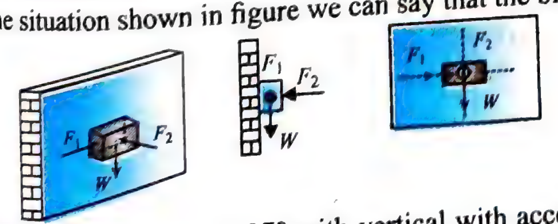
45. Block A, as shown in figure, weighs 2.0 N and block B weighs 6.0 N. The coefficient of kinetic friction between all surfaces is 0.25. Find the magnitude of the horizontal force necessary to drag block B to the left at constant speed if A and B are connected by a light, flexible cord passing around a fixed, frictionless pulley.



(1) 2 N (2) 3 N

(3) 5 N (4) 6 N

46. A block of weight 20 N is pushed against a rough vertical wall with a force of $F_2 = 40$ N, coefficient of static friction being 0.5. Another horizontal force of $F_1 = 15$ N, is applied on the block in a direction parallel to the wall. Regarding the situation shown in figure we can say that the block



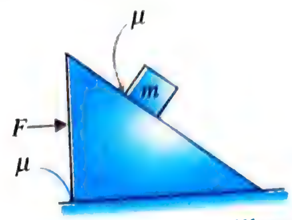
(1) will move at angle 37° with vertical with acceleration 5/2 m s^{-2} downward.

(2) will move at angle 37° with vertical with acceleration 5/2 m s^{-2} upward.

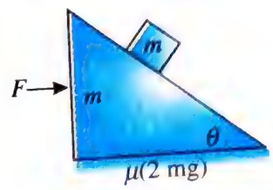
(3) will move at angle 45° with vertical with acceleration 5/2 m s^{-2} downward.

(4) none of these

47. In the situation shown in figure a wedge of mass m is placed on a rough surface, on which a block of equal mass is placed on the inclined plane of wedge. Friction coefficient between plane and the block and the ground and the wedge (μ). An external force F is applied horizontally on the wedge. Given that m does not slide on incline due to its weight.



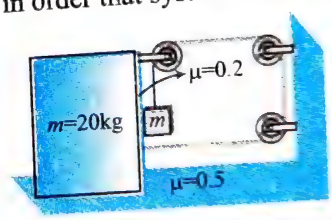
The value of F at which wedge will start slipping is:



(1) $= \mu mg$ (2) $= (3/2)\mu mg$

(3) $> 2\mu mg$ (4) $< \mu mg$

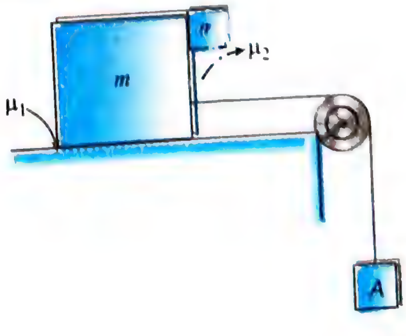
48. In the arrangement shown in figure a, block of mass $M = 20$ kg is placed on rough horizontal surface with $\mu = 0.5$. A block of mass m is arranged as shown. The maximum value of 'm' in order that system remains an equilibrium:



(1) $\frac{10}{3}$ kg (2) 10 kg

(3) $\frac{40}{3}$ kg (4) $\frac{20}{3}$ kg

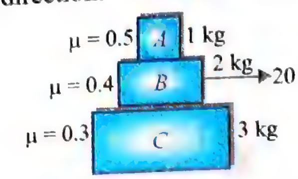
49. In the figure, a block of mass m is placed on a table with rough surface having coefficient of friction μ_1. Another block of same mass is placed on vertical rough surface of this block with coefficient of μ_2 (μ_2 may be greater than 1). If system is released from rest, find the maximum mass of block A, so that the block of mass m does not slip.



(1) $\frac{(M+m)(\mu_1\mu_2+1)}{(\mu_2-1)}$ (2) $\frac{(M+2m)(\mu_1\mu_2+1)}{(\mu_2-1)}$

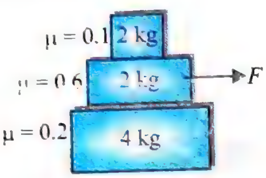
(3) $\frac{(M+m)(\mu_1\mu_2+1)}{(\mu_2+1)}$ (4) None of these

50. Three blocks A, B and C of masses 1 kg, 2 kg and 3 kg respectively are arranged as shown in the figure. The coefficient of friction between different surfaces are shown in figure. A force of magnitude 20 N acts on block B in horizontal direction. The acceleration of block A is



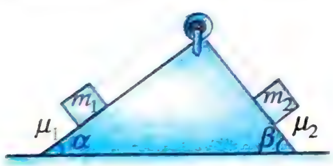
(1) 5 m/s^2 (2) 8/3 m/s^2
(3) zero (4) none of these

51. In the situation shown in figure, for what value of force F (in Newton) sliding between middle and lower block will start? (Take $g = 10$ m/s^2)



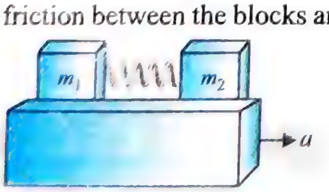
(1) 25 N (2) 30 N
(3) 15 N (4) 18 N

52. Two blocks of masses m_1 and m_2 connected by a string are placed gently over a fixed inclined plane, such that the tension in the connecting string is initially zero. The coefficient of friction between $m1$ and inclined plane is μ_1; between μ_2 and the inclined plane is μ_2. The tension in the string shall continue to remain zero if



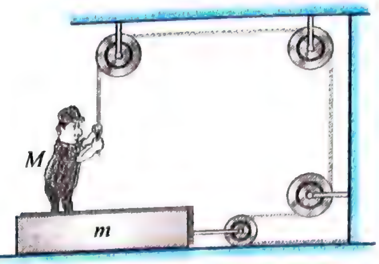
(1) $\mu_1 > \tan\alpha$ and $\mu_2 < \tan\beta$
(2) $\mu_1 < \tan\alpha$ and $\mu_2 > \tan\beta$
(3) $\mu_1 > \tan\alpha$ and $\mu_2 > \tan\beta$
(4) $\mu_1 > \tan\alpha$ and $\mu_2 > \tan\beta$

53. Two blocks of masses m_1 and m_2 are connected with a massless unstretched spring and placed over a plank moving with an acceleration 'a' as shown in figure. The coefficient of friction between the blocks and platform is μ.

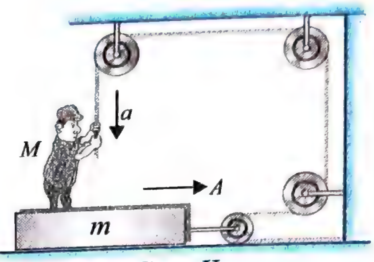


(1) spring will be stretched if $a > \mu g$
(2) spring will be compressed if $a \le \mu g$
(3) spring will neither be compressed nor be stretched for $a \le \mu g$
(4) spring will be in its natural length under all conditions

54. A board of mass $m = 11$ kg is placed on the floor and a man of mass $M = 70$ kg is standing on the board as shown (Case-I). The coefficient of friction between the board and the floor is $\mu = 0.25$. The maximum force that the man can exert on the rope so that the board does not slip on the floor is



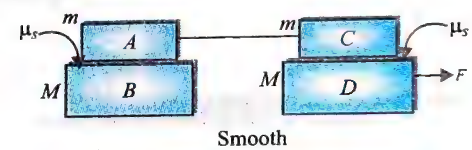
Case-I



Case-II

(1) 125 N (2) 90 N
(3) 162 N (4) None of these

55. Four blocks are arranged on a smooth horizontal surface as shown. The masses of the blocks are given (see the diagram). The coefficient of static friction between the top and the bottom blocks is μs. What is the maximum value of the horizontal force F, applied to one of the bottom blocks as shown, that makes all four blocks move with the same acceleration?



(1) $F_{\max} = 2\mu_S mg\left(\dfrac{2m+M}{m+M}\right)$

(2) $F_{\max} = \mu_S mg\left(\dfrac{m+M}{2m+M}\right)$

(3) $F_{\max} = 2\mu_S mg\left(\dfrac{m+M}{2m+M}\right)$

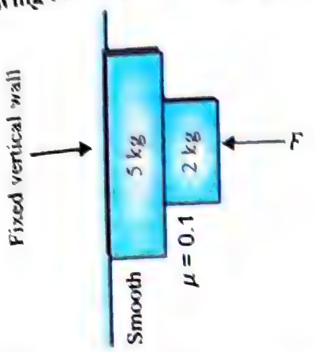
(4) $F_{\max} = \mu_S mg\left(\dfrac{2m+M}{m+M}\right)$

56. Two blocks of equal masses (M) are connected by a string and kept on rough horizontal surface as shown in the figure. The coefficient of friction between the blocks and the surface is μ. If $0 < F_1 - F_2 < 2\mu Mg$, then choose the correct statement.

(1) The direction of friction on block A is towards right.
(2) The direction of friction on block B is either towards left or right
(3) Tension in the string must be zero
(4) Friction force on block B must be zero

57. In the arrangement shown in figure the wall is smooth and friction coefficient between the blocks is $\mu = 0.1$. A

...rizontal force $F = 1000$ N is applied on the 2 kg block. Study following statements regarding this situation:



(i) The normal interaction force between the blocks is 1000 N

(ii) The friction force between the blocks is zero

(iii) Both the blocks accelerate downward with acceleration g m/s^2

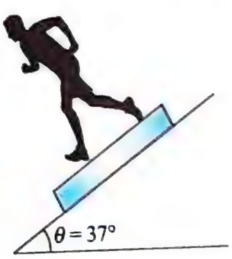
(iv) Both the blocks remain at rest

The correct statements are:

(1) only (ii) and (iv) (2) only (i), (ii) and (iii)

(3) only (ii) and (iii) (4) only (i) and (iv)

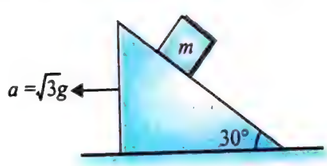
58. A plank of mass $3m$ is placed on a rough inclined plane and a man of mass m walks down the board. If the coefficient of friction between the board and inclined plane is $\mu = 0.5$, the minimum acceleration of does not slide is:



(1) 8 m/s^2 (2) 4 m/s^2

(3) 6 m/s^2 (4) 3 m/s^2

59. If the acceleration is towards right the frictional force exerted by wedge on the block will be: (coefficient of friction between wedge and block $= \dfrac{\sqrt{3}}{2}$)



(1) mg (2) $\dfrac{3mg}{2}$

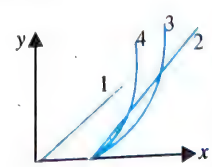
(3) $2mg$ (4) $\dfrac{mg}{2}$

Multiple Correct Answers Type

1. Mark the correct statement(s) regarding friction.
   (1) Friction force can be zero, even though the contact surface is rough.
   (2) Even though there is no relative motion between surfaces, frictional force may exist between them.
   (3) The expressions $f_s = \mu_s N$ or $f_k = \mu_k N$ are approximate expressions.
   (4) The expression $f_s = \mu_s N$ tells that the directions of f_s and N are the same.

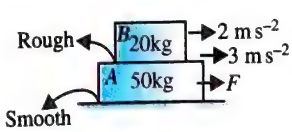
2. A 3-kg block of wood is on a level surface where $\mu_s = 0.25$ and $\mu_k = 0.2$. A force of 7 N is being applied horizontally to the block. Mark the correct statement(s) regarding this situation.
   (1) If the block is initially at rest, it will remain at rest and friction force will be about 7 N.
   (2) If the block is initially moving, then it will continue its motion forever if the force applied is in the direction of the motion of the block.
   (3) If the block is initially moving and the direction of applied force is same as that of motion of block, then block moves with an acceleration of 1/3 m s^{-2} along its initial direction of motion.
   (4) If the block is initially moving and direction of applied force is opposite to that of initial motion of the block, then the block decelerates, comes to a stop, and starts moving in the opposite direction.

3. A block is resting over a rough horizontal floor. At $t = 0$, a time-varying force starts acting on it, the force is described by equation $F = kt$, where k is constant and t is in seconds. Mark the correct statement(s) for this situation.



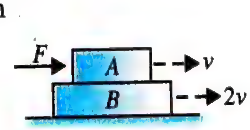
   (1) Curve 1 shows acceleration–time graph.
   (2) Curve 2 shows acceleration–time graph.
   (3) Curve 3 shows velocity–time graph.
   (4) Curve 4 shows displacement–time graph.

4. A 20-kg block is placed on top of a 50-kg block as shown in figure. An horizontal force F acting on A causes an acceleration of 3 m s^{-2} to A and 2 m s^{-2} to B as shown in the figure. For this situation, mark out the correct statement(s).



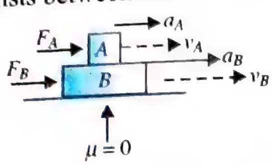
   (1) The friction force between A and B is 40 N.
   (2) The net force acting on A is 150 N.
   (3) The value of F is 190 N.
   (4) The value of F is 150 N.

5. Two blocks A and B of masses m_A and m_B have velocity v and $2v$, respectively, at a given instant. A horizontal force F acts on the block A. There is no friction between ground and block B and coefficient of friction between A and B is μ. The friction



(1) on A supports its motion.
(2) on B opposes its motion relative to A.
(3) on B opposes its motion.
(4) opposes the motion of both.

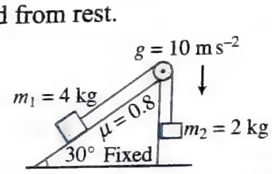
6. Two rough blocks A and B, A placed over B, move with accelerations $\vec{a}_A$ and $\vec{a}_B$, velocities $\vec{v}_A$ and $\vec{v}_B$ by the action of horizontal forces $\vec{F}_A$ and $\vec{F}_B$, respectively. When no friction exists between the blocks A and B,



(1) $v_A = v_B$ (2) $a_A = a_B$

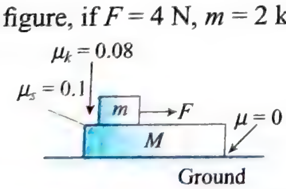
(3) Both (1) and (2) (4) $\frac{F_A}{m_A} = \frac{F_B}{m_B}$

7. Two blocks of masses m_1 and m_2 are connected through a massless inextensible string. A block of mass m_1 is placed at the fixed rigid inclined surface while the block of mass m_2 hanging at the other end of the string, which is passing through a fixed massless frictionless pulley shown in the figure. The coefficient of static friction between the block and the inclined plane is 0.8. The system of masses m_1 and m_2 is released from rest.



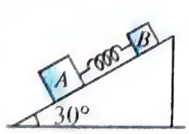
(1) The tension in the string is 20 N after releasing the system.
(2) The contact force by the inclined surface on the block is along normal to the inclined surface.
(3) The magnitude of contact force by the inclined surface on the block m_1 is $20\sqrt{3}$ N.
(4) None of these

8. In the given figure, if $F = 4$ N, $m = 2$ kg, $M = 4$ kg, then



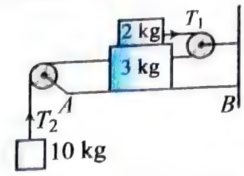
(1) The acceleration of m w.r.t. ground is $2/3\ \text{m s}^{-2}$.
(2) The acceleration of m w.r.t. ground is $1.2\ \text{m s}^{-2}$.
(3) Acceleration of M is $0.4\ \text{m s}^{-2}$.
(4) Acceleration of M w.r.t. ground is $2/3\ \text{m s}^{-2}$.

9. Two blocks A and B of masses 5 kg and 2 kg, respectively, connected by a spring of force constant $= 100\ \text{N m}^{-1}$ are placed on an inclined plane of inclination 30° as shown in figure. If the system is released from rest



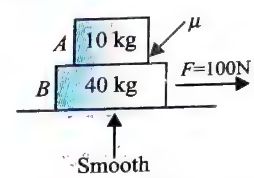
(1) There will be no compression or elongation in the spring if all surfaces are smooth.
(2) There will be elongation in the spring if A is rough and B is smooth.
(3) Maximum elongation in the spring 35 cm if all surfaces are smooth.
(4) There will be elongation in the spring if A is smooth and B is rough.

10. The coefficient of friction between the two blocks is 0.3. whereas the surface AB is smooth. Then



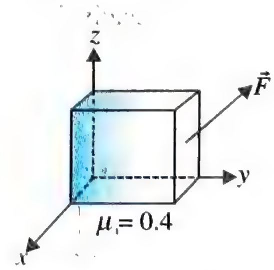
(1) Acceleration of the system of masses is $88/15\ \text{m s}^{-2}$.
(2) Net force acting on 3 kg mass is greater than that on 2 kg mass.
(3) Tension $T_2 > T_1$
(4) Since 10 kg mass is accelerating downwards, so the net force acting on it should be greater than any of the two blocks shown in figure.

11. A 10-kg block is placed on top of a 40-kg block as shown in figure. A horizontal force F acting on B causes an acceleration of $2\ \text{m s}^{-2}$ to B. For this situation, mark out the correct statement(s).



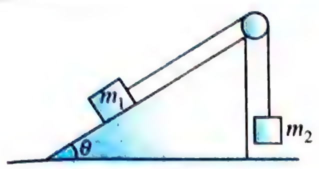
(1) The acceleration of A may be $2\ \text{m s}^{-2}$ or less than $2\ \text{m s}^{-2}$.
(2) The acceleration of A must also be $2\ \text{m s}^{-2}$.
(3) The coefficient of friction between the blocks may be 0.2.
(4) The coefficient of friction between the blocks must be 0.2 only.

12. A solid cube of mass 5 kg is placed on a rough horizontal surface, in xy-plane as shown. The friction coefficient between the surface and the cube is 0.4. An external force $\vec{F} = 6\hat{i} + 8\hat{j} + 20\hat{k}$ N is applied on the cube. Then (use $g = 10\ \text{m/s}^2$)



(1) The block starts slipping over the surface
(2) The friction force on the cube by the surface is 10 N
(3) The friction force acts in xy-plane at angle 127° with the positive x-axis in clockwise direction
(4) The contact force exerted by the surface on the cube is $10\sqrt{10}\ N$

13. Two blocks of masses m_1 and m_2 are connected by a string of negligible mass which passes over a frictionless pulley fixed on the top of an inclined plane as shown in figure. The coefficient of friction between m_1 and plane is μ. Then



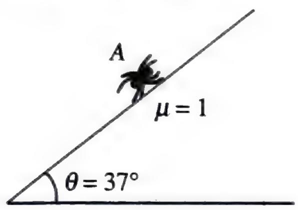
(1) If $m_1 = m_2$ the mass m_1 first begin to move up inclined plane when the angle of inclination is θ, then $\mu = \tan\theta$.

(2) If $m_1 = m_2$ the mass m_1 first begin to move up the inclined plane when the angle of inclination is θ, then $\mu = \sec\theta - \tan\theta$.

(3) If $m_1 = 2m_2$ the mass m_1 first begin to slide down the plane if $\mu = 2\tan\theta$.

(4) If $m_1 = 2m_2$ the mass m_1 first begin to slide down the plane if $\mu = \tan\theta - \dfrac{1}{2}\sec\theta$.

14. An insect of mass m, starts moving on a rough inclined surface from point A. As the surface is very sticky, the coefficient of friction between the insect and the incline is $\mu = 1$. Assume that it can move in any direction; up the incline or down the incline. Then



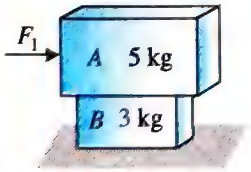
(1) The maximum possible acceleration of the insect can be 14 m/sec^2

(2) The maximum possible acceleration of the insect can be 2 m/sec^2

(3) The insect can move with a constant velocity

(4) The insect cannot move with a constant velocity

15. A block A (5 kg) rests over another block B (3 kg) placed over a smooth horizontal surface. There is friction between A and B. A horizontal force F_1 gradually increasing from zero to a maximum is applied to A so that the blocks move together without relative motion. Instead of this another horizontal force F_2, gradually increasing from zero to a maximum is applied to B so that the blocks move together without relative motion. Then



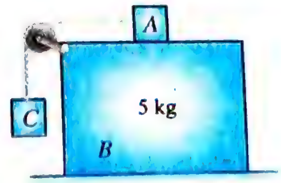
(1) $F_{1(\max)} = F_{2(\max)}$

(2) $F_{1(\max)} > F_{2(\max)}$

(3) $F_{1(\max)} < F_{2(\max)}$

(4) $F_{1(\max)} : F_{2(\max)} = 5 : 3$

16. All the blocks shown in the figure are at rest. The pulley is smooth and the strings are light. Coefficient of friction at all the contacts is 0.2. A frictional force of 10 N acts between A and B. The block A is about to slide on block B.The normal reaction and frictional force exerted by the ground on the block B is



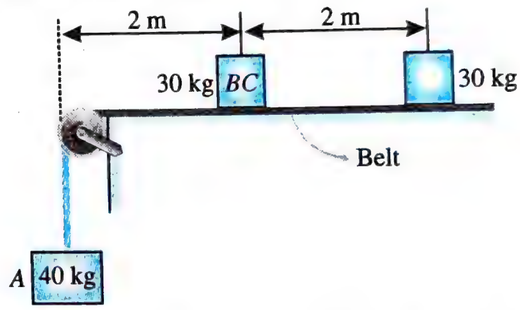
(1) The normal reaction exerted by the ground on the block B is 110 N

(2) The normal reaction exerted by the ground on the block B is 50 N

(3) The frictional force exerted by the ground on the block B is 20 N

(4) The frictional force exerted by the ground on the block B is zero

17. Two 30-kg blocks rest on a massless belt which passes over a fixed pulley and is attached to a 40 kg block. If coefficient of friction between the belt and the table as well as between the belt and the blocks B and C is μ and the system is released from rest from the position shown, the speed with which the block B falls off the belt is



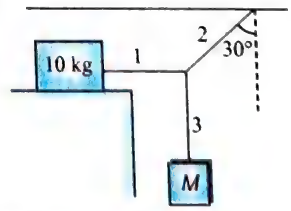
(1) $2\sqrt{2}$ m/s if $\mu = 0.2$

(2) $\sqrt{2}$ m/s if $\mu = 0.2$

(3) 2 m/s if $\mu = 0.5$

(4) 2.5 m/s if $\mu = 0.5$

18. A 10-kg block is at rest as shown on horizontal surface having a coefficient of static friction of 0.7. String-1 is horizontal and string-2 makes an angle of 30° with the vertical. A mass M hangs from string-3. Which of the following statement(s) about this situation is/are true? ($g = 10$ m/s^2)



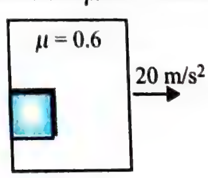
(1) The largest possible tension in string 1 is 70 N

(2) The ratio T_1/T_3 is equal to tan 30°

(3) The largest possible value of M is approximately 12 kg

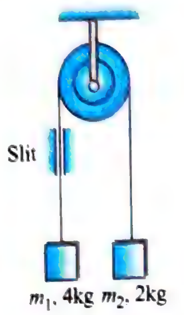
(4) It is impossible to determine the largest possible value of M

19. A box is accelerating with acceleration = 20 m/s^2. A block of mass 10 kg placed inside the box and is in contact with the vertical wall as shown. The friction coefficient between the block and the wall is $\mu = 0.6$ and take g = 10 m/s^2. Then



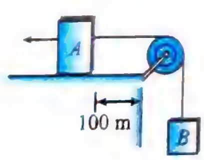
(1) The acceleration of the block will be 20 m/s^2
(2) The friction force acting on the block will be 100 N
(3) The contact force between the vertical wall and the block will be $100\sqrt{5}$ N
(4) The force between the vertical wall and the block is only electromagnetic in nature

20. Two masses $m_1 = 4$ kg and $m_2 = 2$ kg are connected with an inextensible, massless string that passes over a frictionless pulley and through a slit, as shown. The string is vertical on both sides and the string on the left is acted upon by a constant friction force 10 N by the slit as it moves. Then



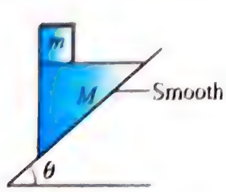
(1) Acceleration of mass m_1 is 5/3 m/s^2, downwards
(2) Tension in the string is same throughout
(3) Force exerted by the string on mass m_2 is 70/3 N
(4) If position of both the masses are interchanged, then 2 kg mass moves up with an acceleration 10/3 m/s^2

21. In the arrangement as shown, block A of mass 3 kg moves towards left with velocity 10 m/s. Initially block A is 100 m from pulley on a smooth surface. Block B is of mass 2 kg:



(1) At $t = 1$ sec, velocity of A will be 6 m/s towards left
(2) A will stop at $t = 2.5$ s
(3) Block A will be at a distance 108 m from pulley at $t = 4$ s
(4) Block A will again be at a distance of 100 m from pulley at $t = 5$ s

22. A triangular prism of mass M with a block of mass m placed on it is released from rest on a smooth inclined plane of inclination θ. The block does not slip on the prism. Then

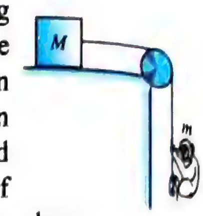


(1) the acceleration of the prism is $g \cos\theta$
(2) the acceleration of the prism is $g \sin\theta$
(3) the minimum coefficient of friction between the block and the prism is $\mu_{min} = \cot\theta$
(4) the minimum coefficient of friction between the block and the prism is $\mu_{min} = \tan\theta$

Linked Comprehension Type

For Problems 1 and 2

A monkey of mass m clings to a rope slung over a fixed pulley. The opposite end of the rope is tried to a weight of mass M lying on a horizontal table. The coefficient of friction between the weight and the table is μ. Find the acceleration of weight and the tension of the rope for two cases. The monkey moves downwards with respect to the rope with an acceleration b.



1. The acceleration of weight is

(1) $\dfrac{2m(g+b)-\mu Mg}{M+2m}$ (2) $\dfrac{m(g+b)-\mu Mg}{2(M+m)}$

(3) $\dfrac{m(g+b)-3\mu Mg}{M+3m}$ (4) $\dfrac{m(g-b)-\mu Mg}{M+m}$

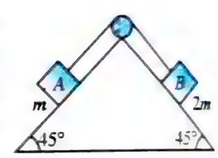
2. The tension of rope is

(1) $\dfrac{Mm(\mu g+g+b)}{M+m}$ (2) $\dfrac{Mm(\mu g-g+b)}{M+m}$

(3) $\dfrac{Mm(\mu g+g-b)}{M+m}$ (4) $\dfrac{Mm(\mu g-g-b)}{M+m}$

For Problems 3–5

Block A of mass m and block B of mass $2m$ are placed on a fixed triangular wedge by means of a massless inextensible string and a frictionless pulley as shown in figure. The wedge is inclined at 45° to the horizontal on both sides. The coefficient of friction between block A and the wedge is 2/3 and that between block B and the wedge is 1/3. If the system of A and B is released from rest, find the following.



3. The acceleration of A is

(1) $\dfrac{g}{3\sqrt{2}}$ (2) Zero

(3) $\dfrac{g}{\sqrt{7}}$ (4) $\dfrac{g}{2\sqrt{3}}$

4. The tension in the string is

(1) $\dfrac{3}{\sqrt{5}}mg$ (2) $\dfrac{5}{3\sqrt{2}}mg$

(3) $\dfrac{2\sqrt{2}\,mg}{3}$ (4) $\dfrac{mg}{3}$

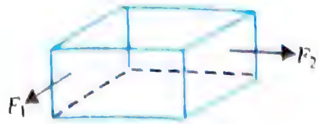
5. The magnitude and direction of the force of friction acting on A are

(1) mg, down the plane (2) $\dfrac{mg}{2}$, up the plane

(3) $\dfrac{mg}{\sqrt{2}}$, up the plane (4) $\dfrac{mg}{3\sqrt{2}}$, down the plane

For Problems 6 and 7

A block of mass 10 kg is kept on a rough floor. The coefficients of friction between floor and block are $\mu_s = 0.4$ and $\mu_k = 0.3$. Forces $F_1 = 5$ N and $F_2 = 4$ N are applied on the block as shown in figure.

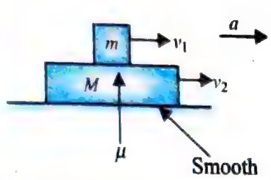


6. Determine the magnitude of friction force.
   (1) $\sqrt{31}$ N (2) $\sqrt{26}$ N
   (3) $\sqrt{41}$ N (4) $\sqrt{36}$ N

7. If $F_1 = 5$ N and $F_2 = a$ N, for what maximum value of a, the motion of block impends?
   (1) $\sqrt{1575}$ N (2) $\sqrt{1225}$ N
   (3) $\sqrt{1664}$ N (4) $\sqrt{875}$ N

For Problems 8–10

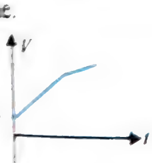
A small block of mass m is placed over a long plank of mass M, the coefficient of friction between them is μ. Ground is smooth. At $t = 0$, m is given a velocity v_1 and M a velocity v_2 ($> v_1$) as shown in figure. After this M is maintained at constant acceleration a ($< \mu g$).

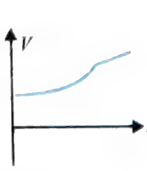


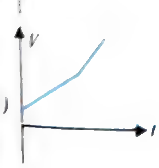
Initially there will be some relative motion between the block and the plank, but after some time relative motion will cease and velocities of both will become same.

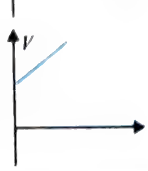
8. Find the time t_0 when the velocities of both block and plank become same.
   (1) $\dfrac{v_2 - v_1}{\mu g + a}$ (2) $\dfrac{v_2 + v_1}{\mu g - a}$
   (3) $\dfrac{v_2 - v_1}{\mu g - a}$ (4) $\dfrac{v_2 + v_1}{\mu g + a}$

9. Draw the variation of velocity of block as a function of time.

(1)


(2)


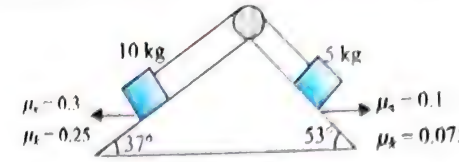
(3)


(4)


10. Find the forward force acting on the plank before and after t_0, respectively.
   (1) Ma, $(M + m)a$ (2) $\mu mg + Ma$, $(M + m)a$
   (3) $\mu Mg + ma$, Ma (4) $(M + m)a$, $\mu mg + Ma$

For Problems 11–13

A system of two blocks and a light string are kept on two inclined faces (rough) as shown in figure. All the required data are mentioned in the diagram. Pulley is light and frictionless. (Take $g = 10$ ms^{-2}, $\sin 37° = 3/5$)



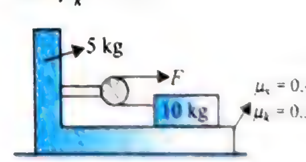
11. If the system is released from rest, then the acceleration of the system is
   (1) $\dfrac{7}{15}$ ms^{-2} (2) Zero
   (3) $\dfrac{47}{15}$ ms^{-2} (4) $\dfrac{2.25}{15}$ ms^{-2}

12. If a system is initially moving in such a way that a block of 10 kg is coming down the incline with a speed of 2 ms^{-1}, then how much time the system takes to come to a stop? [Assume the length of incline to be large enough]
   (1) 13.33 s (2) 80 s
   (3) Infinite (4) Question is irrelevant

13. In the above question, the motion of the system would be best described by
   (1) The system first decelerates, comes to a stop, and then continues to move in the opposite direction.
   (2) The system will continuously move with constant speed.
   (3) The system first decelerates and then comes to a stop.
   (4) The system accelerates and its speed increases with time.

For Problems 14 and 15

A 10-kg block rests on a 5-kg bracket as shown in figure. The 5 kg bracket rests on a horizontal frictionless surface. The coefficients of friction between the 10 kg block and the bracket on which it rests are $\mu_s = 0.40$ and $\mu_k = 0.30$.

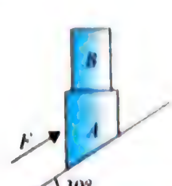


14. The maximum force F that can be applied if the 10 kg block is not to slide on the bracket is
   (1) 32 N (2) 24 N
   (3) 18 N (4) 48 N

15. If 10 kg block does not slide on the bracket, the corresponding acceleration of the 5 kg bracket is
   (1) 1.6 ms^{-2} (2) 0.8 ms^{-2}
   (3) 1.2 ms^{-2} (4) 2.4 ms^{-2}

For Problems 16 and 17

Block A has mass 40 kg and B has mass 15 kg and F is 500 N parallel to smooth inclined plane. The system is moving together.



16. The acceleration of the system is

(1) $\frac{45}{11}\,\text{m s}^{-2}$ (2) $\frac{23}{11}\,\text{m s}^{-2}$

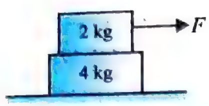
(3) $\frac{13}{7}\,\text{m s}^{-2}$ (4) $\frac{8}{3}\,\text{m s}^{-2}$

17. The least coefficient of friction between A and B is

(1) $\frac{5\sqrt{2}}{12}$ (2) $\frac{9\sqrt{3}}{53}$

(3) $\frac{9\sqrt{2}}{28}$ (4) $\frac{5\sqrt{3}}{18}$

For Problems 18–20



A sufficiently long plank of mass 4 kg is placed on a smooth horizontal surface. A small block of mass 2 kg is placed over the plank and is being acted upon by a time-varying horizontal force $F = (0.5t)$, where F is in newton and t is in seconds as shown in figure. The coefficient of friction between the plank and the block is given as $\mu_s = \mu_k = \mu$. At time $t = 12$ s, the relative slipping between the plank and the block is just likely to occur.

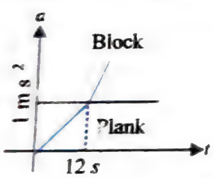
18. The coefficient of friction μ is equal to

(1) 0.10 (2) 0.15

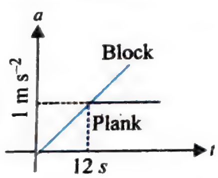
(3) 0.20 (4) 0.30

19. The acceleration a versus time t graph for the plank and the block shown in figures below is correctly represented in

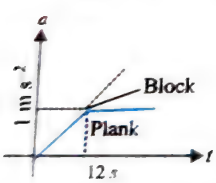
(1)



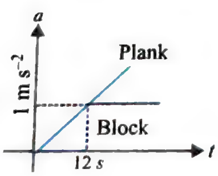
(2)



(3)



(4)

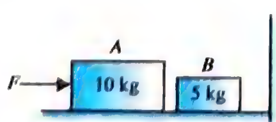


20. The average acceleration of the plank in the time interval 0 to 15 s in the figure will be

(1) $0.20\,\text{m s}^{-2}$ (2) $0.30\,\text{m s}^{-2}$

(3) $0.40\,\text{m s}^{-2}$ (4) $0.60\,\text{m s}^{-2}$

For Problems 21–23



Two bodies A and B of masses 10 kg and 5 kg are placed very slightly separated as shown in figure. The coefficient of friction between the floor and the blocks are as $\mu_s = 0.4$. Block A is pushed by an external force F. The value of F can be changed. When the welding between block A and ground breaks, block A will start pressing block B and when welding of B also breaks, block B will start pressing the vertical wall.

21. If $F = 20$ N, with how much force does block A press block B?

(1) 10 N (2) 20 N

(3) 30 N (4) Zero

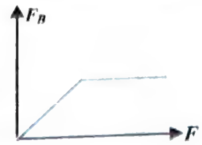
22. If $F = 50$ N, the friction force acting between block B and ground will be

(1) 10 N (2) 20 N

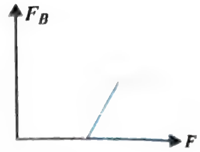
(3) 30 N (4) None

23. The force of friction acting on B varies with the applied force F according to curve

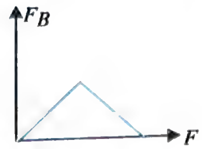
(1)



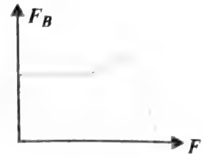
(2)



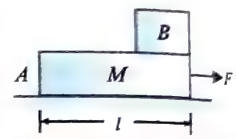
(3)



(4)



For Problems 24 and 25



A plank A of mass M rests on a smooth horizontal surface over which it can move without friction. A cube B of mass m lies on the plank at one edge. The coefficient of friction between the plank and the cube is μ. The size of cube is very small in comparison to the plank.

24. At what force F applied to the plank in the horizontal direction will the cube begin to slide towards the other end of the plank?

(1) $F > \mu(m + M)g$ (2) $F > 0.5\,\mu(m + M)g$

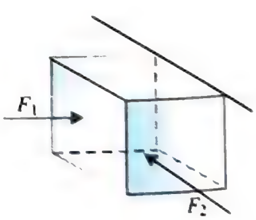
(3) $F = 0.5\,\mu(m + M)g$ (4) $F = \mu(m + M)g$

25. In what time will the cube fall from the plank if the length of the latter is l?

(1) $\sqrt{\frac{Ml}{F - \mu g(M+m)}}$ (2) $\sqrt{\frac{2Ml}{F - \mu g(M+m)}}$

(3) $\sqrt{\frac{Ml}{F + \mu g(M+m)}}$ (4) $\sqrt{\frac{2Ml}{F + \mu g(M+m)}}$

For Problems 26–29



A block of mass 4 kg is pressed against a rough wall by two perpendicular horizontal forces F_1 and F_2 as shown in figure. Coefficient of static friction between the block and wall is 0.6 and that of kinetic friction is 0.5.

26. For $F_1 = 300$ N and $F_2 = 100$ N, find the direction and magnitude of friction force acting on the block.

(1) 180 N, vertically upwards

(2) 40 N, vertically upwards

(3) 107.7 N making an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction.

(4) 91.6 N, making an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction.

For the above data, what is the acceleration of block?

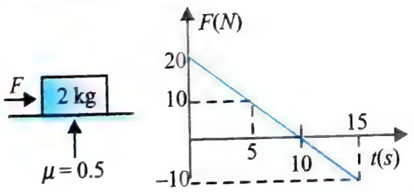
27. (1) Zero
(2) $\frac{140}{4}$ m s^{-2}, upwards
(3) $\frac{180}{4}$ m s^{-2}, upwards
(4) $\frac{107.7}{4}$ m s^{-2} at an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction

28. For $F_1 = 150$ N and $F_2 = 100$ N, find the direction and magnitude of friction force acting on the block.
(1) 90 N, making an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction.
(2) 75 N, making an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction.
(3) 107.7 N, making an angle of $\tan^{-1}\left(\frac{2}{5}\right)$ with the horizontal in the upward direction.
(4) Zero

29. For the data of Q. 28, find the magnitude of acceleration of the block.
(1) Zero (2) 22.5 m s^{-2}
(3) 26.925 m s^{-2} (4) 8.175 m s^{-2}

For Problems 30–32

On a stationary block of mass 2 kg, a horizontal force F starts acting at $t = 0$ whose variation with time is shown in figure. The coefficient of friction between the block and ground is 0.5. Now answer the following questions:



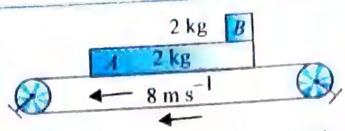
30. Find the time when acceleration of the block is zero.
(1) At 5 s only
(2) At 10 s only
(3) Both at 5 s and 10 s
(4) At a time after $t = 10$ s only.

31. Find the velocity of the block when for the first time its acceleration becomes zero.
(1) 12.5 m s^{-1} (2) 25 m s^{-1}
(3) 10 m s^{-1} (4) None of these

32. Find the velocity of the block at $t = 12$ s.
(1) 20 m s^{-1} (2) –12 m s^{-1}
(3) +6 m s^{-1} (4) Zero

For Problems 33–35

A long conveyer belt moves with a constant velocity of 8 m s^{-1}. Two blocks A and B each of mass 2 kg are placed gently on the belt with B on A. Initial velocity of both blocks is zero. Coefficient of friction between A and belt is 0.1. There is no friction between A and B. Length of A is 4 m.



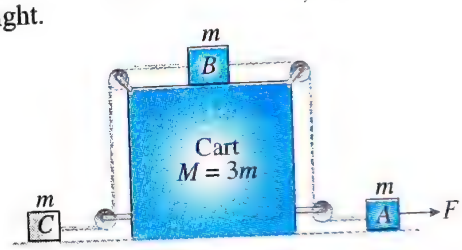
33. Find the time when B falls off A. Initially, B is on the right end of A. Ignore the dimensions of B.
(1) 1 s (2) 3 s
(3) 2 s (4) 4 s

34. Find the velocity of A when B falls off A.
(1) 2 m s^{-1} (2) 4 m s^{-1}
(3) 6 m s^{-1} (4) 8 m s^{-1}

35. If the coefficient of friction between the block B and belt is 0.4, find the separation between the two blocks when B comes to rest w.r.t. belt.
(1) 8 m (2) 6 m
(3) 2 m (4) None of these

For Problems 36–38

A block B is placed over a cart which in turn lies over a smooth horizontal floor. Block A and block C are connected to block B with light inextensible strings passing over light frictionless pulleys fixed to the cart as shown. Initially the blocks and the cart are at rest. All the three blocks have mass m and the cart has mass $M (M = 3m)$. Now a constant horizontal force of magnitude F is applied to block A towards right.



Fixed horizontal floor

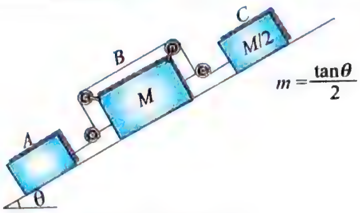
36. Assuming friction to be absent everywhere, the magnitude of acceleration of cart at the shown instant is
(1) $\frac{F}{6m}$ (2) $\frac{F}{4m}$
(3) $\frac{F}{3m}$ (4) 0

37. Taking friction to be absent everywhere the magnitude of tension in the string connecting block B and block C is
(1) $\frac{F}{9}$ (2) $\frac{F}{6}$
(3) $\frac{F}{3}$ (4) $\frac{2F}{3}$

38. Let the coefficient of friction between block B and cart is μ ($\mu > 0$) and friction is absent everywhere else. Then the maximum value of force F applied to block A such that there is no relative acceleration between block B and cart is
(1) μmg (2) $2\mu mg$
(3) $3\mu mg$ (4) $4\mu mg$

For Problems 39–41

As shown in the figure, blocks of masses $M/2$, M and $M/2$ are connected through a light string as shown, pulleys are light and smooth. Friction is only between block C and floor. System is released from rest.



39. Find the acceleration of block B.

(1) $> g \sin\theta$ (2) $< g \sin\theta$

(3) $g \sin\theta$

(4) any of the above can be possible

40. Regarding accelerations of A and C, we can say that:

(1) accelerations of both will be same

(2) acceleration of A will be greater than that of C

(3) acceleration of C will be greater than that of A

(4) any of the above is possible

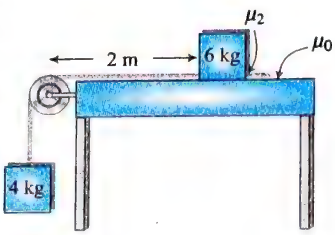
41. Find the tension in the string.

(1) $5\,Mg \sin\theta/8$ (2) $Mg \sin\theta/4$

(3) $Mg \sin\theta/6$ (4) $Mg \sin\theta/8$

For Problems 42–44

A 4 kg block is tied at one end of a massless belt as shown. The belt can slide on table. A 6 kg block is kept on the belt. The coefficient of friction between belt and table (μ_1) and between 6 kg block and belt (μ_2) is same and equal to 0.2. The system is released from rest.



Answer the following questions: (Take $g = 10\ \text{m/s}^2$)

42. Find the acceleration of 4 kg block.

(1) zero (2) $2\ \text{m/s}^2$

(3) $4\ \text{m/s}^2$ (4) $2.8\ \text{m/s}^2$

43. Find the acceleration of 6 kg block.

(1) zero (2) $2\ \text{m/s}^2$

(3) $4\ \text{m/s}^2$ (4) $2.8\ \text{m/s}^2$

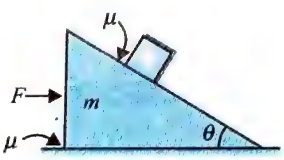
44. Find the displacement of block w.r.t. belt on the table during the time the block reaches pulley.

(1) 2 m towards right (2) 4 m towards left

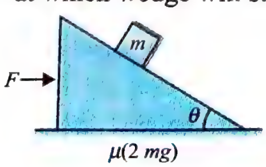
(3) 2 m towards left (4) zero

For Problems 45–48

In the situation shown in figure a wedge of mass m is placed on a rough surface, on which a block of equal mass is placed on the inclined plane of wedge. Friction coefficient between plane and the block and the ground and the wedge (μ). An external force F is applied horizontally on the wedge. Given that m does not slide on incline due to its weight.



45. The value of F at which wedge will start slipping is:



(1) $= \mu mg$ (2) $= \left(\frac{3}{2}\right)\mu mg$

(3) $> 2\mu mg$ (4) $< \mu mg$

46. The value of F at which no friction will act on block on inclined plane, is:

(1) $2\mu mg$ (2) $2\mu mg + mg\tan\theta$

(3) $2\mu mg + 2mg\tan\theta$ (4) $2\mu mg + mg\sin\theta$

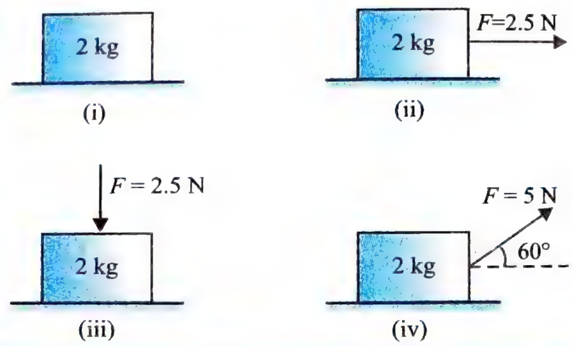
47. The minimum value of acceleration of wedge for which the block starts sliding on the wedge, is:

(1) $g\left(\frac{\cos\theta + \mu\sin\theta}{\sin\theta - \mu\cos\theta}\right)$ (2) $g\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)$

(3) $g\left(\frac{\sin\theta - \mu\cos\theta}{\sin\theta + \mu\cos\theta}\right)$ (4) $g\left(\frac{\cos\theta - \mu\sin\theta}{\sin\theta + \mu\cos\theta}\right)$

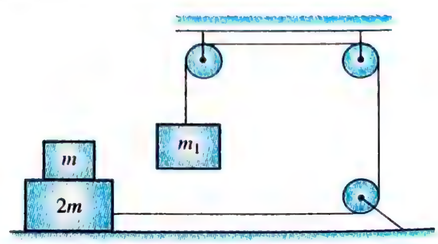
Matrix Match Type

1. The coefficient of friction between the block and the surface in each of the given figures is 0.4. Match Column I with that of Column II



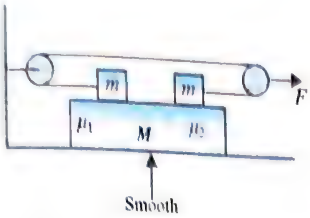
Column I	Column II
i. Force of friction is zero in	**a.** Fig. (i)
ii. Force of friction is 2.5 N in	**b.** Fig. (ii)
iii. Acceleration of the block is zero in	**c.** Fig. (iii)
iv. Normal force is not equal to 2g in	**d.** Fig. (iv)

2. The coefficient of friction between the masses $2m$ and m is 0.5. All other surfaces are frictionless and pulleys are massless. Column I gives the different values of m_1 and Column II gives the possible acceleration of $2m$ and m. Match the columns.



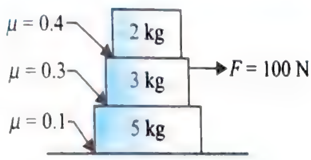
Column I	Column II
i. $m_1 = 2m$	a. Accelerations of $2m$ and m are same.
ii. $m_1 = 3m$	b. Accelerations of $2m$ and m are different.
iii. $m_1 = 4m$	c. Acceleration of $2m$ is greater than m.
iv. $m_1 = 6m$	d. Acceleration of m is less than $0.6g$.

For figure, both the pulleys are massless and frictionless. A force F (of any possible magnitude) is applied in horizontal direction. There is no friction between M and ground. μ_1 and μ_2 are the coefficients of friction as shown between the blocks. Column I gives the different relations between μ_1 and μ_2, and Column II is regarding the motion of M. Match the columns:



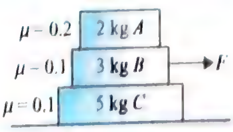
Column I	Column II
i. If $\mu_1 = \mu_2 = 0$	a. May accelerate towards right
ii. If $\mu_1 = \mu_2 \neq 0$	b. May accelerate towards left
iii. If $\mu_1 > \mu_2$	c. Does not accelerate
iv. If $\mu_1 < \mu_2$	d. May or may not accelerate

For the situation shown in figure in Column I, the statements regarding friction forces are mentioned, while in Column II some information related to friction forces are given. Match the entries of Column I with the entries of Column II.



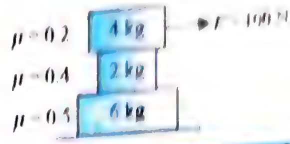
Column I	Column II
i. Total friction force on 3-kg block is	a. Towards right
ii. Total friction force on 5-kg block is	b. Towards left
iii. Friction force on 2-kg block due to 3-kg block is	c. Zero
iv. Friction force on 3-kg block due to 5-kg block is	d. Non-zero

5. For the situation shown in figure, match the entries of Column I with the entries of Column II.



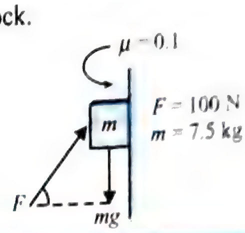
Column I	Column II
i. If $F = 12$ N, then	a. There is relative motion between A and B
ii. If $F = 15$ N, then	b. There is relative motion between B and C
iii. If $F = 25$ N, then	c. There is relative motion between C and the ground
iv. If $F = 40$ N, then	d. Relative motion is not there at any of the surface.

6. For the situation shown in figure, in Column I, the statements regarding friction forces are mentioned, while in Column II, some information related to friction forces are given. Match the entries of Column I with the entries of Column II.



Column I	Column II
I. Total friction force on 4 kg block is	a. Towards right
II. Total friction force on 2 kg block is	b. Towards left
III. Friction force on 6 kg block due to 2 kg block is	c. Zero
IV. Total Friction force on 6 kg block is	d. Non-zero

7. Column I gives the angle θ at which a force F is applied on a block as shown in figure. Column II gives the resulting friction on block.



Column I	Column II
i. $\theta = 37°$	a. friction by wall on block is upwards
ii. $\theta = 45°$	b. friction by wall on block is downwards
iii. $\theta = 53°$	c. friction by wall on block is static
	d. friction by wall on block is kinetic

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.
(1)	b, d	a, c	d
(2)	a, d	a, c	b, c
(3)	a, d	c	b, d
(4)	b, d	a, b	a, d

8. A block of mass m is placed on a plank, which is pivoted at one end. The plank is slowly turned as shown in figure. The friction coefficient between block and plank is 0.75.

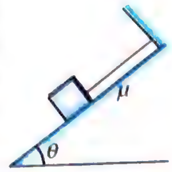
Column I Angle between ground and plank	Column II Friction force between block and plank
I. 30°	a. zero
II. 37°	b. $0.5mg$
III. 60°	c. $0.6mg$
IV. 90°	d. $3mg/8$

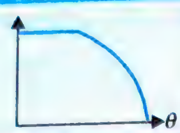
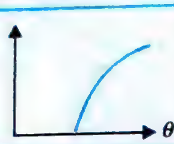
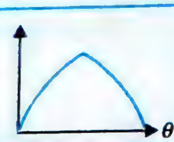
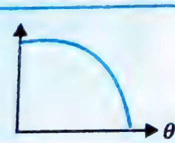
Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	a	d	b	c
(2)	d	c	a	b
(3)	a	b	c	d
(4)	b	c	d	a

9. A block of mass m is put on a rough inclined plane of inclination θ, and is tied with a light thread shown. Inclination θ is increased gradually from $\theta = 0°$ to $\theta = 90°$. Match the column according to corresponding curve.



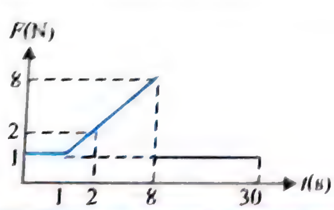
Column I	Column II
i. Tension in the thread versus θ	a. 
ii. Normal reaction between the block and the incline versus θ	b. 
iii. friction force between the block and the incline versus θ	c. 
iv. Net interaction force between the block and the incline versus θ	d. 

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	b	d	c	a
(2)	c	d	b	a
(3)	a	b	c	d
(4)	b	d	c	a

10. A block of mass 1 kg is placed on a rough horizontal surface of coefficient of friction $\mu = 0.2$. A force is applied on the block horizontally whose variation with time is shown in the figure.



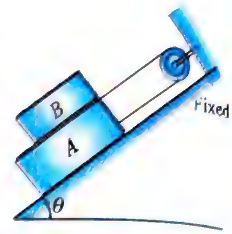
Column I	Column II
i. Velocity of particle is zero at	a. $t = 1$ s
ii. Acceleration of particle is non-zero at	b. $t = 4$ s
iii. Friction force = 1 N	c. $t = 20$ s
iv. Friction force < F	d. $t = 30$ s

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	a, d	b, c	a, d	a, d
(2)	b, c	a, c	a, d	a, d
(3)	a, c, d	b, d	c, d	a, d
(4)	b, d	c, d	b, d	a, c, d

11. A and B are two blocks of respective masses m_A and m_B, connected with each other through a massless and tight string passing over a smooth and massless pulley, as shown. The coefficient of friction between A and B is μ and between A and surface is zero:



Column I	Column II
i. The blocks will remain stationary, if	a. $m_B > m_A$
ii. Force of friction between A and B is zero, if	b. $\mu \geq \dfrac{(m_B - m_A)\tan\theta}{2m_B}$ and $m_B > m_A$
iii. Block B will move upwards if	c. $m_A = m_B$
iv. Force of friction between A and B will not be zero, if	d. $\mu \leq \dfrac{(m_A - m_B)\tan\theta}{2m_B}$

Now match the given columns and select the correct option from the codes given below.

Codes:

	i.	ii.	iii.	iv.
(1)	a, b, d	b	c	d
(2)	b	a, c, d	a, b, c	a
(3)	b	c	d	a, b, d
(4)	c	d	b	a, c, d

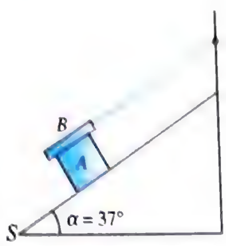
Numerical Value Type

1. Block B, of mass $m_B = 0.5$ kg, rests on block A, with mass $m_A = 1.5$ kg, which in turn is on a horizontal tabletop (as shown in figure). The coefficient of kinetic friction between block A and the tabletop is $\mu_k = 0.4$ and the coefficient of static friction between block A and block B is $\mu_s = 0.6$. A light string attached to block A passes over a frictionless, massless pulley and block C is suspended from the other end of the string. What is the largest mass m_C (in kg) that block C can have so that blocks A and B still slide together when the system is released from rest?

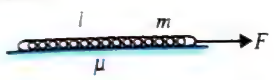
A block of mass $m = 2$ kg is resting on a rough inclined plane of inclination 30° as shown in figure. The coefficient of friction between the block and the plane is $\mu = 0.5$. What minimum force F (in newton) should be applied perpendicular to the plane on the block, so that the block does not slip on the plane?

A block A of mass m is placed over a plank B of mass $2m$. Plank B is placed over a smooth horizontal surface. The coefficient of friction between A and B is 0.4. Block A is given a velocity v_0 towards right. Find acceleration (in ms^{-2}) of B relative to A.

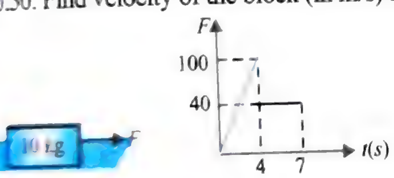
A block A, of weight W, slides down an inclined plane S of slope 37° at a constant velocity, while the plank B, also of weight W, rests on top of A. The plank B is attached by a cord to the top of the plane. The coefficient of kinetic friction μ is the same between the surfaces A and B and between S and A. Determine the value of $1/\mu$.



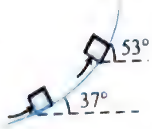
A chain of mass 10 kg and length 8 m is resting on a rough horizontal surface ($\mu = 0.2$). A force $F = 15$ N is applied as shown. Find the length (in m) of the chain on which no friction force acts.



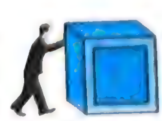
A 10 kg block is resting on the horizontal surface when the force F is applied to it for 7 seconds. The variation of F with time is shown. The coefficients of static and kinetic friction are both 0.50. Find velocity of the block (in m/s) at $t = 4$ sec.



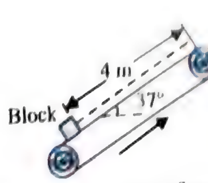
A man can just push a box on 37° slope. When he keeps it at the point where the angle increases to 53°, he can just hold it from sliding back. If the coefficient of friction between the box and the slope is μ, find $1/\mu$ Assume that the man is applying same magnitude of force along the tangent to the curve only.



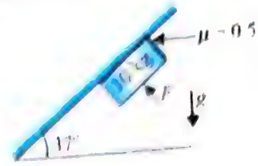
A man of mass 75 kg is pushing a heavy box on a flat floor. The coefficient of kinetic and static friction between the floor and the box is 0.20, and the coefficient of static friction between the man's shoes and the floor is 0.80. If the man pushes horizontally (see figure), what is the maximum mass (in $\times 10^2$ kg) of the box he can move?



9. The following figure shows an accelerating conveyor belt inclined at an angle 37° above horizontal. The coefficient of friction between the belt and block is 1. Find the least time (in sec) in which block can reach the top, starting from rest at the bottom.

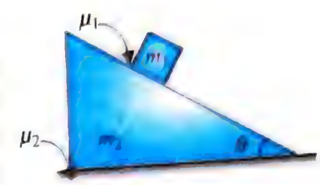


10. In the figure shown, find the minimum force F (in $\times 10^2$N) to be applied perpendicular to the incline so that the block does not slide.

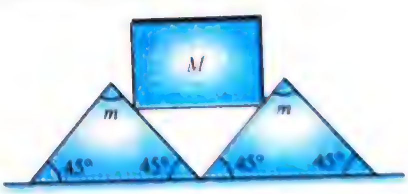


11. A block of 7 kg is placed on a rough horizontal surface and is pulled through a variable force F (in N) = $5t$, where t is time in second, at an angle of 37° with the horizontal as shown in figure. The coefficient of static friction of the block with the surface is one. If the force starts acting at $t = 0$ s, if the time (in sec) at which the block starts to slide is found to be $10n$. Find the value of n. (Take $g = 10$ m/s^2):

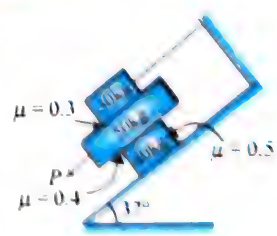
12. A block of mass $m_1 = 4$ kg is placed on a wedge of an angle $\theta = 45°$, as shown. The block is moving over the inclined surface of the wedge of mass $m_2 = 5$ kg. Friction coefficient between the block and the wedge is $\mu_1 = 1/2$, whereas it is μ_2 between the wedge and the horizontal surface. If minimum value of μ_2 so that the wedge remains stationary on the surface is found to be $1/n$. Find the value of n. Take $g = 10$ m/s^2



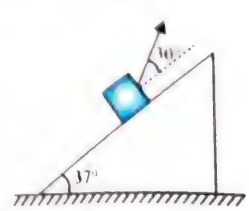
13. Two wedges, each of mass m, are placed next to each other on a flat horizontal floor. A cube of mass M is balanced on the wedges as shown in figure. Assume no friction between the cube and the wedges, but a coefficient of static friction $\mu = 1/3$ between the wedges and the floor. What is the largest ratio of M/m so that the system can be balanced without motion of the wedges?



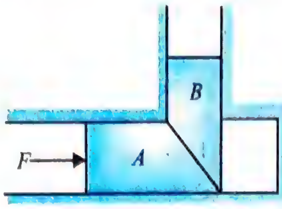
14. The three flat blocks in the figure are positioned on the 37° incline and a force parallel to the inclined plane is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The masses of three blocks in kg and coefficient of static friction for each of the three pairs of contact surfaces are shown in the figure. If the maximum value which force P may have before slipping take place anywhere is found to be $4x$ N. Find the value of x.



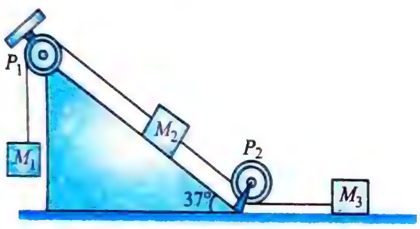
15. A block of mass $m = 4$ kg is placed over a rough inclined plane as shown in figure. The coefficient of friction between the block and the plane is $\mu = 0.5$. A force $F = 10$ N is applied on the block at an angle of 30°. Find the contact force (in N) between the block and the plane.



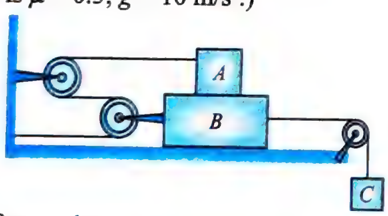
16. A side view of a simplified form of vertical latch B of mass $m = 0.6$ kg is as shown in figure. The lower member A can be pushed forward in its horizontal channel. The sides of the channels are smooth, but at the interfaces of A and B, which are at 45° with the horizontal, there exists a static coefficient of friction $\mu = 0.4$. What is the minimum force F(in N) that must be applied horizontally to A to start motion of the latch B.



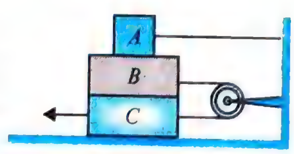
17. Masses M_1, M_2 and M_3 are connected by light strings which pass over pulleys P_1 and P_2 as shown. The masses move such that the string between P_1 and P_2 is parallel to incline and the string between P_2 and M_2 is horizontal, $M_2 = M_3 = 4$ kg. The coefficient of kinetic friction between masses and the surface is 0.25. The angle of inclination of plane is 37° to the horizontal. If the mass M_1 moves downward with uniform velocity, find the tension in the horizontal string (in N). (Given g = 10 m/s²)



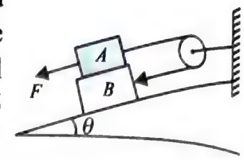
18. The maximum value of mass of block C(in kg) so that neither A nor B moves is (Given that mass of A is 100 kg and that of B is 140 kg. Pulleys are smooth and friction coefficient between A and B and between B and horizontal surface is $\mu = 0.3$; $g = 10$ m/s².)



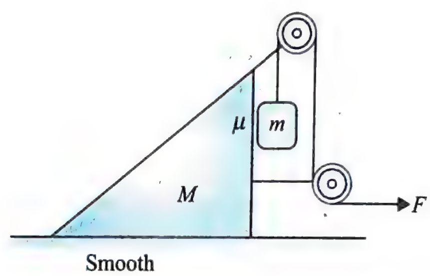
19. In the figure shown, block A, B and C weigh 3 kg, 4 kg and 8 kg, respectively. The coefficient of sliding friction between any two surfaces is 0.25. A is held at rest by a massless rigid rod fixed to the wall while B and C are connected by a string passing round a frictionless pulley. Find the force needed (in N) to drag C along the horizontal surface to left at constant speed. Assume the arrangement shown in figure is maintained all through.



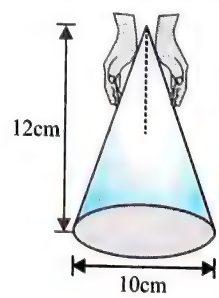
20. Block A has a mass of 30 kg and block B a mass of 15 kg. The coefficients of friction between all surfaces of contact are $\mu_s = 0.15$ and $\mu_k = 0.10$. Knowing that $\theta = 30°$ and that the magnitude of the force F applied to block A is 250 N, determine the acceleration of block A (in m/s²).



21. Mass of the wedge shown in figure is $M = 4$ kg and that of the block is $m = 1$ kg. Horizontal surface beneath the wedge is smooth while the coefficient of friction between vertical surface of the wedge and block is equal to $m = 0.1$. Taking $g = 9.8$ ms^{-2} and assuming pulleys to be massless and frictionless, calculate maximum possible value of force F (in N), upto which the block will remain stationary relative to the wedge.



22. With two hands, you hold a cone motionless upside down, as shown in figure. The mass of the cone is m = 1kg, and the coefficient of static friction between your fingers and the cone is $\mu = 0.5$. What is the minimum normal force (in N) you must apply with each hand in order to hold up the cone? Consider only translational equilibrium.



Archives

JEE Main

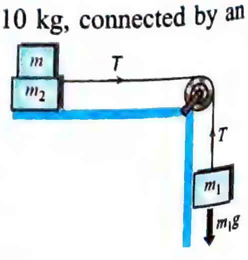
Single Correct Answer Type

1. Given in the figure are two blocks A and B of weight 20 N and 100 N, respectively. These are being pressed against a wall by a force F as shown. If the coefficient of friction between the blocks is 0.1 and between block B and the wall is 0.15, the frictional force applied by the wall on block B is

(1) 100 N (2) 80 N
(3) 120 N (4) 150 N

(JEE Main 2015)

2. Two masses m_1 = 5 kg and m_2 = 10 kg, connected by an inextensible string over a frictionless pulley, are moving as shown in the figure. The coefficient of friction of horizontal surface is 0.15. The minimum weight m that should be put on top of m_2 to stop the motion is:



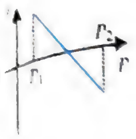
(1) 10.3 kg (2) 18.3 kg
(3) 27.3 kg (4) 43.3 kg

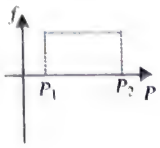
(JEE Main 2018)

ADVANCED

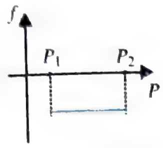
Single Correct Answer Type

1. A block of mass m is on an inclined plane of angle θ. The coefficient of friction between the block and the plane is μ and $\tan\theta > \mu$. The block is held stationary by applying a force P parallel to the plane. The direction of force pointing up the plane is taken to be positive. As P is varied from $P_1 = mg(\sin\theta - \mu\cos\theta)$ to $P_2 = mg(\sin\theta + \mu\cos\theta)$, the frictional force f versus P graph will look like

(1) 

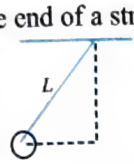
(2) 

(3) 

(4) 

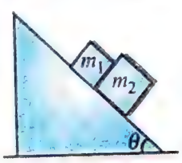
(IIT-JEE 2010)

2. A ball of mass (m) 0.5 kg is attached to the end of a string having length (L) 0.5 m. The ball is rotated on a horizontal circular path about vertical axis. The maximum tension that the string can bear is 324 N. The maximum value of angular velocity of ball (in rad s^{-1}) is



(1) 9 (2) 18
(3) 27 (4) 36 **(IIT-JEE 2011)**

3. A block of mass $m_1 = 1$ kg and another mass $m_2 = 2$ kg are placed together (see figure) on an inclined plane with angle of inclination θ. Various values of θ are given in List I. The coefficient of friction between the block m_1 and the plane is always zero. The coefficient of static and dynamic friction between the block m_2 and the plane are equal to $\mu = 0.3$. In List II expressions for the friction on the block m_2 are given. Match the correct expression of the friction in List II with the angles given in List I, and choose the correct option. The acceleration due to gravity is denoted by g.



[**Useful information:** $\tan(5.5°) \approx 0.1$; $\tan(11.5°) \approx 0.2$; $\tan(16.5°) \approx 0.3$]

List I	List II
P. $\theta = 5°$	1. $m_2 g\sin\theta$
Q. $\theta = 10°$	2. $(m_1 + m_2)g\sin\theta$
R. $\theta = 15°$	3. $\mu m_2 g\cos\theta$
S. $\theta = 20°$	4. $\mu(m_1 + m_2)g\cos\theta$

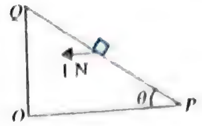
Code:

(1) P-1, Q-1, R-1, S-3 (2) P-2, Q-2, R-2, S-3
(3) P-2, Q-2, R-2, S-4 (4) P-2, Q-2, R-3, S-3

(JEE Advanced 2014)

Multiple Correct Answers Type

1. A small block of mass of 0.1 kg lies on a fixed inclined plane PQ which makes an angle θ with the horizontal. A horizontal force of 1 N acts on the block through its center of mass as shown in the figure. The block remains stationary if (take $g = 10$ m s^{-2})



(1) $\theta = 45°$
(2) $\theta > 45°$ and a frictional force acts on the block towards P
(3) $\theta > 45°$ and a frictional force acts on the block towards Q
(4) $\theta < 45°$ and a frictional force acts on the block towards Q **(IIT-JEE 2012)**

Linked Comprehension Type

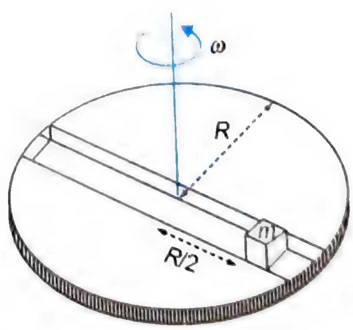
Paragraph-1

A frame of reference that is accelerated with respect to an inertial frame of reference is called a noninertial frame of reference. A coordinate system fixed on a circular disc rotating about a fixed axis with a constant angular velocity ω is an example of a non-inertial frame of reference. The relationship between the force $\vec{F}_{rot}$ experienced by a particle of mass m moving on the rotating disc and the force in $\vec{F}_{in}$ experienced by the particle in an inertial frame of reference is,

$$\vec{F}_{\text{rot}} = \vec{F}_{\text{in}} + 2m\left(\vec{v}_{\text{rot}} \times \vec{\omega}\right) + m(\vec{\omega} \times \vec{r}) \times \vec{\omega}$$

where $\vec{v}_{\text{rot}}$ the velocity of the particle in the rotating is frame of reference and $2\theta \le \tan^{-1}(\mu/2)$ is the position vector of the particle with respect to the centre of the disc.

Now consider a smooth slot along a diameter of a disc of radius R rotating counter-clockwise with a constant angular speed ω about its vertical axis through its center. We assign a coordinate system with the origin at the center of the disc, the x-axis along the slot, the y-axis perpendicular to the slot and the z-axis along the rotation axis ($\vec{\omega} = \omega\hat{k}$). A small block of mass m is gently placed in the slot at $\vec{r} = (R/2)\hat{i}$ at $t = 0$ and is constrained to move only along the slot.



1. The distance r of the block at time t is:

(1) $\dfrac{R}{2}\cos 2\omega t$ (2) $\dfrac{R}{4}(e^{2\omega t} + e^{-2\omega t})$

(3) $\dfrac{R}{2}\cos\omega t$ (4) $\dfrac{R}{4}(e^{\omega t} + e^{-\omega t})$

(JEE Advanced 2016)

2. The net reaction of the disc on the block is :

(1) $m\omega^2 R\sin\omega t\,\hat{j} - mg\hat{k}$

(2) $-m\omega^2 R\cos\omega t\,\hat{j} - mg\hat{k}$

(3) $\frac{1}{2}m\omega^2 R(e^{\omega t} - e^{-\omega t})\hat{j} + mg\hat{k}$

(4) $\frac{1}{2}m\omega^2 R(e^{2\omega t} - e^{-2\omega t})\hat{j} + mg\hat{k}$

(JEE Advanced 2016)

Numerical Value Type

1. A block is moving on an inclined plane making an angle 45° with the horizontal and the coefficient of friction is μ. The force required to just push it up the inclined plane is three times the force required to just prevent it from sliding down. If we define $N = 10\mu$, then N is

(IIT-JEE 2011)

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	**2.** (3)	**3.** (4)	**4.** (3)	**5.** (1)
6. (3)	**7.** (1)	**8.** (1)	**9.** (1)	**10.** (3)
11. (1)	**12.** (2)	**13.** (3)	**14.** (3)	**15.** (3)
16. (1)	**17.** (1)	**18.** (4)	**19.** (1)	**20.** (2)
21. (3)	**22.** (2)	**23.** (2)	**24.** (4)	**25.** (2)
26. (4)	**27.** (3)	**28.** (2)	**29.** (4)	**30.** (1)
31. (4)	**32.** (1)	**33.** (1)	**34.** (3)	**35.** (3)
36. (3)	**37.** (3)	**38.** (2)	**39.** (4)	**40.** (1)
41. (1)	**42.** (3)	**43.** (2)	**44.** (1)	**45.** (2)
46. (1)	**47.** (3)	**48.** (4)	**49.** (1)	**50.** (2)
51. (2)	**52.** (3)	**53.** (4)	**54.** (2)	**55.** (3)
56. (2)	**57.** (2)	**58.** (1)	**59.** (1)	

Multiple Correct Answer Type

1. (1),(2),(3)	**2.** (1),(2),(3)	**3.** (2),(3),(4)
4. (1),(2),(3)	**5.** (1),(2),(3)	**6.** (1),(2),(3),(4)
7. (1),(2),(3)	**8.** (2),(3)	**9.** (1),(4)
10. (1),(2),(3)	**11.** (2),(3)	**12.** (2),(3),(4)
13. (2),(4)	**14.** (1),(3)	**15.** (2, 4)
16. (1, 4)	**17.** (1),(3)	**18.** (1),(2),(3)
19. (1),(2),(3),(4)	**20.** (1),(3)	**21.** (1),(2),(3),(4)
22. (2),(4)		

Linked Comprehension Type

1. (4)	**2.** (3)	**3.** (2)	**4.** (3)	**5.** (4)
6. (3)	**7.** (1)	**8.** (3)	**9.** (1)	**10.** (2)
11. (2)	**12.** (1)	**13.** (3)	**14.** (2)	**15.** (1)
16. (1)	**17.** (2)	**18.** (3)	**19.** (1)	**20.** (4)
21. (4)	**22.** (1)	**23.** (2)	**24.** (4)	**25.** (2)
26. (3)	**27.** (1)	**28.** (2)	**29.** (4)	**30.** (3)
31. (1)	**32.** (4)	**33.** (3)	**34.** (2)	**35.** (3)
36. (4)	**37.** (3)	**38.** (2)	**39.** (3)	**40.** (1)
41. (4)	**42.** (3)	**43.** (2)	**44.** (1)	**45.** (3)
46. (3)	**47.** (2)			

Matrix Match Type

1. i → a, c; ii → b, d; iii → a, b, c, d; iv → c, d.
2. i → a, d; ii → a, d; iii → b, c, d; iv → b, c, d.
3. i → c; ii → c; iii → b, d; iv → a, d.
4. i → b, d; ii → c; iii → a, d; iv → b, d.
5. i → c; ii → c, iii → b, c; iv → a, b, c.
6. i. → b, d, ii → c, iii → a, d, iv → c.

7. (2)	**8.** (4)	**9.** (1)	**10.** (1)	**11.** (3)

Numerical Value Type

1. (5)	**2.** (8)	**3.** (6)	**4.** (4)	**5.** (2)
6. (5)	**7.** (7)	**8.** (3)	**9.** (2)	**10.** (2)
11. (1)	**12.** (8)	**13.** (1)	**14.** (3)	**15.** (30.2)
16. (14)	**17.** (10.5)	**18.** (162)	**19.** (80)	**20.** (5.2)
21. (10)	**22.** (65)			

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3) **2.** (3)

JEE Advanced

Single Correct Answer Type

1. (1) **2.** (4) **3.** (4)

Multiple Correct Answers Type

1. (1, 3)

Linked Comprehension Type

1. (4) **2.** (3)

Numerical Value Type

1. (5)

8 Work, Energy and Power

INTRODUCTION

Almost all the terms we have used thus far—velocity, acceleration, force, and so on—convey a similar meaning in physics as they do in everyday life. Now, however, we encounter a term whose meaning in physics is distinctly different from its everyday meaning: work.

In science and technology, the conservation of energy (energy balance) plays a significant role. While designing a machine (motor, generator, automobiles etc), the design engineer and R&D scientists must account for all sorts of energy transformations happening within the system. The loss of mechanical energy is accountable for the efficiency of the system.

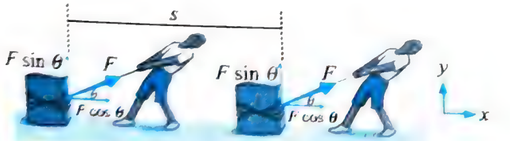
At last, we define power which explains us why it is easier to do any work taking more time rather than doing the total work rapidly (in lesser time). Power supplied (input power), power consumed (output-power), and power loss (radiation, etc.) are the key points (factors) to be accounted for the world of science and technology.

WORK DONE BY A FORCE

Let a constant force $\vec{F}$ be applied on the body such that it makes an angle θ with the horizontal and body is displaced through a distance s. By resolving force $\vec{F}$ into two components:

1. $F\cos\theta$ in the direction of displacement of the body.
2. $F\sin\theta$ in the perpendicular direction of displacement of the body.

The work done on a system, W, by an agent exerting a constant force on the system is the product of the magnitude F of the force, the magnitude s of the displacement of the point of application of the force, and $\cos\theta$, where θ is the angle between the force and displacement vectors:



Since the body is being displaced in the direction of $F\cos\theta$, therefore work done by the force in displacing the body through a distance s is given by

$$W = (F\cos\theta)s = Fs\cos\theta$$

When a constant force $\vec{F}$ acts on a particle while the particle moves through a displacement $\vec{s}$ (for a rigid body, $\vec{s}$ is the displacement of point of application of force on body w.r.t. the frame, in which we have to find the work), the force is said to do work W on the particle given by

$$W = \vec{F}.\vec{s}$$

$\vec{F}.\vec{s}$, the scalar (dot) product of $\vec{F}$ and $\vec{s}$ can be evaluated as:

$$W = \vec{F}.\vec{s} = Fs\cos\theta \quad \text{...(i)}$$

$$W = Fs\cos\theta = F(s\cos\theta) = fs_{\parallel}$$

= Magnitude of the force × Component of the displacement in the direction of the force

$$W = (F\cos\theta)s = F_{\parallel}s$$

= Component of force in the direction of displacement × Magnitude of the displacement

Notice also that the displacement in Eq. (i) is that of *the point of application of the force.* If the force is applied to a particle or a rigid object that can be modeled as a particle, this displacement is same as that of the particle. For a deformable system, however, these displacements are not the same. For example, imagine pressing in on the sides of a balloon with both hands.

The center of the balloon may move through zero displacement. The points of application of the forces from your hands on the sides of the balloon, however, do indeed move through a displacement as the balloon is compressed. We will see other examples of deformable systems, such as springs and samples of gas contained in a vessel. Thus, the work done by a force is equal to the scalar (or dot product) of the force and the displacement of the point of application of force.

Note: If a number of forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots, \vec{F}_n$ are acting on a body and they shift from position vector $\vec{r}_1$ to position vector $\vec{r}_2$, then

$$W = (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots \times \vec{F}_n).(\vec{r}_2 - \vec{r}_1) = (\Sigma\vec{F})\cdot(\Delta\vec{r})$$

In terms of rectangular components, the force and displacement vectors can be written as: $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ and $\vec{s} = x\hat{i} + y\hat{j} + z\hat{k}$.

Therefore,

Work done, $W = \vec{F}\cdot\vec{s}$

$$= (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= F_x x + F_y y + F_z z$$

Important Points:

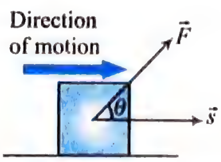
- We can calculate the work done by a force on an object, but that force is *not* necessarily the cause of the object's displacement. For example, if you lift an object (negative), work is done on the object by the gravitational force, although gravity is not the cause of the object moving upward.

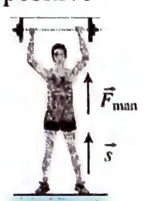
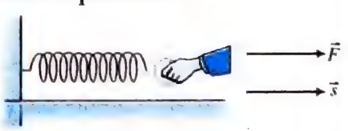
- Work is defined for an interval or displacement. There is no term such as instantaneous work similar to instantaneous velocity.
- For a particular displacement, the work done by a force is independent of the type of motion, i.e., whether it moves with constant velocity, constant acceleration or retardation, etc.
- For a particular displacement work is independent of time. Work will be same for same displacement whether the time taken is small or large.
- When several forces act, the work done by a force for a particular displacement is independent of other forces.

NATURE OF WORK DONE

POSITIVE WORK ($0° \le \theta < 90°$)

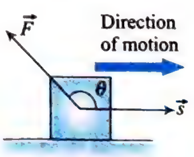
Positive work means that force (or its component) is parallel to displacement. The positive work signifies that the external force favors the motion of the body.

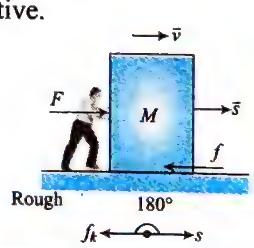


When a person lifts a body from the ground, the work done by the (upward) lifting force is positive	When a spring is stretched, work done by the external (stretching) force is positive.
	

NEGATIVE WORK ($90° < \theta \le 180°$)

Negative work means that force (or its component) is opposite to displacement, i.e., the negative work signifies that the external force opposes the motion of the body.



When a person lifts a body from the ground, the work done by the (downward) force of gravity is negative.	When a body is made to slide over a rough surface, the work done by the frictional force is negative.
	

ZERO WORK

Under three conditions, work done becomes zero:

If the force is perpendicular to the displacement ($\vec{F} \perp \vec{s}$)

Example:

1. When a coolie travels on a horizontal platform with a load on his head, the work done against gravity by the coolie is zero.
2. When a body moves in a circle, the work done by the centripetal force is always zero.

If there is no displacement ($s = 0$)

Example:

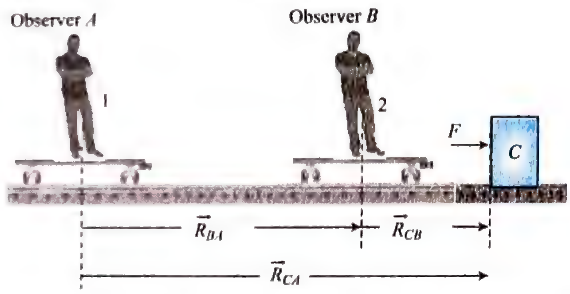
1. When a person tries to displace a wall or heavy stone by applying a force and it does not move, then the work done is zero.
2. A weight lifter does work in lifting the weight off the ground but does not work in holding it up.

If there is no force acting on the body ($F = 0$)

Example: Motion of an isolated body in free space.

WORK DEPENDS ON THE FRAME OF REFERENCE

With change of the frame of reference (inertial), force does not change while displacement may change. So the work done by a force will be different in different frames.



Let us now observe how a force $\vec{F}$ acting on a block C performs works W_1 and W_2 relative to two different observers A and B fixed with two reference frames (1) and (2) respectively, as shown in figure.

During certain time interval, let the displacement of the block relative to the observers A and B be $\Delta\vec{r}_{CA}$ and $\Delta\vec{r}_{CB}$, respectively. Then the corresponding work done can be given as

$$W_1 = \vec{F}.\Delta\vec{r}_{CA} \text{ and } W_2 = \vec{F}.\Delta\vec{r}_{CB}$$

The above expression tells us that same force (F) may perform different works (W_1 and W_2) relative to different observers (reference frames) A and B. Hence, the work done by a force depends on the reference frames. With respect to different reference frames, work done may be different.

ILLUSTRATION 8.1

A constant force $\vec{F} = (3\hat{i} + 2\hat{j} + 2\hat{k})$ N acts on a particle displacing it from a position $\vec{r}_1 = (-\hat{i} + \hat{j} - 2\hat{k})$ m to a new position $\vec{r}_2 = (\hat{i} - \hat{j} + 3\hat{k})$m. Find the work done by the force.

Sol. The displacement vector, $\vec{s} = \vec{r}_2 - \vec{r}_1$

$$\vec{s} = (1+1)\hat{i} + (-1-1)\hat{j} + (3+2)\hat{k} = 2\hat{i} - 2\hat{j} + 5\hat{k}$$

From $W = \vec{F}\cdot\vec{s}$, we have

$$W = (3\hat{i} + 2\hat{j} + 2\hat{k})\cdot(2\hat{i} - 2\hat{j} + 5\hat{k}) = 6 - 4 + 10 = 12 \text{ J}$$

ILLUSTRATION 8.2

Three constant forces $\vec{F}_1 = 2\hat{i} - 3\hat{j} + 2\hat{k}$, $\vec{F}_2 = \hat{i} + \hat{j} - \hat{k}$, and $\vec{F}_3 = 3\hat{i} + \hat{j} - 2\hat{k}$ in newtons displace a particle from (1, −1, 2) to (−1, −1, 3) and then to (2, 2, 0) (displacement being measured in metres). Find the total work done by the forces.

Sol. Net (resultant) force, $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = (6\hat{i} - \hat{j} - \hat{k})\,\text{N}$

and net displacement, $\vec{s} = (2-1)\hat{i} + (2+1)\hat{j} + (0-2)\hat{k}$

$$= \hat{i} + 3\hat{j} - 2\hat{k}\ \text{m}$$

Therefore, work done,

$$W = \vec{F}\cdot\vec{s} = (6\hat{i} - \hat{j} - \hat{k})\cdot(\hat{i} + 3\hat{j} - 2\hat{k})\ \text{N m}$$

$$= (6 - 3 + 2) \times 10^{-5}\,\text{J} = 5 \times 10^{-5}\,\text{J}$$

ILLUSTRATION 8.3

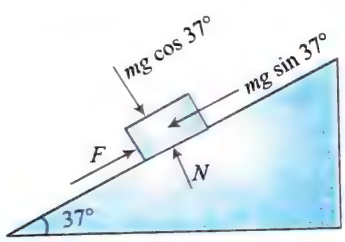
A block of mass 10 kg is slowly slid up on a smooth incline of inclination 37° by a person. Calculate the work done by the person in moving the block through a distance of 2.0 m, if the driving force is applied

(a) parallel to the incline

(b) in the horizontal direction

Sol.

(a) When force is applied parallel to incline: As block moves slowly,



$$F = mg\sin 37° = \frac{3}{5} \times 10 \times 10 = 60\ \text{N}$$

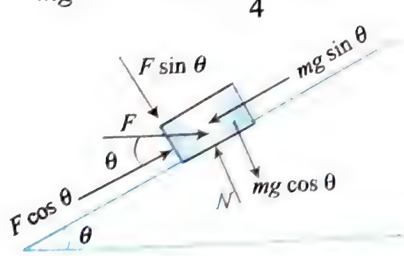
Work done, $W = \vec{F}.\vec{d} = F.d.\cos 0°$

$$W = 60 \times 2 = 120\ \text{J}$$

(b) When force is applied horizontally, As block moves slowly,

$$F\cos\theta = mg\sin\theta$$

$$F = mg\tan\theta = 100 \times \frac{3}{4} = 75\,\text{N}$$



Work done by F

$$W_F = F.d\cos\theta$$

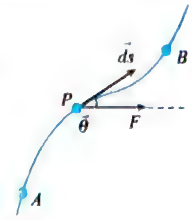
$$= 75 \times 2 \times \frac{4}{5} = 120\,\text{J}$$

WORK DONE BY A VARIABLE FORCE

Force is a vector quantity. When either its magnitude of direction or both magnitude and direction change, we say that the force is varying. For work done by a variable force, the work done for an infinitesimal displacement $\overrightarrow{ds}$ is given by

$$dW = \vec{F}.\overrightarrow{ds}$$

The total work done in going from A to B as shown in figure is



$$W_{AB} = \int_A^B \vec{F}.\overrightarrow{ds} = \int_A^B F\,ds\cos\theta$$

In terms of rectangular components,

$$\overrightarrow{ds} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Work done by a variable force between points A and B,

$$W = \int \vec{F}.\overrightarrow{ds}$$

$$W_{AB} = \int (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_{x_A}^{x_B} F_x\,dx + \int_{y_A}^{y_B} F_y\,dy + \int_{z_A}^{z_B} F_z\,dz$$

ILLUSTRATION 8.4

A force $\vec{F} = 6x\hat{i} + 2y\hat{j}$ displaces a body from $\vec{r}_1 = 3\hat{i} + 8\hat{j}$ to $\vec{r}_2 = 5\hat{i} - 4\hat{j}$. Find the work done by the force.

Sol. Given force $\vec{F} = 6x\hat{i} + 2y\hat{j}$

Initial position $\vec{r}_1 = 3\hat{i} + 8\hat{j}$ and final position $\vec{r}_2 = 5\hat{i} - 4\hat{j}$

We know work done, $W = \int_{s_1}^{s_2} \vec{F}\cdot d\vec{s} = \int_{s_1}^{s_2} \vec{F}\cdot(dx\hat{i} + dy\hat{j})$

$$W = \int_{s_1}^{s_2} (6x\hat{i} + 2y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_3^5 6x\,dx + \int_8^{-4} 2y\,dy$$

$$= \left[3x^2\right]_3^5 + \left[y^2\right]_8^{-4} = [75 - 27] + [16 - 64] = 0\ \text{J}$$

ILLUSTRATION 8.5

A force $F = a + bx$ acts on a particle in x-direction, where a and b are constants. Find the work done by this force during the displacement from x_1 to x_2.

Sol. Let the particle at any instant be at a position x. Let, under the action of force, $F = a + bx$, it describes a small displacement dx.

Work done during the displacement dx will be

$$dW = F\,dx = (a + bx)dx$$

Total work done can be obtained by "summing up" the work done in individual elemental displacements (i.e., by integrating)

Then, $W = \int dW$

$$= \int_{x_1}^{x_2} (a+bx)\,dx \text{ [Here } x \text{ varies from } x_1 \text{ to } x_2]$$

$$= \left[ax + \frac{bx^2}{2}\right]_{x_1}^{x_2}$$

$$= a(x_2 - x_1) + \frac{b}{2}(x_2^2 - x_1^2)$$

$$= \frac{x_2 - x_1}{2}[2a + b(x_1 + x_2)]$$

ILLUSTRATION 8.6

The displacement of a particle of mass 1 kg on a horizontal smooth surface is a function of time given by $x = \frac{1}{3}t^3$. Find out the work done by the external agent for the first one second.

Sol. Given that $x = \frac{1}{3}t^3$. The velocity of the particle at any instant t is

$$v = \frac{dx}{dt} = t^2$$

The acceleration of the particle at any instant t is

$$a = \frac{d^2x}{dt^2} = 2t$$

Therefore, work done by the force imposed is

$$W = \int F\,dx = \int F\frac{dx}{dt}.\,dt = \int m(a)\left(\frac{dx}{dt}\right)(dt)$$

Putting the values of $m = 1$ kg, $\frac{dx}{dt}$, and a, we obtain

$$W = \int_0^{t=1} (1)(t)^2(2t)\,dt = 2\int_0^1 t^3\,dt = 2\left(\frac{t^4}{4}\right)_0^1 = 0.5 \text{ J}$$

ILLUSTRATION 8.7

A chain of length L and mass M is held on a frictionless table with $(1/n)^{th}$ of its length hanging over the edge (figure). Calculate the work done in pulling the chain slowly on the table against gravity.

Sol. Let $\lambda = M/L$ = mass per unit length of the chain and y is the length of the chain hanging over the edge. So the mass of the chain of length y will be λy and the force acting on it due to gravity will be mgy.

The work done in pulling the dy length of the chain on the table.

$dW = F(-dy)$ [dy is negative as y is decreasing]

As the chain is pulled slowly,

F = Weight of the hanging chain = λyg

i.e., $dW = (\lambda yg)(-dy)$

So the work done in pulling the hanging portion on the table,

$$W = -\int_{L/n}^{0} \lambda gy\,dy = -\lambda g\left[\frac{y^2}{2}\right]_{L/n}^{0} = \frac{\lambda g L^2}{2n^2} = \frac{MgL}{2n^2}$$

[as $\lambda = M/L$]

Alternative method

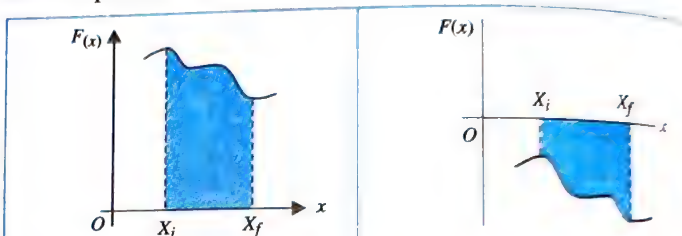
If a point mass m is pulled through a height h then work done, $W = mgh$.

Similarly, for a chain, we can consider its center of mass at the middle point of the hanging part, i.e., at a height of $L/(2n)$ from the lower end as shown in figure and mass of the hanging part of chain = M/n

So the work done to raise the center of mass of the chain on the table is given by $W = \frac{M}{n} \times g \times \frac{L}{2n} = \frac{MgL}{2n^2}$ [as $W = mgh$]

GRAPHICAL INTERPRETATION OF WORK DONE

Generally, the work done by a variable force $F(x)$ from an initial position x_i to final position x_f is interpreted as the area under the force-displacement curve.

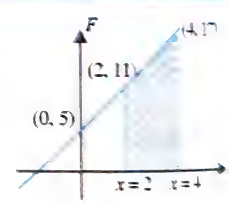


(a) Positive work (b) Negative work

ILLUSTRATION 8.8

Consider a variable force $F = (3x + 5)$ N acting on a body and if it is displaced from $x = 2$ m to $x = 4$ m, calculate the work done by this force.

Sol. If we plot the force in the function of displacement, the work done can be given by the shaded area shown in figure. Thus, work done by this force is



W = Area of shaded trapezium

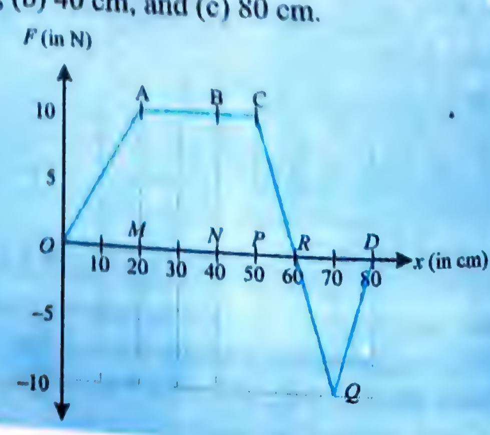
$$= \frac{1}{2} \times 2 \times (11 + 17) = 28$$

If we find the same using integration, we have

$$W = \int_2^4 (3x+5)\,dx = \left(\frac{3x^2}{2} + 5x\right)_2^4 = 28 \text{ J}$$

ILLUSTRATION 8.9

From the figure, find the work done at the end of displacements: (a) 20 cm, (b) 40 cm, and (c) 80 cm.



Sol.

(a) Work done at the end of displacement 20 cm,

= Area of triangle OAM

$$= \frac{1}{2} \times (20 \times 10^{-2}) \times 10 = 1 \text{ J}$$

(b) Work done at the end of displacement 40 cm,

= Area of $OABN$

= Area of OAM + Area of rectangle $ABMN$

$= 1 + 20 \times 10^{-2} \times 10 = 3$ J

(c) Work done at the end of displacement 80 cm,

= Area of trapezium $OACR$ – Area of triangle RQD

$$= \frac{1}{2}(60 + 30) \times 10^{-2} \times 10 - \frac{1}{2} \times (20 \times 10^{-2}) \times 10$$

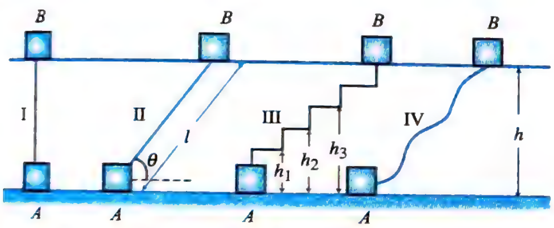
$= 3.5$ J

Net area from 50 cm to 60 cm will be zero.

WORK DONE BY DIFFERENT FORCES

WORK DONE BY GRAVITY

A body of mass m lifted to height h from the ground level by different path as shown in figure.



Work done through different paths,

$$W_{\text{I}} = F.s = mg \times h = mgh$$

$$W_{\text{II}} = F.s = mg\sin\theta \times l = mg\sin\theta \times \frac{h}{\sin\theta} = mgh$$

$$W_{\text{III}} = mgh_1 + 0 + mgh_2 + 0 + mgh_3 + 0 + mgh_4$$

$$= mg(h_1 + h_2 + h_3 + h_4) = mgh$$

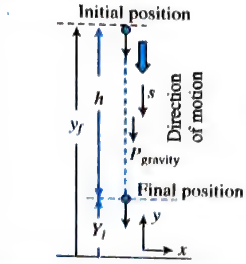
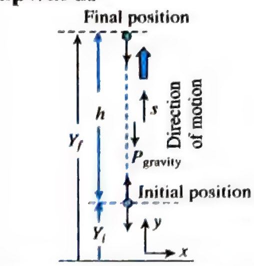
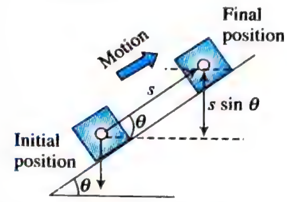
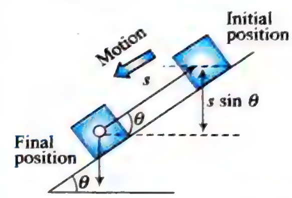
$$W_{\text{IV}} = \int \vec{F}.d\vec{s} = mgh$$

It is clear that $W_{\text{I}} = W_{\text{II}} = W_{\text{III}} = W_{\text{IV}} = mgh$.

Further if the body is brought back to its initial position A, similar amount of work (energy) is released from the system. It means $W_{AB} = mgh$ and $W_{BA} = -mgh$. Hence, the net work done against gravity over a round trip is zero.

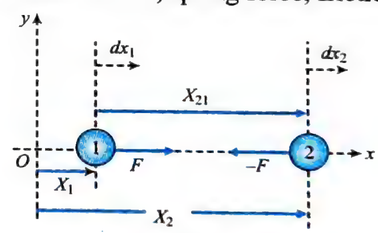
$$W_{\text{Net}} = W_{AB} + W_{BA} = mgh + (-mgh) = 0$$

The work done by the force of gravity on a particle depends only on the initial and final vertical coordinates (because gravity is a vertical force). It does not depend on the path taken or on the speed of the particle. The work done by gravity is zero for any path that returns to its initial point.

A body is thrown vertically downwards  Gravity does positive work during downward motion.	**A body is thrown vertically upwards**  Gravity does negative work during upward motion.
A body is thrown vertically upwards $W = mgs \sin\theta$  Upward motion: Negative work done by gravity.	**A body is thrown vertically downwards** $W = mgs \sin\theta$  Downward motion: Positive work by gravity.

WORK DONE BY A PAIR OF INTERACTING FORCES

Now we will discuss the work done by interacting forces such as gravity, electrostatic force, spring force, friction, tension, etc.



Let two particles (1) and (2) interact with each other by a pair of action–reaction forces $\vec{F}$ and $-\vec{F}$, as shown in figure. Assuming the elementary displacements dx_1 and dx_2 of the particles, the corresponding work done are

Work done on particle (1): $dW_1 = Fdx_1$

Work done on particle (2): $dW_2 = -Fdx_2$

The sum of these elementary work done is

$$dW = dW_1 + dW_2$$

$$= Fdx_1 - Fdx_2$$

$$= F(dx_1 - dx_2) = Fd(x_1 - x_2) = -Fdx_{12}$$

where dx_{12} is the elementary displacement of (2) relative to (1) along the line of their separation.

- When the particles do not move relative to each other. $dx_{12} = 0$. Hence, $dW = 0$. That means, the sum of the work done by these forces is zero.
- If two bodies move under constraint forces (tension of inextensible strings, reaction forces offered by hard, rigid surfaces), the relative displacement between the points of application of the constraint forces along the line of interaction is zero. Then, as a whole, the sum of work done

by all constraint forces along the line of interaction is zero. Then, as a whole, the sum of work done by all constraint forces is zero.

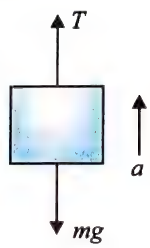
- When the particles move relative to each other, $dx_{12} \neq 0$. For example, two charged particles move due to mutual attractive/repulsive force. Hence, the interacting forces perform a non-zero work as a whole.

ILLUSTRATION 8.10

A block of mass 5 kg is being raised vertically upwards by the help of a string attached to it. It rises with an acceleration of 2 m s^{-2}. Find the work done by the tension in the string if the block rises by 2.5 m. Also find the work done by the gravity and the net work done.

Sol. Let us first calculate the tension. From figure

$T - mg = 5a$; $T = 5(10 + 2) = 60$ N



As T and displacement are in same direction (upwards), work done by the tension T is W.

$W = Ts = 60(2.5) = 150$ J

Work done by gravity

$= -mgs = -5(10)(2.5) = -125$ J

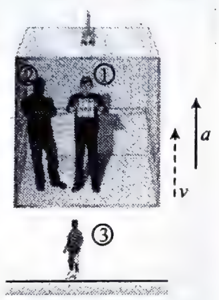
Net work done on the block

= Work done by T + Work done by mg

$= 150 + (-125) = 25$ J

ILLUSTRATION 8.11

A man (1) of mass m stands on an elevator moving with upward acceleration a. A man (2) is standing on the elevator. Elevator starts with initial velocity v_0 at time $t = 0$. Consider time interval t from beginning.

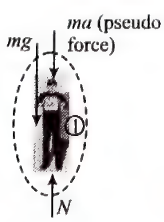


(a) What is the work done by normal contact force and gravity on the man (1) as observe by man (2) standing on the elevator and man (3) standing on ground?

(b) What is the net work done by normal contact force between man (1) and elevator?

Sol.

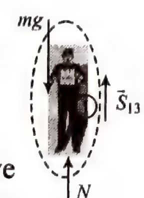
(a) Observation of man (1) from man (2)



Forces on man are $mg\downarrow$ and $N\uparrow$ and pseudo force $ma\downarrow$

Since the displacement of the man(1) relative to man (2) is zero, the forces do not perform work relative to the man (2).

Observation of man (1) from man (3)



Forces on man (1) as seen from are man (3) are $mg\downarrow$ and $N\uparrow$

However, the displacement of man (1) relative to man (3) (ground) is not zero. Hence, the forces will perform non-zero work relative to ground (man 3).

Work done by normal reaction:

$$W_N = \vec{N}.\vec{s} \quad \text{...(i)}$$

Force equation: $N - mg = ma \Rightarrow N = m(g + a)$ ...(ii)

Displacement of man (1) as seen from man (3),

$$\vec{s} = \vec{v}_{13}t + \frac{1}{2}\vec{a}_{13}t^2 = (v_0 t + \frac{1}{2}at^2) \quad \text{...(iii)}$$

Using Eqs (i), (ii), and (iii), we have

$$W_N = m(g + a)(v_0 t + \frac{1}{2}at^2)$$

Similarly, work done by gravity on man (1) as seen from man (2)

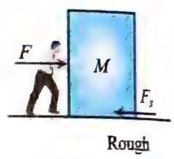
$$W_{gr} = m\vec{g}.\vec{s} = (mg)(s)\cos 180°$$

$$= -mg\left(v_0 t + \frac{1}{2}at^2\right)$$

(b) Here net work done means total work done normal reaction on man (1) ($\uparrow$) and elevator ($\downarrow$). As the relative displacement between the points of application of the normal reaction between person (1) and elevator is zero, then, as a whole, the sum of work done by normal reaction is zero.

WORK DONE BY STATIC FRICTION

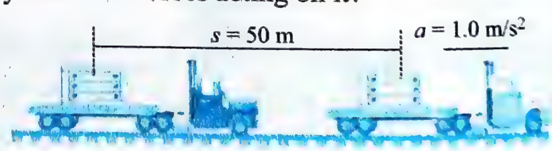
We have learnt about two types of frictional forces, i.e. static friction and kinetic friction. Let us discuss the work done by static friction.



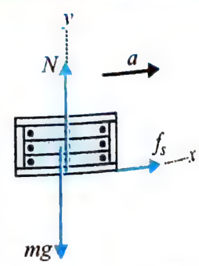
If you push a box by applying a force F, say, let us assume that the box does not move (relative to fixed surface on which it is placed) as shown in figure. Then the static friction f_s does not perform any work as the displacement of the box (displacement of point of application of friction force) is zero ($W_{fs} = 0$).

ILLUSTRATION 8.12

The figure given below shows a 100-kg crate on the flatbed of a truck that is moving with an acceleration of $a = +1.0$ m/s^2 along the positive x axis. The crate does not slip with respect to the truck as the truck undergoes a displacement whose magnitude is $s = 50$ m. What is the total work done on the crate by all of the forces acting on it?



Sol. The free-body diagram in the figure shows the forces that act on the crate:



(1) the weight (mg) of the crate acting vertically downwards,

(2) the normal force (N) exerted by the flatbed acting vertically upwards, and

(3) the static frictional force (f_s) which is exerted by the flatbed in the forward direction and keeps the crate from slipping backward.

The weight (mg) and the normal force (N) are perpendicular to the displacement, so they do no work. Only the static frictional force (f_s) does work, since it acts in the x-direction.

To determine the frictional force, we note that the crate does not slip and, therefore, must have the same acceleration of $a = +1.0\ \text{m/s}^2$ as does the truck. The force creating this acceleration is the static frictional force, and, knowing the mass of the crate and its acceleration, we can use Newton's second law to obtain its magnitude. Then, knowing the frictional force and the displacement, we can determine the total work done on the crate.

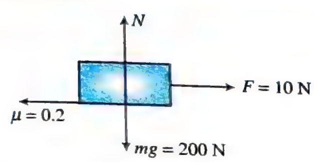
From Newton's second law, we find that the magnitude f_s of the static frictional force, $f_s = ma = (100\,\text{kg})(1.0\,\text{m/s}^2) = 100\,\text{N}$

The total work done by the static frictional force, $W = (f_s \cos\theta)s$
$= (100\,\text{N})(\cos 0^\circ)(50\,\text{m}) = 5.0\times 10^3\ \text{J}$

ILLUSTRATION 8.13

A force of 10 N is acting on a block of 20 kg on a horizontal surface with coefficient of friction $\mu = 0.2$. Calculate the work done by the force.

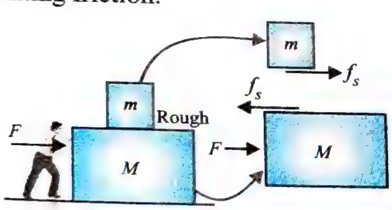
Sol. $f_{\text{max}} = \mu N = 40\ \text{N}$



Driving force, $F < f_{\text{max}}$. Therefore, $s = 0$.

Hence, $W = 0$.

The block does not displace ($s = 0$) because the applied force is less than the limiting friction.



Now take another example. A block of mass m is placed on the block of mass M as shown in the figure above. The horizontal force $\vec{F}$ acts on M. The horizontal surface is smooth, assuming no relative sliding between the blocks. If the lower block moves through a distance x, the upper blocks m will also move through the same distance x. The direction of static friction on the upper block is in forward direction while the direction of static friction on lower block will be in backward direction. Work done by the friction can be given as:

On lower block: $(W_f)_M = -f_s x$

On upper block: $(W_f)_m = f_s x$

The total work done by static friction at the box–surface system (interface) is

$$W_f = (W_f)_M + (W_f)_m = -f_s x + f_s x$$

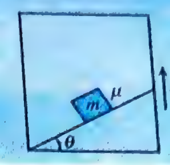
Hence, the total "static frictional work," $W = 0$.

From the above discussion, we conclude the following points: the work done by static friction can be positive, negative, or zero.

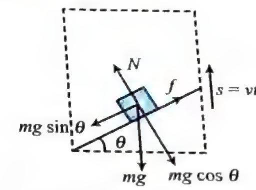
When we consider the net work done by the static friction at the contacting surfaces, since there is no relative displacement between the surfaces, the total work done by the static friction is zero.

ILLUSTRATION 8.14

An inclined plane is moving up with constant velocity v. A block kept on incline is at rest. Calculate the work done by gravity, friction force, and normal reaction on block in time interval of t.

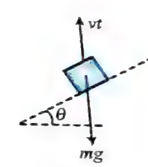


Sol. As the block is at rest w.r.t. inclined plane, hence, the friction between blocks and plane will be of static nature and will act in the direction up the plane.



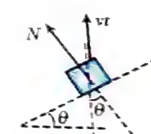
Free-body diagram of block:

(a) For work done by gravity



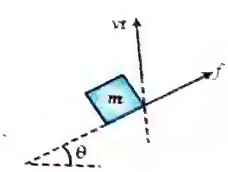
$$W_{\text{gravity}} = \vec{F}.\vec{s}$$
$$= mg\cdot vt\cos 180^\circ$$
$$= mgvt\,(-1)$$
$$= -mg\,vt$$

(b) Work done by normal reaction



$$W_{\text{normal}} = N\cdot vt\cdot\cos\theta$$
$$= (mg\cos\theta)\,vt\cos\theta$$
$$= mgvt\cos^2\theta$$

(c) Work done by friction force



$$W_{\text{friction}} = F\cdot vt\cos(90-\theta)$$
$$= (mg\sin\theta).vt\sin\theta$$
$$= mgvt\sin^2\theta$$

ILLUSTRATION 8.15

A block of mass m is kept over another block of mass M and the system rests on a horizontal surface. A constant horizontal force F acting on the lower block produces an acceleration $\dfrac{F}{2(m+M)}$ in the system. The two blocks always move together. Consider displacement d of the system.

(a) Find the work done by friction on bigger block.

(b) Find the coefficient of kinetic friction between the bigger block and the horizontal surface.

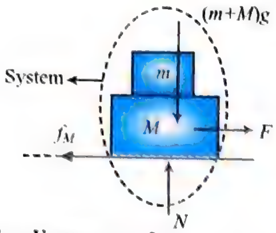
(c) Find the frictional force acting on the smaller block.

(d) Find the work done by the force of friction on the smaller block by the bigger block.

(e) Find the work done by static friction on bigger block.

Sol. As both blocks move together, there will be static friction between m and M. The friction between M and ground will be kinetic in nature. Common acceleration, $a = \dfrac{F}{2(m+M)}$

(a) As both block moves together, we can take m and M as a system. Let friction between M and ground is f_M.



From free body diagram of $m + M$

Equation of motion: $F - f_M = (m + M)a$

$$F - f_M = (m+M)\frac{F}{2(m+M)}$$

$$\Rightarrow \quad f_M = \frac{F}{2}$$

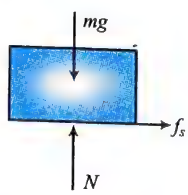
(b) Work done by friction on bigger block,

$$(W_f)_M = \vec{f_M}\cdot\vec{d} = -f_M d = -\frac{F}{2}\cdot d$$

As friction between M and ground is kinetic, then $f_M = \mu N$

$$\frac{F}{2} = \mu(m+M)g \Rightarrow \mu = \frac{F}{2(m+M)g}$$

(c) Considering the free body diagram of m



$$f_s = ma$$

$$f_s = m\frac{F}{2(m+M)} = \frac{Fm}{2(m+M)}$$

(d) Work done by friction force on smaller block by bigger block

$$(W_{fs})_m = \vec{f_s}\cdot d$$

$$= f_s \cdot d$$

$$= \left(\frac{Fm}{2(m+M)}\right)d$$

$$= \frac{Fdm}{2(m+M)}$$

(e) Net work done by static friction on a system is zero.

$$(W_{fs})_m + (W_{fs})_M = 0$$

$$\left[\frac{Fdm}{2(m+M)}\right] + (W_{fs})_M = 0 \Rightarrow (W_{fs})_M = \frac{-Fdm}{2(m+M)}$$

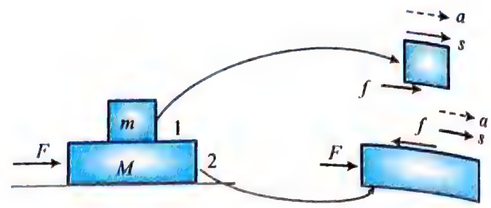
ILLUSTRATION 8.16

A block of mass m is placed on the block of mass M as shown in Figure. The horizontal force $\vec{F}$ acts on M during time interval t. If the horizontal surface is smooth, assuming no relative sliding between the blocks, find the

(a) work done by friction on the blocks

(b) work done by $\vec{F}$ on the lower block

Sol.



(a) As there is no relative sliding between the blocks, they move with a common acceleration,

$$a = \frac{F}{M+m}$$

Then the static frictional force on the block m is

$f = ma$, where $a = \dfrac{F}{M+m}$. Then

$$f = m\frac{F}{M+m}$$

Work done by static frictional force on the block m:

Since $(W_f)_m = \vec{f}.\vec{s} = fs\cos 0° = fs$

where $s = \dfrac{1}{2}at^2$. We have

$$(W_f)_m = f\left(\frac{1}{2}at^2\right)$$

Substituting $f = \dfrac{mF}{M+m}$ and $a = \dfrac{F}{M+m}$, we have

$$(W_f)_m = \frac{mF^2t^2}{2(M+m)^2}$$

As no relative sliding between m and M, hence, net work done by static friction is zero.

Since $(W_f)_m + (W_f)_M = 0$

Hence, the work done by the static frictional force on the block m. We have

$$(W_f)_m = \frac{mF^2t^2}{2(M+m)^2}$$

(b) The displacement of lower block during time t,

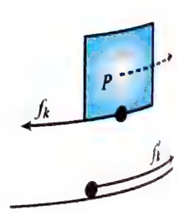
$$\Rightarrow \quad s = \frac{1}{2}\left(\frac{F}{M+m}\right)t^2$$

Hence, work done by F, $W_F = \vec{F}\cdot\vec{s}$

$$W_F = F\cdot s = F\left[\frac{1}{2}\left(\frac{F}{M+m}\right)t^2\right] = \frac{F^2t^2}{2(M+m)}$$

WORK DONE BY KINETIC FRICTION

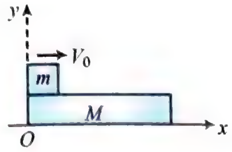
Object is sliding over a fixed surface: As you know, kinetic friction acts on a particle when it slides on a surface. If the surface is fixed, the displacement of the particle is non-zero. That means, kinetic friction f_k will perform certain work. Since $\vec{f_k}$ points opposite to the displacement $\vec{s}$, the work done by kinetic friction is negative.



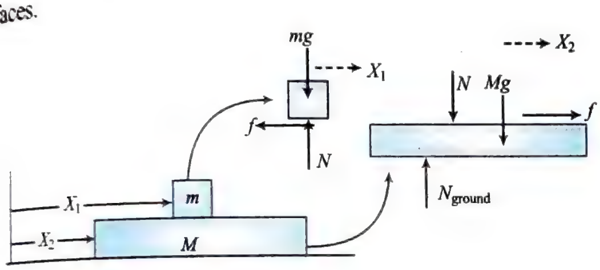
or $W_{f_k} = \vec{f_k}.\vec{s} = f_k s\cos 180° \Rightarrow W_{f_k} = -f_k s$

As the surface is assumed to be fixed ($s = 0$), the work done by the kinetic friction f'_k on the surface is zero. In this way, the total work done by kinetic friction on both the surfaces is negative.

Object is sliding over a moving surface: Now consider that the surface is moving and we analyze the work done by kinetic friction. Let us discuss this through the following situation:



A block of mass m is projected with a velocity v_0 on a plank of mass M such that the block slides through a distance x relative to the plank. If the coefficient of kinetic between block and plank is μ, assuming smooth horizontal surface, let us calculate the total work done by the kinetic friction between the contacting surfaces.



Frictional forces on m and M are $f\leftarrow$ and $f\rightarrow$, respectively. Let the block and plank move through distances x_1 and x_2, respectively.

Work done by friction force on m,

$$(W_f)_m = \vec{f}\cdot\vec{x}_1 = fx_1\cos 180° = -fx_1$$

The work done by friction f is $\vec{f}$ opposes $\vec{x}_1$

Work done by friction force on M,

$(W_f)_M = fx_2$; W because $\vec{f}$ and $\vec{x}_2$ are unidirectional.

Then the total work done by the two frictional forces is

$$W = (W_f)_m + (W_f)_M = -fx_1 + fx_2$$

$$W = (W_f)_m + (W_f)_M = -fx_1 + fx_2 = -f(x_1 - x_2)$$

We have $W = -\mu mg(x_1 - x_2)$ ...(i)

Since the block moves through a distance x relative to the plank. Hence, we can write

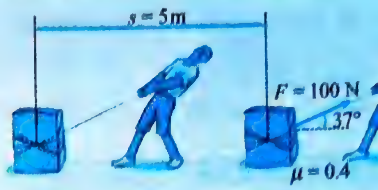
$x_1 - x_2 = x$ ...(ii)

Using Eqs. (i) and (ii), we have $W = -\mu mgx$

From the above discussion, we conclude that when two surfaces slide relative to each other, kinetic friction does a positive work on one surface and more negative work on the other surface. Hence, the total work done by the kinetic friction on the contracting surfaces is negative. If there is no relative sliding, the friction changes from kinetic to static. Hence, the overall work done by static friction is zero. That means static friction performs a positive work on one surface and equal negative work on the other surface, summing up we find zero as a whole. However, the total work done by friction does not depend upon the choice of reference frame.

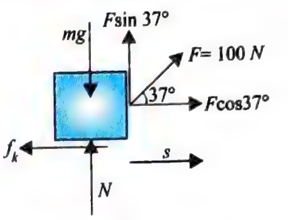
ILLUSTRATION 8.17

A 10-kg block is placed on a rough horizontal floor. The block is being pulled by a constant force 100 N acting at angle 37°over displacement of 5 m. If the coefficient of kinetic friction between the block and the floor is 0.4, find the work done by



(a) work done by the gravity **(b)** the normal reaction
(c) the applied force
(d) the force of kinetic friction

Sol. Let us draw the F.B.D. of the block



(a) Work done by gravity; $W_{gr} = mg\cdot s\cdot\cos 90° = 0\,\text{J}$

(b) Work done by normal reaction

Hence $W_N = N\cdot s\cdot\cos 90° = 0\,\text{J}$

(c) Work done by applied force

$$W_{app} = F\cdot s\cdot\cos 37° = 100\times 5\times\frac{4}{5} = 400\,\text{J}$$

(d) Work done by kinetic friction,

$$W_{fk} = f_k\cdot s\cdot\cos 180° = (\mu N)\cdot s\cdot(-1)$$

$$N = mg - F\sin 37° = 10\times 10 - 100\times\frac{3}{5} = 40\,\text{J}$$

$$\Rightarrow W_{fk} = -\mu N\cdot s = -0.4\times 40\times 5 = -80\,\text{J}$$

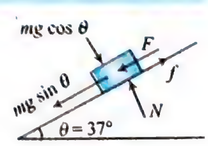
ILLUSTRATION 8.18

A block of mass 2.0 kg is pushed down an inclined plane of inclination 37° with a force of $F = 20$ N acting parallel to the incline. It is found that the block moves down the incline with an acceleration of 10 ms^{-2}. If the block started from rest, find the work done

(a) by the applied force in the first second
(b) by the weight of the block in the first second
(c) by the frictional force acting on the block in the first second

Sol. Displacement of block in 1 s

$$s = \frac{1}{2}\times 10\times 1^2 = 5\,\text{m}$$



The work done by the applied force in the first second,

$$W_F = \vec{F}\cdot\vec{s} = F\cdot s\cos\theta = 20\times 5 = 100\,\text{J}$$

The work done by the weight of the block in the first second = Component of weight in the direction of displacement × Displacement

$$W_{weight} = (mg\sin\theta)\times d = \left(2\times 10\times\frac{3}{5}\right)\times 5 = 60\,\text{J}$$

Now we need to calculate friction force acting on block. For this, we need to write

Equation of motion of block, $mg\sin\theta + F - f = ma$

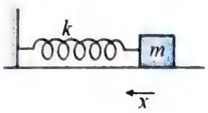
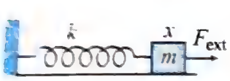
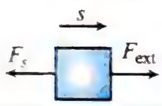
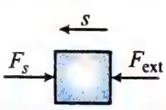
$$2\times10\times\frac{3}{5}+20-f=2\times10 \Rightarrow f=12\text{ N}$$

The work done by the frictional force acting on the block in the first second.

$$W_{\text{friction}} = \vec{f}\cdot\vec{d} = f\cdot d\cos180^\circ = -f\times d = -60\text{ J}$$

Work Done By Spring Force

Whenever a spring is stretched or compressed, the spring force always tends to restore it to the relaxed position.

A spring stretched from its equilibrium position.	A spring is compressed from its equilibrium position.
	
	
F_s and s are antiparallel. F_{ext} and s are parallel.	F_s and s are antiparallel. F_{ext} and s are parallel.
	

If x be the displacement of the free end of the spring from its equilibrium position, then the magnitude of the spring force is

$$F_s = -kx$$

The negative sign indicates that the force is restoring.

The work done by the spring force for a displacement from x_i to x_f is given by

$$W_s = \int \vec{F}_s d\vec{x} = -\int_{x_i}^{x_f} kx\,dx = -\frac{1}{2}k(x_f^2 - x_i^2)$$

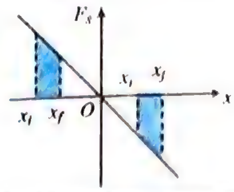
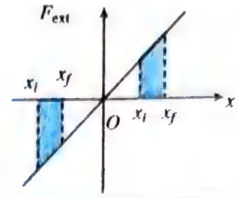
where x_i and x_f are the initial and final deformations of the spring. Hence, when a spring is deformed from $x = x_i$ to $x = x_f$,

- If $x_f > x_i$, W_{sp} is negative.
- If $x_f = x_i$, $W_{sp} = 0$. If $x_f < x_i$, W_{sp} is positive.
- That means a spring can perform positive, negative and zero work depending upon the initial and final deformations.
- When the spring is undeformed, we calculate the displacement of the free end P of the spring from the relaxed position of the spring. Substituting $x = 0$ and $x = x$, we have

$$W_{sp} = -\frac{1}{2}kx^2$$

The above expression tells us that a spring always performs negative work when deformed (compressed or elongated) from its relaxed (undeformed) position.

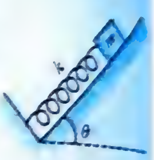
- The graph plotted between a spring force and the displacement from the equilibrium position is a straight line with negative slope.

Work done by spring force 	The work done by spring force is negative both in compression and extension.
Work done by external force 	The work done by external force is positive both in compressing or stretching the spring.

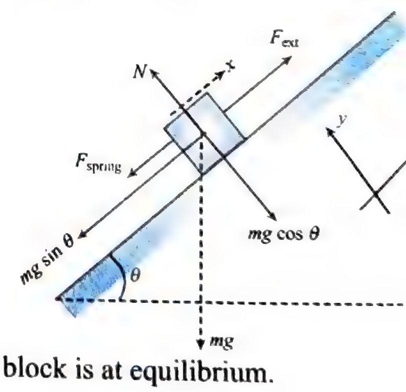
Note:
- Like gravity, the work done by spring force only depends on the initial and final positions.
- Also, the net work done by the spring force is zero for any path that returns to the initial position.

ILLUSTRATION 8.19

A block of mass m welded with a light spring of stiffness k is in equilibrium on a smooth inclined plane with angle of inclination θ. If a variable external force is applied slowly till the spring comes to its relaxed position, find the work done by spring force.



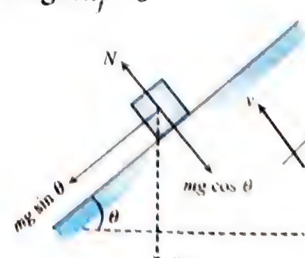
Sol. Referring to the FBD as shown in figure, we have four forces (N, mg, F_{spring} and F_{ext}) acting on the particle.



Initially, the block is at equilibrium.

$$kx_i = mg\sin\theta \Rightarrow x_i = \frac{mg\sin\theta}{k}$$

When the block is pulled up by an external force to bring the spring to relaxed length $x_f = 0$



Hence, work done by spring force,

$$W_{sp} = -\frac{1}{2}k(x_f^2 - x_i^2)$$

$$= -\frac{1}{2}\left[0-\left(\frac{mg\sin\theta}{k}\right)^2\right] = \frac{1}{2}\frac{(mg\sin\theta)^2}{2k}$$

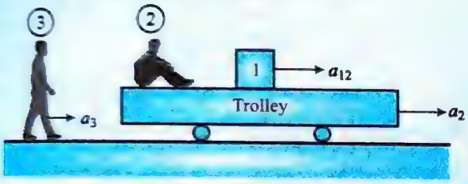
Work Done by a Pseudo (Inertial) Force

When we observe an object of mass m from an accelerated reference frame, a trolley car, we impose a pseudo force $F_{pseudo} = ma$, where a is the acceleration of the reference frame in the direction opposite to the direction of acceleration of observer (reference frame). We treat it similar to a real force to find its work.

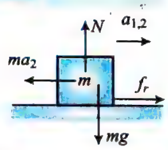
Work done by pseudo force is always calculated relative to the observer. Sometimes the observer is fixed with the ground and sometimes it is fixed with some other reference frame. If the particle does not move relative to the observer, in that case, pseudo force is observed, but $W_{pseudo} = 0$. Hence, while calculating W_{pseudo}, remember that F_{pseudo} changes from observer to observer unlike the real forces.

Illustration 8.20

A block of mass m_1 moves with an acceleration a_{12} relative to a trolley as shown in figure. The block is being observed by two observers (2) and (3). The observer (2) is at rest with respect to trolley which is moving with acceleration a_2 while the observer (3) is moving on ground with acceleration a_3. What is the work done by the pseudo force as observed by the observers (2) and (3) on the block during time t? Assume zero initial velocities of the bodies and observers.



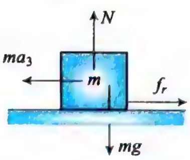
Sol. The observer (2) will observer a pseudo force of magnitude ma_2 in the direction opposite to the acceleration of observer (2) as shown in the figure.



The displacement of the block with respect to trolley (observer 2),

$$d_{12} = \frac{1}{2}a_{12}t^2$$

Hence, work done by pseudo force on block as shown by observer (2).



$$W_2 = -(ma_2)\cdot(d_{12})$$

$$= -ma_2\frac{1}{2}a_{12}t^2 = -\frac{1}{2}ma_2\cdot a_{12}t^2$$

The observer (3) will observe a pseudo force ma_3 (←).

For calculating the work done by pseudo force as seen from observer (3) (W_3), we need to calculate displacement of the block w.r.t. observer (3) (i.e., s_{13}).

$$\vec{s}_{13} = \frac{1}{2}\vec{a}_{13}t^2$$

$$a_{13} = \vec{a}_1 - \vec{a}_3 = (\vec{a}_{12} + \vec{a}_2) - \vec{a}_3 = (a_{12} + a_2 - a_3)$$

$$\Rightarrow s_{13} = \frac{1}{2}(a_{12} + a_2 - a_3)t^2$$

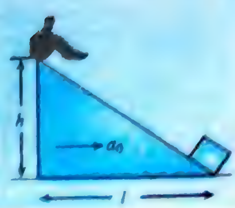
Hence, $W_3 = -(ma_3)s_{13} = -(ma_3)\left[\frac{1}{2}(a_{12} + a_2 - a_3)t^2\right]$

Using the above equations, we have

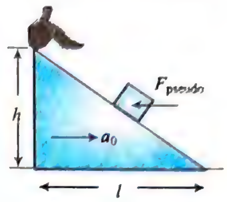
or $$W_3 = -\frac{ma_3}{2}(a_{12} + a_2 - a_3)t^2$$

Illustration 8.21

A smooth block of mass m moves up from bottom to top of a wedge which is moving with an acceleration a_0. Find the work done by the pseudo force measured by the person sitting at the edge of the wedge.



Sol. The observer moving with wedge will observe a pseudo force ma_0 in the direction opposite to $\vec{a}_0$.

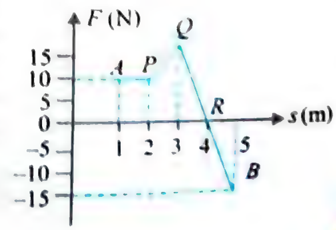


Since $\vec{F}_{ps} = -\vec{m}a_0$ is a constant force, therefore,

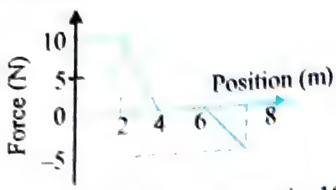
$$W_{ps} = \vec{F}_{ps}\cdot\vec{S} = -(ma_0)l\cdot\cos 180° = ma_0 l$$

Concept Application Exercise 8.1

1. A body constrained to move along the z-axis of a coordinate system is subjected to a constant force given by $\vec{F} = -\hat{i} + 2\hat{j} + 3\hat{k}$ N, where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the x-, y-, and z-axes of the system, respectively. What is the work done by this force in moving the body through a distance of 4 m along the z-axis?
2. An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})$ m to $\vec{r}_2 = (4\hat{i} + 6\hat{j})$ m under a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})$ N. Find the work done by the force.
3. A force $\vec{F} = (3xN)\hat{i} + (4N)\hat{j}$, with x in meter, acts on a particle, changing only the kinetic energy of the particle. How much work is done on the particle as it moves from coordinates (2 m, 3 m, 5 m) to (3 m, 0 m, 6 m)? Does the speed of the particle increase, decrease, or remain the same?
4. An object is displaced from a point A (0, 0, 0) to B (1 m, 1 m, 1 m) under a force $\vec{F} = (y\hat{i} + x\hat{j})$ N. Find the work done by this force in this process.
5. An object is displaced from point A (2 m, 3 m, 4 m) to a point B (1 m, 2 m, 3 m) under a constant force $\vec{F} = (2\hat{i} + 3\hat{j} + 4\hat{k})$. Find the work done by this force in this process.
6. A body moves from point A to B under the action of a force, varying in magnitude as shown in figure. Obtain the work done. Force is expressed in newton and displacement in meter.



7. A 10-kg block moves in a straight line under the action of a force that varies with position as shown in figure. How much work does the force do as the block moves from origin to $x = 8$ m?



8. A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposite to the motion.
   (a) How much work does the road do on the cycle?
   (b) How much work does the cycle do on the road?

9. A body is thrown on a rough surface such that the friction force acting on it is linearly varying with the distance travelled by it as $f = ax + b$. Find the work done by the friction on the box if before coming to rest the box travels a distance s.

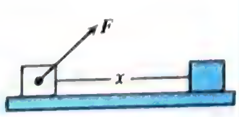
10. One end of a fixed spring is pulled by an average force of 10 N through a distance of 5 cm, find the work done by the spring.

11. The work done by an external agent in pulling a spring from a deformation of 10 cm to 15 cm is W_1. When pulled, the spring from a deformation of 15 cm to 20 cm, the work done is W_2. Find W_1/W_2.

12. A block of mass m is kept on a rough plank which moves with a horizontal acceleration a. If the plank was at rest at $t = 0$, and the block does not slide relative to the plank, find the work done by friction on the (a) block, (b) plane, (c) system, (block + plank) during time t.

13. In the previous question, if $a > \mu_s g$, find the work done by friction on the block during time t.

14. A block of mass m is pulled slowly by a minimum constant force (F) on a horizontal surface through a distance x. The coefficient of kinetic friction is μ. Find the work done by the force (F).



ANSWERS

1. 12 J 2. 83 J 3. – 4.5 J 4. 1 J 5. – 9 J

6. 22.5 J 7. 25 J 8. (a) –2000 J (b) zero

9. $-\frac{1}{2}as^2 - bs$ 10. – 0.5 11. $\frac{7}{5}$

12. (a) $\frac{1}{2}ma^2t^2$ (b) $-\frac{1}{2}ma^2t^2$ (c) 0

13. $\frac{m}{2}\mu_k^2 g^2t^2$ 14. $\frac{\mu mgx}{1+\mu^2}$

KINETIC ENERGY

The energy possessed by a body by virtue of its motion is called kinetic energy.

A particle of mass m moving in a reference frame (1) with velocity $\vec{v}_1$ is said to have kinetic energy given by

$$K_1 = \frac{1}{2}mv_1^2$$

Let another reference frame (2) moves with velocity $\vec{v}_2$. The velocity of the particle as seen from reference frame (2) is

$$\vec{v}_{1,2} = \vec{v}_1 - \vec{v}_2$$

Kinetic energy of the particle as seen from reference frame (2),

$$K_2 = \frac{1}{2}m(|\vec{v}_1 - \vec{v}_2|)^2$$

Note that $K_2 = \frac{1}{2}m(|\vec{v}_1 - \vec{v}_2|)^2$ is not necessarily same as $\frac{1}{2}m(v_1 - v_2)^2$. Instead, $|\vec{v}_1 - \vec{v}_2| = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos\theta}$, where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$.

Note:

- In vector form, $\text{KE} = \frac{1}{2}m(\vec{v}.\vec{v})$

  As m and $\vec{v}\cdot\vec{v}$ are always positive, kinetic energy is always positive scalar, i.e., kinetic energy can never be negative.

- **Relation of kinetic energy with linear momentum:** As we know,

  $$E = \frac{1}{2}mv^2 = \frac{1}{2}\left[\frac{P}{v}\right]v^2 \qquad [\text{As } P = mv]$$

  $$\therefore\ E = \frac{1}{2}Pv = \frac{P^2}{2m} \qquad \left[\text{As } v = \frac{P}{m}\right]$$

  So we can say that kinetic energy,

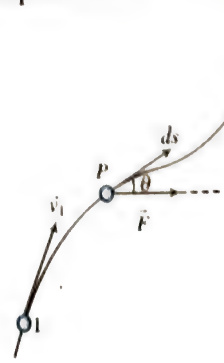
  $$E = \frac{1}{2}mv^2 = \frac{1}{2}Pv = \frac{P^2}{2m}$$

  and momentum, $P = \frac{2E}{v} = \sqrt{2mE}$

  From the above relation, it is clear that a body cannot have kinetic energy without having momentum and vice-versa

WORK–ENERGY THEOREM

One force on one particle: For the sake of simplicity, let us assume a single force F acting on a particle P of mass m. If the particle moves in any arbitrary path, the work done by the force in an elementary displacement $d\vec{s}$ of the particle is



$$dW = \vec{F}.d\vec{s}$$

Substituting $\vec{F} = m\vec{a}$, we have

$$dW = m\vec{a}.d\vec{s}$$

Then the total work done is $W = \int dW = m\int \vec{a}.d\vec{s}$

$$W = m\int_{v_1}^{v_2} v\,dv$$

This gives $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

where $\frac{1}{2}mv_1^2 = KE_1$ and $\frac{1}{2}mv_2^2 = KE_2$

Then, we have $W = KE_2 - KE_1 = \Delta KE$

When only a single force acts on a particle, *work done by the force is equal to the change in the kinetic energy of the particle.*

Many forces on one particle: If more than one forces act on a particle, we know that the sum of work done by all forces is equal to the work done by the resultant force acting on the particle. Since the resultant force $\vec{F}_{net}$ (say) decides the acceleration $\vec{a}$ of the particle, the total work done can be given as:

$$W_{total} = \Sigma W_{F_i} = W_F,\ \text{where}\ W_F = \vec{F}_{net}.d\vec{s}$$

Then, $W_{total} = \int \vec{F}_{net}.d\vec{s}$

Substituting $\vec{F}_{net} = m\vec{a}$ and $\vec{a}.d\vec{s} = v\,dv$, we have

$$W_{total} = m\int_{v_1}^{v_2} v\,dv$$

which gives the same expression.

$$W_{total} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \Delta KE$$

$$\sum W_{F_i} = \Delta K$$

The sum of work done by all forces acting on each particle of a system of particles is equal to the sum of change in the KE of each particle of the system. This is what we call work–energy theorem which signifies the "work" as "energy transfer."

Note:

For a system of particles (or system of rigid bodies),

$$\sum W_{ext} + \sum W_{int} = \Delta K$$

where ΣW_{ext} is the work done by the external forces on the system.

and ΣW_{int} is the work done by the internal forces on the system.

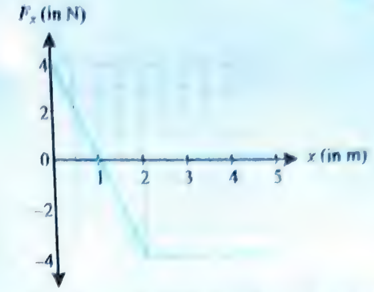
Newton's third law of motion gives you that $\Sigma\vec{F}_{int} = 0$. But the work done by the internal force may or may not be equal to zero. Now it is more useful to write the work–energy theorem in non-inertial frame as

$$W_{external} + W_{internal} + W_{pseudo} + W_{other} = \Delta KE$$

- Positive work increases the kinetic energy and negative work decreases the kinetic energy.
- Work can be converted into kinetic energy and kinetic energy can be converted into work.

ILLUSTRATION 8.22

The only force acting on a 2.0-kg body as it moves along the x-axis varies as shown in figure. The velocity of the body at $x = 0$ is 4.0 m s^{-1}.



(a) What is the kinetic energy of the body at $x = 3.0$ m?

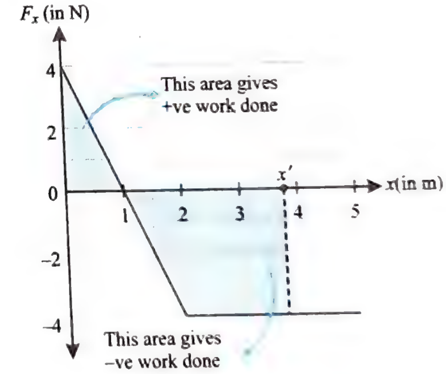
(b) At what value of x will the body have a kinetic energy of 8.0 J?

(c) What is the maximum kinetic energy attained by the body between $x = 0$ and $x = 5.0$ m?

Sol. Kinetic energy of the body at $x = 0$ is

$$K_1 = \frac{1}{2}\times 2.0\times(4.0)^2 = 16.0\ \text{J}$$

Work done by the force on the body is given by the area bounded by the curve and x-axis in figure. From $x = 0$ to $x = 1.0$ m, the force decreases linearly from $F_x = 4$ N to 0, but it is directed along positive x-axis. Work done by the force is positive. For 1.0 m $< x \le 5.0$ m, the force is negative, so it is directed along the negative x-axis for any interval in this region. the work done by the force will be negative.



(a) From $x = 0$ to 3.0 m, work done by the force on the body = Area of F_x–x diagram between $x = 0$ to $x = 3.0$ m region in figure, i.e.,

$$\frac{1}{2}\times 4\times 1.0 - \frac{1}{2}\times 4\times 1.0 - 4\times 1.0 = -4.0\ \text{Nm} = -4.0\ \text{J}$$

Let the kinetic energy of the body at $x = 3.0$ m be K_3. From the work–energy theorem, $W = K$.

$$-4.0 = K_3 - 16.0\ \text{J} \Rightarrow K_3 = 12.0\ \text{J}$$

(b) Let the kinetic energy of the body be 8.0 J at $x = x'$. From $x = 0$ to x', work done by the force = Area of the graph between $x = 0$ to x' shown in the figure, i.e.,

$$\frac{1}{2}\times 4\times 1.0 - \frac{1}{2}\times 4\times 1.0 - 4\times(x' - 2.0)\ \text{Nm} = -4\times(x' - 2.0)\ \text{J}$$

Using the work–energy theorem, we get

$$-4\times(x' - 2.0) = 8.0 - 16.0\ (\Sigma W = K_2 - K_1)$$

$$x' = 4.0\ \text{m}$$

Incidentally, in the diagram, x' has been drawn between 3.0 m and 4.0 m. You must not infer from this that x' lies between 3.0 m and 4.0 m.

(c) From $x = 0$ to 5.0 m, when the work done by the force is positive, kinetic energy of the block increases. Subsequently, when the work done by the force is negative, kinetic energy of the block decreases. Then, clearly, the kinetic energy of the block is maximum at $x = 1.0$ m. Using the work–energy theorem again, we get

$$\frac{1}{2}\times 4\times 1.0 = K_{max} - K_1$$

$$K_{max} = K_1 + 2.0 = 16.0 + 2.0 = 18.0 \text{ J}$$

ILLUSTRATION 8.23

A bullet leaving the muzzle of a rifle barrel with a velocity v penetrates a plank and loses one-fifth of its velocity. It then strikes second plank, which it just penetrates through. Find the ratio of the thickness of the planks, supposing the average resistance to the penetration is same in both the cases.

Sol. Let R be the resistance force offered by the planks, t_1 be the thickness of first plank, and t_2 be the thickness of second plank.

For first plank:

Loss in KE = work against resistance

$$\frac{1}{2}mv^2 - \frac{1}{2}m\left(\frac{4}{5}v\right)^2 = Rt \Rightarrow \frac{1}{2}mv^2\left(\frac{9}{25}\right) = Rt_1 \quad \text{...(i)}$$

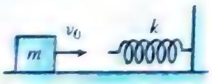
For second plank:

$$\frac{1}{2}m\left(\frac{4}{5}v\right)^2 - 0 = Rt_2 \Rightarrow \frac{1}{2}mv^2\left(\frac{16}{25}\right) = Rt_2 \quad \text{...(ii)}$$

Dividing Eq. (i) by Eq. (ii), we get $\dfrac{t_1}{t_2} = \dfrac{9}{16}$

ILLUSTRATION 8.24

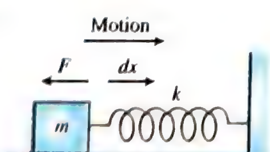
A block of mass m is moving with an initial velocity v_0 towards a stationary spring of stiffness k attached to the wall as shown in figure.



(a) Find the maximum compression in the spring.

(b) Is the work done by the spring negative or positive?

Sol. The force exerted by the spring is opposite to the displacement of the block. Therefore, the work done by the spring force is negative.



Let x_0 be the maximum compression in the spring, then work done by spring force on the block

$$W = -\int_0^{x_0} kx\,dx = -\frac{1}{2}kx_0^2$$

If we consider a block in the system. The external force acting on the block is spring force. Using work–energy theorem,

$$W = \Delta K = K_f - K_i$$

$$\Rightarrow \quad -\frac{1}{2}kx_0^2 = 0 - \frac{1}{2}mv_0^2 \quad \Rightarrow \quad x_0 = v_0\sqrt{\frac{m}{k}}$$

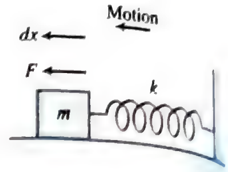
ILLUSTRATION 8.25

In the previous illustration, consider the situation when the string is completely compressed. Then it begins to relax and will come to its original length.

(a) What is the work done by the spring during the period?

(b) Is the work done by the spring positive or negative?

Sol. Work done by the spring force.



$$W_{spring} = -\frac{1}{2}k(x_f^2 - x_i^2)$$

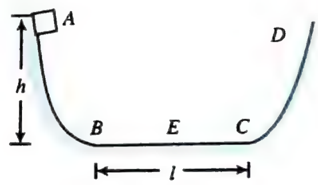
$$= -\frac{1}{2}k(0 - x_0^2) = \frac{1}{2}kx_0^2$$

From previous problem, $x_0 = v_0\cdot\sqrt{\dfrac{m}{k}}$

Hence, $W_{spring} = \dfrac{1}{2}k\left(v_0\sqrt{\dfrac{m}{k}}\right)^2 = \dfrac{1}{2}mv_0^2$

ILLUSTRATION 8.26

A particle slides along a track with elevated ends and a flat central part as shown in figure. The flat part has a length $l = 3$ m. The curved portions of the track are frictionless. For the flat part, the coefficient of kinetic friction is $\mu_K = 0.2$. The particle is released at point A which is at height $h = 1.5$ m above the flat part of the track. Where does the particle finally come to rest?



Sol. The particle will finally come to rest on the flat part. Hence, displacement of the particle along vertical is h. If W_g be the work done on the particle by the gravity, then

$$W_g = mgh \quad \text{...(i)}$$

where m is the mass of the particle.

If distance travelled by the particle on the flat part is x, the work done on the particle by the friction is

$$W_f = -\mu mgx \quad \text{...(ii)}$$

Since, initially, particle was at rest and finally it comes to rest again. Hence, change in its KE is zero.

From work–energy theorem, $W_g + W_f = \Delta\text{KE}$

$$\Rightarrow \quad mgh - \mu mgx = 0$$

$$\Rightarrow \quad x = \frac{h}{\mu} = \frac{1.5}{0.2}\text{ m} \Rightarrow x = 7.5 \text{ m}$$

Since $x > l$, the particle will reach C and then will rise up till the remaining KE at C is converted into potential energy. It will then again descend to C and will have the same kinetic energy as it had when ascending but now will move from C to B. At B, same thing will be repeated (because $7.5 > 2l$), and finally, the particle will stop at E such that

$$BC + CB + BE = 7.5$$

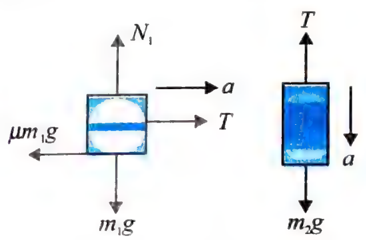
$$BE = 7.5 - 6 = 1.5 \text{ m}$$

ILLUSTRATION 8.27

Two blocks are connected by a string, as shown in figure. They are released from rest. Show that after they have moved a distance L, their common speed is given by $v = \sqrt{2(m_2 - \mu m_1)gL/(m_1 + m_2)}$, in which μ is the coefficient of kinetic friction between the upper block and the surface. Assume that the pulley is massless and frictionless.

Sol. First we will solve the problem by the force method and then by the work–energy method. This will give you an opportunity to compare these two methods and sharpen your skills in selecting the method to be used for solving a problem.

Force method: Let the acceleration of m_1 be a horizontally towards right. The acceleration of m_2 will be a vertically downwards (pulley constraint). Application of $\sum \vec{F} = m\vec{a}$ to m_1 and m_2 (see the free body diagrams for m_1 and m_2 in figure) shows



$$T - \mu m_1 g = m_1 a \text{ and } m_2 g - T = m_2 a$$

Solving these equations for a, we get $a = \dfrac{(m_2 - \mu m_1)g}{m_1 + m_2}$

Using $s = ut + (1/2)at^2$, we find that the blocks take time,

$$t = \sqrt{\frac{2L(m_1 + m_2)}{(m_2 - \mu m_1)g}}$$

to cover the distance L; and from $v = u + at$, they will have the speed

$$v = 0 + \frac{(m_2 - \mu m_1)g}{m_1 + m_2} \times \sqrt{\frac{2L(m_1 + m_2)}{(m_2 - \mu m_1)g}}$$

$$= \sqrt{\frac{2Lg(m_2 - \mu m_1)}{m_2 + m_1}}$$

Work–energy method: The essence of the process in terms of energy considerations is that m_2 loses potential energy of amount m_2gL while coming down by L, and the loss of potential energy of m_2 appears partly in m_1 and m_2 in the form of kinetic energy and is partly used up as work done against friction.

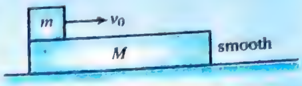
$$m_2 gL = \frac{1}{2}(m_1 + m_2)v^2 + \mu m_1 gL$$

$$\Rightarrow \quad (m_2 - \mu m_1)gL = \frac{1}{2}(m_1 + m_2)v^2$$

$$\Rightarrow \quad v = \sqrt{\frac{2(m_2 - \mu m_1)gL}{m_1 + m_2}}$$

ILLUSTRATION 8.28

A plank of mass M and length L is placed at rest on a smooth horizontal surface. A small block of mass m is projected with a velocity v_0 from the left end of it as shown in the figure. The coefficient of friction between the block and the plank is μ, and its value is such that the block becomes stationary with respect to the plank before it reaches the other end.



(a) Find the time and common velocity when relative sliding between the block and the plank stops.

(b) Find the work done by the friction force on the block during the period it slides on the plank. Is the work positive or negative?

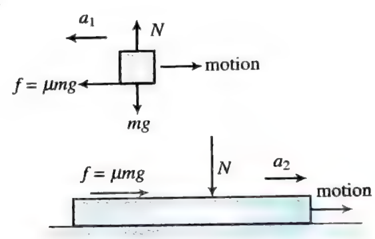
(c) Calculate the work done on the plank during the same period. Is the work positive or negative?

(d) Also, determine the net work done by friction. Is it positive or negative?

Sol.

(a) Initially, the block will slide on plank. The friction will be of kinetic nature. The friction will decrease the speed of the block and starts motion of the plank.

The free body diagrams of the block and the plank are shown in figure.



Equation of motion

For block: $a_1 = \dfrac{f}{m} = \mu g$

Instantaneous velocity,

$$v_1 = v_0 - \mu g t$$

For plank: $a_2 = \dfrac{f}{M} = \dfrac{\mu mg}{M}$

Instantaneous velocity, $v_2 = \dfrac{\mu mgt}{M}$

Finally, both the block and the plank start moving together, i.e., $v_1 = v_2$, then

$$v_0 - \mu g t = \frac{\mu mgt}{M} \text{ or } t = \frac{Mv_0}{(M+m)\mu g}$$

and the final common velocity is $V = \dfrac{mv_0}{M+m}$.

(b) If we take the block as system, friction is the only force which do work.

The work done by friction on the block is equal to its change in kinetic energy, i.e.,

$$W_1 = K_f - K_i = \frac{1}{2}mV^2 - \frac{1}{2}mv_0^2$$

$$W_1 = \frac{1}{2}m\left(\frac{mv_0}{m+M}\right)^2 - \frac{1}{2}mv_0^2$$

$$= -\frac{1}{2}\frac{mM(M+2m)v_0^2}{(M+m)^2}$$

The work done by friction on the block is negative.

(c) If we take plank as system only friction is only force which do work on plank.

The work done by friction on the plank is given by

$$W_2 = K_f - K_i = \frac{1}{2}MV^2 - 0$$

$$W_2 = \frac{1}{2}M\left(\frac{mv_0}{m+M}\right)^2 = \frac{1}{2}\frac{m^2M}{(m+M)^2}v_0^2$$

The work done by friction on the plank is positive.

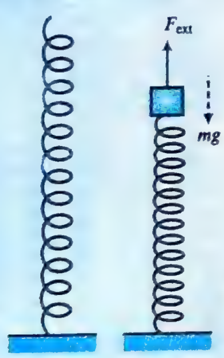
(d) The net work done by friction is

$$W = W_1 + W_2 = -\frac{1}{2}\frac{mM}{M+m}v_0^2$$

The net work done by friction is negative.

ILLUSTRATION 8.29

A block of mass m is slowly lowered from a point where it just touches a vertical fixed spring of stiffness k, till it remains stationary after the applied force is withdrawn. Find the work done by the external agent (a) in compressing the spring by a distance x and (b) bringing the block to its stable equilibrium position.



Sol.

(a) As the block is moving down, the spring pushes it up with a force $F_{sp} = kx$. The work done by the spring is

$$W_{sp} = -\frac{1}{2}kx^2$$

The work done by gravity is $W_{gr} = mgx$

The block does not change its kinetic energy as it is lowered slowly, $\Delta K = 0$

Assuming W_{ext} as the work done by the applied external force by applying work–energy theorem, we have

Work done by total forces is equal to change in kinetic energy of the system.

$$W_{total} = \Delta K$$

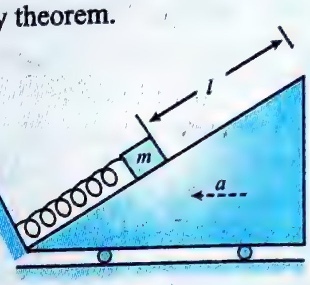
$$W_{ext} + W_{sp} + W_{gr} = \Delta K$$

Substituting $W_{sp} = -\frac{1}{2}kx^2, W_{gr} = mgx$, and $\Delta K = 0$, we have $W_{ext} - \frac{1}{2}kx^2 + mgx = 0$

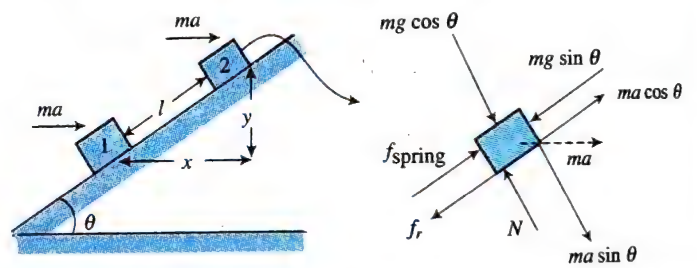
(b) When the block remains at rest under the action of gravity and spring force, as the continuously increasing spring force kx will nullify the gravity force mg, we have $kx = mg$. This gives $x = mg/k$ which corresponds to stable equilibrium position. Now substituting $x = mg/k$ in the expression, $W_{ext} = \frac{1}{2}kx^2 - mgx$, we have $W_{ext} = -\frac{m^2g^2}{2k}$.

ILLUSTRATION 8.30

A block of mass m is welded with a light spring of stiffness k. The spring is initially relaxed. When the wedge fitted moves with an acceleration a, as shown in figure, the block slides through a maximum distance l relative to the wedge. If the coefficient of kinetic friction between the block and the wedge is μ, find the maximum deformation l of the spring by using work–energy theorem.



Sol. In this case, we are given the block slides through a distance l relative to the wedge. We can apply work–energy theorem from the frame of reference of wedge.



While the block is displaced from point 1 to point 2, work done by all the forces are given as following:

Work: $W_{gr} = -mgl\sin\theta$, $W_{sp} = \frac{-Kl^2}{2}$

$$W_f = (\mu N)\cdot l$$

$$N = mg\cos\theta + ma\sin\theta$$

$$W_f = -\mu mgl\cos\theta - \mu ma\sin\theta$$

and $\quad W_{pseudo} = (ma\cos\theta)l = mal\cos\theta$

W–E theorem: Summing up all works, the total done is

$$W_{total} = W_{gr} + W_{sp} + W_f + W_{pseudo}$$

$$W = -mgl\sin\theta - \mu mgl\cos\theta$$

$$+ mal\cos\theta - \frac{1}{2}kl^2 + Fl - \mu mal\sin\theta$$

Since the block was at rest at 1 and will remain at rest relative to the wedge, we can write $\Delta K = 0$ (relative to the wedge). Using work–energy theorem relative to the wedge, we have $W = \Delta K = 0$. Substituting $W = 0$, we have

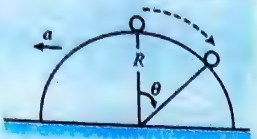
$$-mgl\sin\theta-\frac{1}{2}kl^2-\mu mg\cos\theta-\mu mal\sin\theta+mal\cos\theta=0$$

$$l=\frac{2}{k}[ma(\cos\theta-\mu\sin\theta)-mg(\sin\theta+\mu\cos\theta)]$$

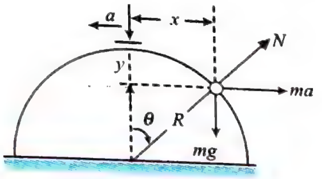
$$=\frac{2m}{k}[a(\cos\theta-\mu\sin\theta)-g(\sin\theta+\mu\cos\theta)]$$

ILLUSTRATION 8.31

A small ball is placed at the top of a smooth hemispherical wedge of radius R. If the wedge is accelerated with an acceleration a, find the velocity of the ball relative to wedge as the function of θ.



Sol. Here we analyze the motion of ball with respect to wedge. As we are analyzing the motion of the ball from accelerating wedge, we impose a pseudo force $ma \rightarrow$ on the ball. The work done by the pseudo force is



$$W_{ps}=max$$

The work done by gravity is $W_{gr}=mgy$

The work done by normal reaction force is zero as there is no relative sliding between the ball and the wedge in the direction of normal reaction.

Then, the total work done is

$$W=max+mgy \quad \text{...(i)}$$

KE: The change in kinetic energy of the ball relative to wedge is

$$K=\frac{1}{2}mv^2 \quad \text{...(ii)}$$

where v is the velocity of the ball relative to wedge.

***W–E* theorem relative to wedge:**

$$W=\Delta K \quad \text{...(iii)}$$

From Eqs (i), (ii), and (iii), we have

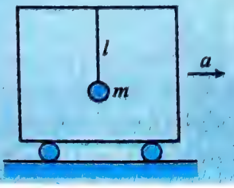
$$max+mgy=\frac{1}{2}mv^2$$

Substituting $x=R\sin\theta$ and $y=R(1-\cos\theta)$, we have

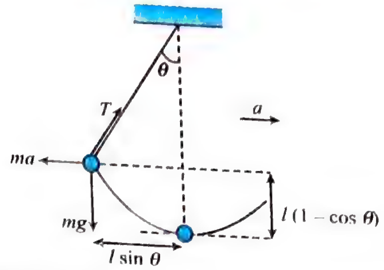
$$v=\sqrt{2\{gR(1-\cos\theta)+aR\sin\theta\}}$$

ILLUSTRATION 8.32

A pendulum of mass m and length l is suspended from the ceiling of a trolley which has a constant acceleration a in the maximum deflection θ of the pendulum from the vertical. Find θ



Sol. The free-body diagram of the pendulum with respect to trolley is shown in figure.



Free-body diagram of the pendulum bob with respect to trolley.

The forces acting on the bob are:

- the gravity, mg
- the pseudo force, ma
- the tension, T

The work done by gravity is $W_g=-mgl(1-\cos\theta)$

$$\Rightarrow \quad W_g=-mgl(1-\cos\theta)=-mgl\left(2\sin^2\frac{\theta}{2}\right)$$

The work done by pseudo force, $W_{ps}=mal\sin\theta$

The work done by tension $W_T=0$

At the position of maximum deflection, the velocity of the bob is zero, $\Delta K=0$.

Applying work-energy theorem, we get

$$W_g+W_{ps}+W_T=\Delta K$$

$$-mg\left[2\sin^2\frac{\theta}{2}\right]+ma\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right]=0$$

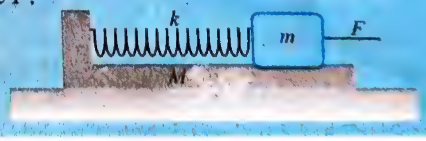
$$\tan\frac{\theta}{2}=\frac{a}{g} \text{ or } \theta=2\tan^{-1}\left(\frac{a}{g}\right)$$

Note: This angle is double to that at the equilibrium which is

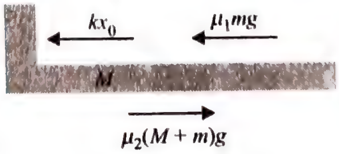
$$\theta=2\tan^{-1}\left(\frac{a}{g}\right);\ \theta=2\theta_0=2\tan^{-1}\left(\frac{a}{g}\right)$$

ILLUSTRATION 8.33

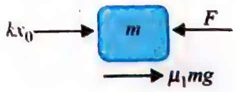
A constant force F pushes the block m till the wedge M starts sliding. If the stiffness of the light spring connecting M and m is K, coefficient of friction between block and wedge is μ_1 and between the wedge and ground is m_2, then find the value of the force F.



Sol. Let x_0 is the compression required so that wedge M just starts moving.



F.B.D. of wedge



F.B.D. of block

From F.B.D of wedge: $\mu_1 mg + kx_0 = \mu_2(M+m)g$

$$\Rightarrow \quad x_0 = \frac{\mu_2(M+m)g - \mu_1 mg}{k}$$

Now for block applying work energy theorem:

$$W_{\text{total}} = W_{\text{external}} + W_{\text{friction}} + W_{\text{spring}} = \Delta K = K_2 - K_1$$

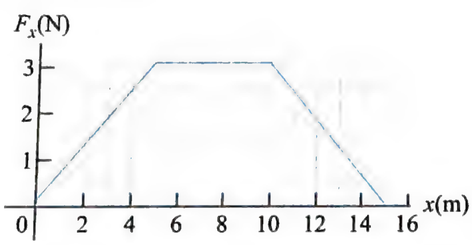
$$\Rightarrow \quad Fx_0 - \mu_1 mgx_0 - \frac{1}{2}kx_0^2 = 0$$

$$\Rightarrow \quad W_F = Fx_0 = \mu_1 mg\, x_0 + \frac{kx_0^2}{2}$$

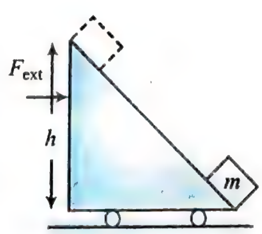
$$\Rightarrow \quad W_F = \left(\mu_1 mg + \frac{kx_0}{2}\right)x_0 = \frac{\mu_2^2(M+m)^2 g^2 - \mu_1^2 m^2 g^2}{2k}$$

CONCEPT APPLICATION EXERCISE 8.2

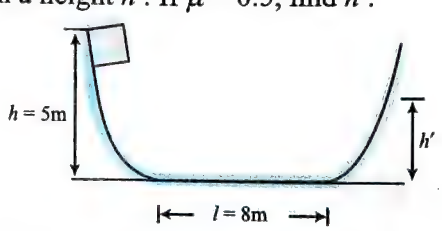
1. A 5-kg ball when falls through a height of 20 m acquires a speed of 10 m/s. Find the work done by air resistance.
2. A 4.0-kg particle is subject to a net force that varies with position as shown in figure. The particle starts from rest at $x = 0$. What is its speed at (a) $x = 5.0$ m, (b) $x = 10.0$ m, and (c) $x = 15.0$ m?



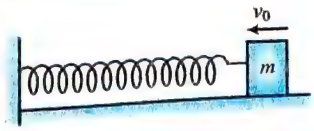
3. A block of mass m is placed at the bottom of a massless smooth wedge which if placed on a horizontal surface. When we push the wedge with a constant force, the block moves up the wedge. Find the work done by the external agent when the block has a speed v and is reaches the top of the wedge.



4. A particle is projected in gravity with a speed v_0. Using W-E theorem, find the speed of the particle as the function of vertical distance y.
5. A block is released from rest from a height $h = 5$ m. After travelling through the smooth curved surface it moves on the rough horizontal surface through a length $l = 8$ m and climbs onto the other smooth curved surface through a height h'. If $\mu = 0.5$, find h'.



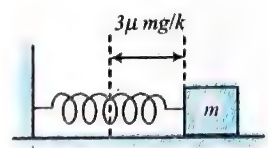
6. A block of mass m is connected with a rigid wall by light spring of stiffness k. Initially the spring is relaxed.



If the block is pushed with a velocity v_0, it oscillates back and forth and stops. Assuming μ as the coefficient of kinetic friction between block and ground, find the work done by friction till the block stops.

7. Two blocks of masses m_1 and m_2 inter connected by a spring of stiffness k are placed on a horizontal surface. If a constant horizontal force F acts on the block m_1 it slides through a distance x whereas m_2 remains stationary. If the coefficient of friction between all contacting surfaces is μ, find the speed of the block m_1 as the function x.

8. A ball of mass m is thrown in air with speed v_1 from a height h_1 and it is at a height h_2 ($>h_1$) when its speed becomes v_2. Find the work done on the ball by the air resistance.
9. A spring block system is placed on a rough horizontal surface having coefficient of friction μ. The spring is given initial elongation $3\mu mg/k$ (where m = mass of block and k = spring constant) and the block is released from rest. For the subsequent motion, find

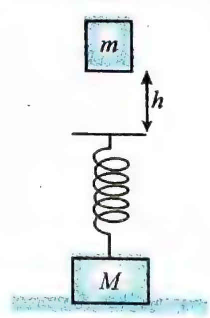


(a) Initial acceleration of block

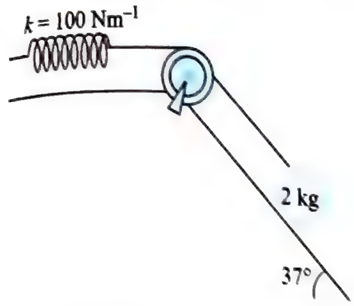
(b) Maximum compression in spring

(c) Maximum speed of the block

10. A block of mass m is dropped onto a spring of constant k from a height h. The second end of the spring is attached to a second block of mass M as shown in the figure. Find the minimum value of h so that the block M bounces off the ground. If the block of mass m sticks to the spring immediately after it comes into contact with it.



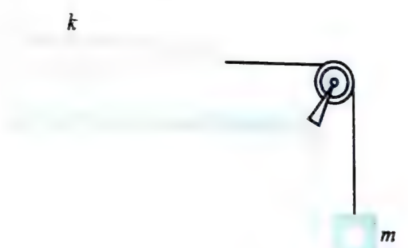
11. **(a)** A 2-kg block situated on a smooth fixed incline is connected to a spring of negligible mass, with spring constant $k = 100$ N m^{-1}, via a frictionless pulley. The block is released from rest when the spring is unstretched. How far does the block move down the incline before coming (momentarily) to rest? What is its acceleration at its lowest point?



(b) The experiment is repeated on a rough incline. If the block is observed to move 0.20 m down along the incline before it comes to instantaneous rest, calculate the coefficient of kinetic friction.

12. Find how much m will rise if $4m$ falls away. Blocks are at rest and in equilibrium.

13. In the given figure, the light spring is of force constant k and is on a smooth horizontal surface. Initially the spring is relaxed. Calculate the work done by an external agent to lower the hanging body of mass M slowly, till it remains in equilibrium.



Answers

1. –750 J

2. (a) 1.94 m/s (b) 3.35 m/s (c) 3.87 m/s

3. $\frac{1}{2}mv^2 + mgy$ 4. $\sqrt{v_0^2 - 2gy}$

5. 1 m 6. $\frac{\mu^2 m^2 g^2}{2k} - \frac{1}{2}mv_0^2$

7. $\sqrt{2\left(\frac{F}{m_1} - \mu g\right) - \frac{k}{m_1}x^2}$ 8. $mg(h_2 - h_1) + \frac{1}{2}m(v_2^2 - v_1^2)$

9. (a) $2\mu g$ (b) $\frac{\mu mg}{k}$ (c) $2\mu g\sqrt{\frac{m}{k}}$

10. $\frac{(M^2 + 2mM)g}{2km}$

11. (a) $x = 0.24$ m, $a = 6$ m/s^2 (b) $\frac{1}{8}$

12. $\frac{8mg}{k}$ 13. $-\frac{m^2g^2}{2k}$

CONSERVATIVE AND NON-CONSERVATIVE FORCES

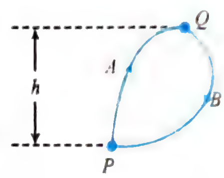
Consider an object moving downward near the surface of the earth. The work done by the gravitational force on the object does not depend on whether it falls vertically or slides down a sloping incline. All that matters is the change in the object's elevation. In other words, the path makes no difference when we consider the work done by the gravitational force, but it does make a difference when we consider energy transformation due to friction forces. We can use this varying dependence on path to classify forces as either conservative or non-conservative.

Conservative Force

Conservative forces have the following two equivalent properties:

(1) The work done by a conservative force on a particle moving between any two points is independent of the path taken by the particle.

(2) The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one for which the beginning point and the endpoint are same)



A particle is taken from point P to point Q via the path PAQ and then placed back to point P via the path QBP in the vertical plane. Let us calculate the work done by gravity on the body over this closed path. Here, displacement of the particle is $\overline{PQ}$; gravity is acting vertically downward. The vertical component of $\overline{PQ}$ is h upward. Hence,

$$W_{(PAQ)} = -mgh \quad \text{...(i)}$$

For the path QBP, component of the displacement along vertical is h (downward).

In this case, $W_{QBP} = mgh$

Therefore, total work done = $W_{PAQ} + W_{QBP} = 0$

For the case of the object–spring system, the work W_s done by the spring force is given by $W_s = \frac{1}{2}Kx_i^2 - \frac{1}{2}kx_f^2$. We see that the spring force is conservative because W_s depends only on the initial and final x coordinates of the object and is zero for any closed path.

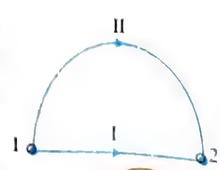
Non-Conservative Force

The work done by a non-conservative force not depends only on the initial and final positions but also on the path followed.

The common examples of such forces are frictional force and drag force in fluids.

The work done by a non-conservative force along a closed path is not equal to zero.

Let us choose two points 1 and 2 on a rough horizontal floor and drag an object slowly and horizontally between these points along paths I and II can observe that, in shifting the object along two paths. We feel



more exhausted when we drag a body through more distance. In other words, work done by friction is more for a longer path. The work done due to friction for the paths I and II can be given as $W_1 = -fl_1$ and $W_2 = -fl_2$, respectively.

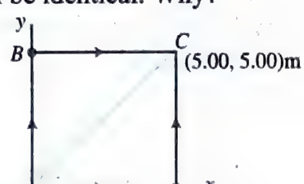
Since the work done by kinetic friction is path-dependent, for a round trip (when we drag the object from position 1 to 2 along the path I and we bring it back slowly to its initial position 1 from the path II), the total work done by friction is

$$W = W_1 + W_2 = -fl_1 + (-fl_2) = -f(l_1 + l_2) = -fl$$

where l is the length of the closed path.

ILLUSTRATION 8.34

A 4.00-kg particle moves from the origin to position C, having coordinate x = 5.00 m and y = 5.00 m. One force on the particle is the gravitational force acting in the negative y direction. Using equation $W = F\Delta r \cos\theta = \vec{F}\cdot\Delta\vec{r}$, calculate the work done by the gravitational force on the particle as it goes from O to C along (a) OAC, (b) OBC, and (c) OC. Your results should all be identical. Why?



Sol. $F_g = mg = (4.00\text{ kg})(10.0\text{ m s}^{-2}) = 40.0\text{ N}$

(a) Work done along OAC

$= $ Work along OA + Work along AC

$= F_g(OA)\cos 90.0^\circ + F_g(AC)\cos 180^\circ$

$= (40.0\text{ N})(5.00\text{ m})\cos 90^\circ + (40.0\text{ N})(5.00\text{ m})\cos 180^\circ$

$= -200$ J

(b) Work done along OBC = W along OB + W along BC

$= (40.0)(5.00\text{ m})\cos 180^\circ + (40.0\text{ N})(5.00\text{ m})\cos 90.0^\circ$

$= -200$ J

(c) Work done along OC

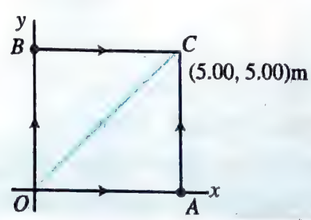
$= F_g(OC)\cos 135^\circ$

$$= (40.0\text{ N})(5.00\times\sqrt{2}\text{ m})\left(-\frac{1}{\sqrt{2}}\right) = -200\text{ J}$$

Work done is same in all cases because gravitational force is a conservative force.

ILLUSTRATION 8.35

A force acting on a particle moving in the x–y plane is given by $\vec{F} = (2y\hat{i} + x^2\hat{j})$N, where x and y are in meters. The particle moves from the origin to a final position having coordinates x = 5.00 m and y = 5.00 m as shown in figure. Calculate the work done by $\vec{F}$ on the particle as it moves along (a) OAC, (b) OBC, and (c) OC. (d) Is $\vec{F}$ conservative or non-conservative? Explain.



Sol.

(a) For each of the paths from O to C, the work done is given by

$$W = \int \vec{F}\cdot\vec{ds}$$

Here $\vec{F} = (2y\hat{i} + x^2\hat{j})$ and $\vec{ds} = (dx\hat{i} + dy\hat{j})$

Hence $W = \int (2y\hat{i} + x^2\hat{j})(dx\hat{i} + dy\hat{j})$

$$\Rightarrow \quad W = \int_0^5 2y\,dx + \int_0^5 x^2\,dy \qquad \text{...(i)}$$

The path OAC can be break up into two paths, from O to A and then from A to C.

First considering path O to A, in this path $y = 0$. Hence, from equation (i) it is clear $W_{OA} = 0$

Now consider path A to C, in this path x is constant (i.e., $dx = 0$) and is equal to 5 m. From equation (i)

$$W_{AC} = x^2\int_0^5 dy = (5)^2[y]_0^5 = 125\text{ J}$$

Hence, total work done $W_{OAC} = 0 + 125 = 125$ J

(b) Now consider the path OBC, this path also break up into two paths O to B and then B to C.

Considering path O to B, in this path $x = 0$. Hence, from equation (i) it is clear $W_{OB} = 0$.

Now considering path B to C, in this path y is constant (i.e. $dy = 0$) from equation (i)

$$W_{BC} = 2y\int_0^5 dx = 2\times 5\times [x]_0^5 = 50\text{ J}$$

Hence total work done in this path $W_{OBC} = 0 + 50 = 50$ J

(c) Motion of the particle from O to C, in this path both x and y are changing. In this path $x = y \Rightarrow dx = dy$

From equation (i);

$$W_{OC} = \int_0^5 (2x + x^2)dx = 2\left[\frac{x^2}{2}\right]_0^5 + \left[\frac{x^3}{3}\right]_0^5$$

$$= 25 + \frac{1}{3}\times 125 = \frac{200}{3}\text{ J}$$

(d) We have calculated the work done from O to C in three different paths which is not same. Hence, the force F is a non-conservative force.

POTENTIAL ENERGY

Potential energy is associated with the configuration of one or more bodies. Every configuration of a system of particle is characterized by an internal potential energy U and the work done by all the internal forces while the configuration changes is equal to the change (decrease) in the potential energy of the system.

The potential energy of a system of particle is the work the systems of bodies can do by virtue of the relative position of its parts, that is, by virtue of its configuration. Physically, the notion of potential energy is applicable only to the class of forces where work against these forces gets *stored up* as energy. as soon as the external constraints are removed, PE, manifests itself as KE.

In mechanics, two types of the potential energy are of particular importance: gravitational potential energy and elastic potential energy.

When we use the word potential energy, we have to attribute it to the "system of interacting bodies" like earth-body, spring-mass etc, but not to any object of the system. For instance, when we release a body it pulls the earth. Hence earth is also accelerated by the body. Since the mass of earth is very large compared to that of the body, acceleration, velocity and displacement of earth is negligible. It means, the change in position (configuration) of the earth-body system occurs mainly due to the displacement of the body. Following the above logic, roughly we may state "potential energy of the body," but in strict sense that should be stated as "potential energy of earth-body system,"

Similarly, in a block-spring system, as the block is rigid, its deformation is neglected. Hence, the elastic potential energy is mainly contributed by the change in configuration (deformation) of spring.

Potential energy is defined as the ability of doing work by a conservative force. It arises from the configuration of the system or position of the particles in the system (conservative force fields like spring, gravity, etc.)

Work-potential energy theorem: The conservative force always does positive work at the expense of its potential energy sored in its field. For instance, when the compression or elongation of the spring decreases, the spring force decreases. Consequently the spring becomes more inactive. It means, the stability of work done by the spring decreases. In this way we can logically interpret that:

Work done by the spring is equal to the loss of its potential energy, $W_{sp} = -\Delta U_{sp}$; negative sign is used for the loss of potential energy.

Same logic is also valid for gravitational field. When the body moves down, its kinetic energy increases and the gravity does a positive work. It means, the gravitational potential energy decreases by an amount which is equal to the gravitational work done on the falling body, $W_{gr} = -\Delta U_{gr}$.

In general, we can state that: any conservative force field performs a work at the expense of its potential energy.

$$W_{conc.} = -\Delta U$$

The loss in potential energy is equal to the total work done by the conservative forces.

Gravitational Potential Energy

If a block of mass m is lifted from earth's surface through a height $h << R$; R = radius of earth. Find the gravitational potential energy of the particle.

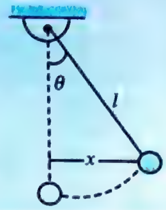
When we lift an object from earth's surface $y_1 = 0$ to a height h, we have $y_2 = h$. Then the work done by gravity is

$$W_{gr} = mg\,(y_1 - y_2) = -\,mgh$$

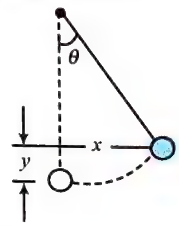
Substituting $W_{gr} = -\,mgh$ in the expression $W_{gr} = -\,\Delta U$, we have $\Delta U = (U_2 - U_1) = mgh$. If we choose arbitrarily $U_1 = 0$ at surface of earth, we have $U_2 (= U) = mgh$.

ILLUSTRATION 8.36

A pendulum initially is at rest in vertical position. The bob is pulled slowly towards right. Find the change in gravitational potential energy of the pendulum bob of mass m as the function of x.



Sol. The change in potential energy of a particle depends upon the vertical displacement of the particle with respect to initial position. It increases for upward displacement and decreases for downward displacement. Here the change in potential energy of the bob at any angular position θ is



$$\Delta U = mgl\,(1 - \cos\theta) \qquad \text{...(i)}$$

$$\cos\theta = \frac{\sqrt{l^2 - x^2}}{l} \qquad \text{...(ii)}$$

Using Eqs (i) and (ii), we get

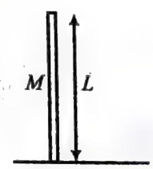
$$U = mgl\left(1 - \sqrt{1 - \frac{x^2}{l^2}}\right)$$

For writing potential energy of point mass, we take earth surface as reference point where we take potential energy zero.

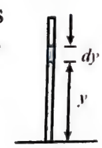
For a regular-shaped body where mass is distributed uniformly, we take as mass concentrated at its geometrical counter. This point is called *center of gravity*. We can replace the body a point mass placed at the center of gravity.

ILLUSTRATION 8.37

A uniform rod of mass M and length L is held vertically upright on a horizontal surface as shown in figure. Assuming zero potential energy at the base of the rod, determine the potential energy of the rod.



Sol. Since parts of the rod are at different levels from the horizontal surface, therefore, we have to use the integration to find its potential energy.



Consider a small element of length dy at a height y from the surface.

Mass of the element is $dm = \frac{M}{L}dy$

Potential energy of the element is

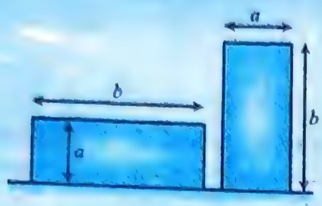
$$dU = (dm)gy = \frac{M}{L}gy\,dy$$

On integrating, we get

$$U = \frac{Mg}{L}\int_0^L y\,dy = \frac{Mg}{L}\left[\frac{y^2}{2}\right]_0^L = \frac{1}{2}MgL$$

ILLUSTRATION 8.38

A plate of mass m, length b, and breadth a is initially lying on a horizontal floor with length parallel to the floor and breadth perpendicular to the floor. Find the work done to erect it on its breadth.

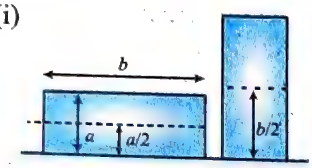


Sol. Taking earth surface as reference point. Gravitational potential energy of the block at the position 1,

$$U_1 = mgh_1 \quad \text{...(i)}$$

Gravitational potential energy of the block at the position 2 is given as

$$U_2 = mgh_2 \quad \text{...(ii)}$$



position-(I) position-(II)

Thus, work done by the external agent

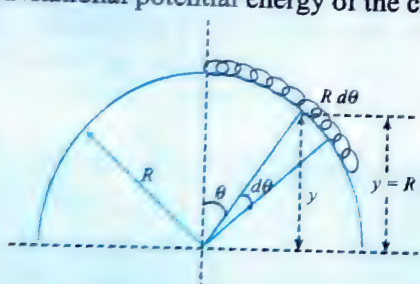
$$W = \Delta U = (U_2 - U_1) = mgh_2 - mgh_1$$

where $h_2 = b/2$ and $h_1 = a/2$ as it is a regular-shaped object.

$$\Rightarrow \quad W = mg\left(\frac{b}{2} - \frac{a}{2}\right) = \frac{mg(b-a)}{2}$$

ILLUSTRATION 8.39

A chain of length l and mass m lies of the surface of a smooth hemisphere of radius $R > l$ with one end tied to the top of the hemisphere. Taking base of the hemisphere as reference line, find the gravitational potential energy of the chain.



Sol. The mass is distributed in chain uniformly along its length. Choose a small element of chain of width $d\theta$ at an angle θ from the vertical.

The mass of the element, $dm = \left(\frac{m}{l}R\,d\theta\right)$

The gravitational potential energy of the element $dU = (dm)gy$

Thus, the gravitational potential energy of whole chain

$$U = \int (dm)gy$$

$$= \int_0^{(l/R)}\left(\frac{m}{l}R\,d\theta\right)g\,(R\cos\theta)$$

$$= \frac{mR^2g}{l}\int_0^{(l/R)}\cos\theta\,d\theta$$

$$= \frac{mgR^2}{l}\left|\sin\theta\right|_0^{l/R} = \frac{mgR^2}{l}\sin\left(\frac{l}{R}\right)$$

ELASTIC POTENTIAL ENERGY STORED IN A SPRING

As the spring deformation changes from x_1 to x_2, the work done by spring is

$$W_{sp} = -\frac{k}{2}(x_2^2 - x_1^2)$$

In this process, the spring potential energy changes from U_1 to U_2. Substituting the above value of W_{spring} in

$$W_{sp} = -\Delta U = -(U_2 - U_1)$$

we have $U_2 - U_1 = \dfrac{kx^2}{2} - \dfrac{1}{2}kx_1^2$

If $x_1 = 0$ and $x_2 = x$, we have $U_1 = 0$ and $U_2 = U = kx^2/2$. Then the potential energy U as the function of deformation x can be given as $U = kx^2/2$.

CHANGE IN POTENTIAL ENERGY

Change in potential energy between any two points is defined in the terms of the work done by the associated conservative force in displacing the particle between these two points without any change in kinetic energy.

$$U_2 - U_1 = -\int_{r_1}^{r_2}\vec{F}.d\vec{r} = -W \quad \text{...(i)}$$

We can define a unique value of potential energy only by assigning some arbitrary value to a fixed point called the reference point. Whenever and wherever possible, we take the reference point at infinity and assume potential energy to be zero there, i.e., if we take $r_1 = \infty$ and $r_2 = r$, then from equation (i),

$$U = -\int_{\infty}^{r}\vec{F}.d\vec{r} = -W$$

In case of conservative force (field) potential energy is equal to negative of work done by conservative force in shifting the body from reference position to given position.

This is why, in shifting a particle in a conservative field (say gravitational or electric), if the particle moves opposite to the field, work done by the field will be negative and so change in potential energy will be positive, i.e., potential energy will increase. When the particle moves in the direction of field, work will be positive and change in potential energy will be negative, i.e., potential energy will decrease.

ILLUSTRATION 8.40

A conservative force held function is given by $F = k/r^2$, where k is a constant.

(a) Determine the potential energy function $U(r)$ assuming zero potential energy at $r = r_0$.

(b) Also, determine the potential energy at $r = \infty$.

Sol. (a) Using the definition of potential energy function,

$$U(r)-U(r_0)=-\int_{r_0}^{r} F\,dr$$

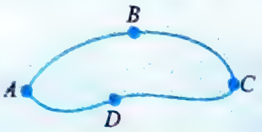
$$U(r)-U(r_0)=-k\int_{r_0}^{r}\frac{dr}{r^2}=k\left[\frac{1}{r}\right]_{r_0}^{r}=k\left[\frac{1}{r}-\frac{1}{r_0}\right]$$

Since at $r=r_0$, $U(r_0)=0$, therefore, $U(r)=\frac{k}{r}-\frac{k}{r_0}$

(b) Potential energy at $r=\infty$ is $U_\infty=-k/r_0$

ILLUSTRATION 8.41

A particle moves along the loop A-B-C-D-A while a conservative force acts on it. Work done by the force along the various sections of the path are $W_{AB}=-50$ J; $W_{BC}=25$ J; $W_{CD}=60$ J. Assume that potential energy of the particle is zero at A. Write the potential energy of particle when it is at B and D.



Sol. We know the work done by a conservative force in a closed path should be zero.

$$W_{AB}+W_{BC}+W_{CD}+W_{DA}=0$$

$$\Rightarrow\quad -50+25+60+W_{DA}=0$$

$$\Rightarrow\quad W_{DA}=-35\text{ J}$$

The change in potential energy of the particle while moving from A to B.

$$U_B-U_A=-W_{A\to B}$$

$$U_B-0=50\text{ J}\Rightarrow U_B=50\text{ J}$$

and the change in potential energy of the particle while moving from A to D, $U_D-U_A=-W_{A\to D}$

$$U_D-0=-(+35)\Rightarrow U_D=-35\text{ J}$$

Relationship between conservative forces and potential energy: If the point of application of the force undergoes an infinitesimal displacement dr, we can express the infinitesimal change in the potential energy of the system dU as

$$dU=-F_r\,dr$$

Therefore, the conservative force is related to the potential energy function through the relationship

$$F_r=-\frac{dU}{dr}$$

That is the component of a conservative force in the direction of $\vec{r}$ acting on an object within a system equals the negative derivative of the potential energy of the system with respect to r.

THREE-DIMENSIONAL FORMULA FOR POTENTIAL ENERGY

For only conservative fields, $\vec{F}$ equals the negative gradient of the potential energy.

In Cartesian coordinate system, the components of a force F in the axes x, y and z can be given as the negative space derivative (gradient) of the potential energy measured along the corresponding axes.

$$\vec{F}_x=-\frac{\partial U}{\partial x}\hat{i},\ \vec{F}_y=-\frac{\partial U}{\partial y}\hat{j},\text{ and }\vec{F}_z=-\frac{\partial U}{\partial z}\hat{k}$$

Then, the net force $\vec{F}$ can be given as

$$\vec{F}=-\left(\frac{\partial U}{\partial x}\hat{i}+\frac{\partial U}{\partial y}\hat{j}+\frac{\partial U}{\partial z}\hat{k}\right)$$

where $\partial U/\partial x$ is the partial derivative of U w.r.t. x (keeping y and z constant), $\partial U/\partial y$ is the partial derivative of U w.r.t. y (keeping x and z constant), and $\partial U/\partial z$ is the partial derivative of U w.r.t. z (keeping x and y constant).

The above expression is based on the following concept: the component of a conservative force in any direction is equal to negative of its potential energy gradient in that direction.

ILLUSTRATION 8.42

The potential energy of configuration changes in x and y directions as $U=kxy$, where k is a positive constant. Find the force acting on the particle of the system as the function of x and y.

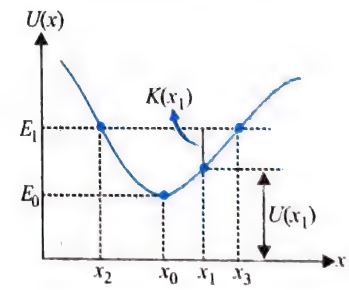
Sol. Substituting $U=kxy$ in the expression

$$F=-\frac{\partial U}{\partial x}\hat{i}-\frac{\partial U}{\partial y}\hat{j}-\frac{\partial U}{\partial z}\hat{k}$$

we have $\vec{F}=-k(y\hat{i}+x\hat{j})$

POTENTIAL ENERGY CURVE

A graph plotted between the potential energy of a particle and its displacement from the center of force is called potential energy curve. The figure below shows a graph of potential energy for one dimensional motion (say along x-axis).



You can consider this curve as a roller coaster riding without friction. We know that the negative gradient of the potential energy gives conservative force, i.e., $F_x=-dU/dx$.

In the region of the graph, where the slope is positive, the force acts in negative x-direction. And in the region in the graph where the slope is negative, the conservative force acts in positive x-direction. Here you can say the range of values of x for which the potential energy curve appears "uphill", the particle slows down i.e., the kinetic energy of the particle decreases and the region where potential energy curve appears "downhill", the magnitude of velocity increases i.e., the kinetic energy of the particle increases.

The total energy (E) of the system is constant. It can be represented by a straight line parallel to x-axis. In given graph, the

horizontal lines represents the total energies of a moving particle moving in a conservative field.

Total energy, $E = U(x) + K(x) \Rightarrow U(x) = E - K(x)$

Here $U(x)$ and $K(x)$ are the potential and kinetic energy of the particle at any position x. As $K(x)$ is always positive, hence $U(x)$ must be less than or equal to E for all situation.

Now let us analyze different points in potential energy curve.

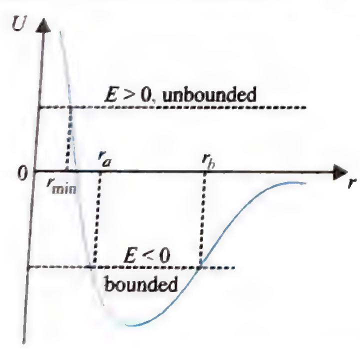
At position where the object has total energy E_0: The potential energy of the particle is minimum. At this position, the total energy of the particle is equal to its potential energy. At this position, kinetic energy of the particle will be zero.

At positions of total energy E_1: At position x_2 and x_3, the total energy of the object will be the potential energy. The kinetic energy of the particle at these position will be zero. The particle will move between the points x_2 and x_3. The points x_2 and x_3 are called turning points of the motion. If we take a point x_1 between x_2 and x_3, then we can write the expression for kinetic energy of the particle as i.e., the kinetic energy at this position is represented by the distance between the line E_1 and $U(x)$ curve.

Energy Diagram for a Typical Attractive Two-Atom System

Let us consider the interaction between two atoms. At large separations, the atoms attract each other weakly with the van der waals force, which varies as $1/r^7$. As the atoms approach, the electron clouds begin to overlap, producing strong forces. In this intermediate region the force is either attractive or repulsive depending on the details of the electron configuration.

In case of the force is attractive, the potential energy decreases with decreasing r. At very short distances the atoms always repel each other strongly, U increases rapidly as r becomes small.



For positive energy, $E > 0$, the motion is unbounded, and the atoms are free to fly apart. As the figure indicates, the distance of closest approach, r_{min}, does not change appreciably as E is increased. The steep slope of the potential energy curve at small r means that the atoms behave like hard spheres and r_{min} is not sensitive to the energy of collision.

The situation is quite different if E is negative. Then the motion is bounded for both small and large separations; the atoms never approach closer than r_a or move farther apart than r_b. A bound system of two atoms is, of course, a molecules energy diagram.

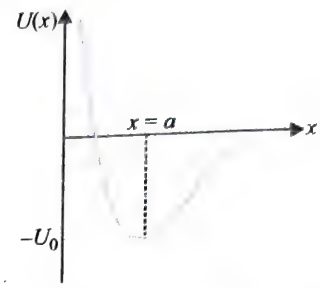
If two atoms collide with positive energy, they cannot form a molecule unless some means is available for losing enough energy to make E negative.

In general, a third body is necessary to carry off the excess energy. Sometimes the third body is a surface, which is the reason surface catalysts are used to speed certain reactions.

A third atom can also carry off the excess energy, but for this to happen the two atoms must collide when a third atom is nearby. This is a rare event at low pressures, but it becomes increasingly important at higher pressures. Another possibility is for the two atoms to lose energy by the emission of light.

ILLUSTRATION 8.43

The force between two atoms in a diatomic molecule can be represented approximately by the potential energy function $U = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right]$, where U_0 and a are constants.



(a) At what value of x is the potential energy zero?

(b) Find the force F_x.

(c) At what value of x is the potential energy a minimum?

Sol.

(a) If the potential energy is zero

$$\Rightarrow \quad 0 = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right] \text{ we get}$$

$$x = \frac{a}{2^{1/6}}$$

(b) The force is negative derivative of potential energy function.

$$F_x = -\frac{dU}{dx} = \frac{12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right] \quad \text{...(i)}$$

(c) The potential energy has its minimum value when $\frac{dU}{dx} = 0$

Equating equation (i) equal to zero, we get $x = a$. The minimum occurs at $x = a$, which is the average spacing between atoms in such a molecule.

ILLUSTRATION 8.44

A single conservative force $F(x)$ acts on a 1.0-kg particle that moves along x-axis. The potential energy $U(x)$ is given by $U(x) = 20 + (x - 2)^2$, where x is in meters. At $x = 5.0$ m, the particle has kinetic energy 16 J.

(a) What is the mechanical energy of the system.

(b) What are the least and greatest values of x between which the particle can move.

(c) The maximum kinetic energy of the particle and value of x at which it occurs.

(d) Determine the equation for $F(x)$ as a function of x.

Sol.

(a) Given potential energy function, $U(x) = 20 + (x-2)^2$ ...(i)

The given potential energy relation at position $x = 5.0$ m, the potential energy of the particle

$$U(5\text{ m}) = 20 + (5-2)^2 = 29\text{ J}$$

The mechanical energy of the particle,

$$E = U + K = 29 + 16 = 45\,\text{J}$$

(b) The least and greatest position should be the turning points of the motion of the particle. The particle will move between these two points. At these points, the kinetic energy of the particle should be zero. Hence, at these positions, the potential energy of the particle should be equal to its mechanical energy.

Hence, $U(x) = E = 45 = 20 + (x-2)^2$

$(x-2)^2 = 25$ or $(x-2) = \pm 5$

It means x_{min} –3.0 m and x_{max} = 7.0 m

(c) The kinetic energy will be maximum when potential energy is minimum. From equation (i), the potential energy of the particle will be minimum at $x = 2$ m, $U_{min} = 20$ J.

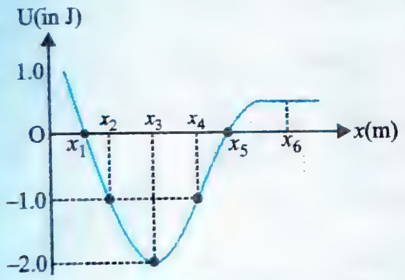
The mechanical energy of the particle is 45 J. Hence, maximum kinetic energy of the particle, $K_{max} = 45 - 20 = 25\,\text{J}$

(d) The force acting on the particle,

$$F = -\frac{dU}{dx} = -\frac{d}{dx}\left[20 + (x-2)^2\right] = -[2(x-2)] = 4 - 2x$$

ILLUSTRATION 8.45

A conservative force has the potential energy function $U(x)$ as shown in the graph. A particle moving in one dimensional under the influence of this force has kinetic energy 1.0 J when it is at position x_3.



(a) Find the mechanical energy of the particle

(b) Can the particle reach either x_1 or x_5?

(c) Find the least and greatest value of x between which the particle can move.

Sol.

(a) The mechanical energy of the particle, $E = U + K$ ...(i)

It is given that at position x_3, the kinetic energy of the particle is 1.0 J. From given graph at O position x_3, the potential energy of the particle is –2.0 J. Hence from (i)

$E = -2.0 + 1.0 = -1.0$ J

(b) At position x_1 and x_5 the potential energy of the particle is zero. The particle has mechanical energy –1.0 J. It means the kinetic energy of the particle at these positions should be –1.0 J, which cannot be possible as the kinetic energy of the particle cannot be negative. It means the particle cannot reach at the positions either x_1 or x_5.

(c) At positions x_2 and x_4, the potential energy of the particle is –1.0 J. As the mechanical energy of the particle is –1.0 J, it means the kinetic energy at these positions will be zero. It suggest that the particle oscillate between the points x_2 and x_4.

Hence $x_{min} = x_2$ and $x_{max} = x_4$.

NATURE OF FORCE

Attractive force	Repulsive force	Zero force
On increasing x, if U increases, dU/dx is positive, then F is in negative direction, i.e., force is attractive in nature. In graph, this is represented in region BC.	On increasing x, if U decreases, dU/dx is negative, then F is in positive direction, i.e. force is repulsive in nature. In graph, this is represented in region AB.	On increasing x, if U does not change, dU/dx is 0, then F is zero, i.e. no force works on the particle. Point B, C, and D represent the point of zero force or these points can be termed as position of equilibrium.

We can easily check equation $F_r = -dU/dr$ for the two examples already discussed.

In the case of the deformed spring, $U_s = \frac{1}{2}kx^2$. Therefore,

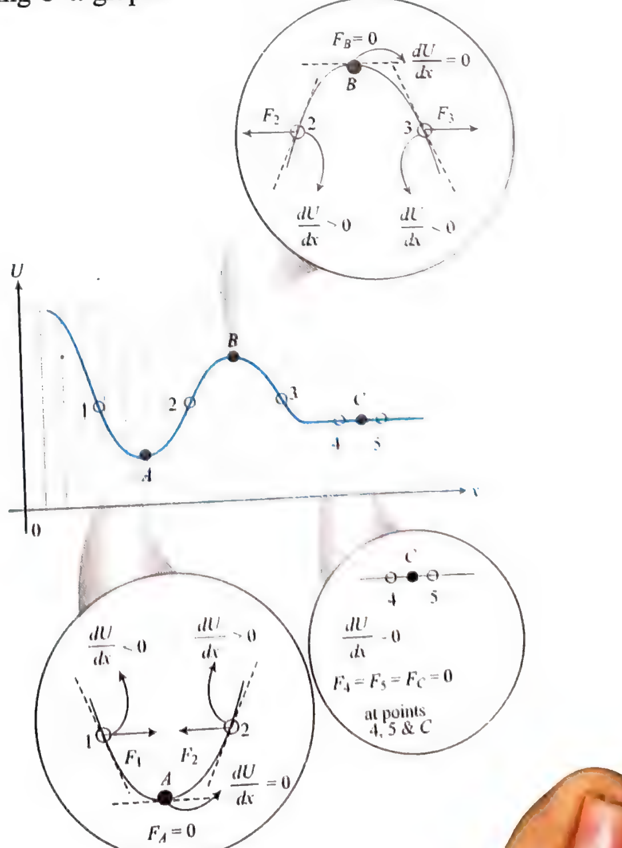
$$F_s = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$$

which corresponds to the restoring force in the spring (Hooke's law). The gravitational potential energy function is $U_g = mgy$, it follows $F_g = -\frac{dU}{dy} = \frac{-d(mgy)}{dy} = -mg$

We now see that U is an important function because a conservative force can be derived from it.

STABILITY

If the potential energy of a particle changes with a distance along an axis, x and the graph of potential energy U versus x is given, we can tell many important facts of the motion of the particles moving under the influence of the conservative forces by analyzing U–x graph.



We assume an arbitrary U–x graph as shown in the figure above. Here we should not confuse x as horizontal—it may be horizontal, vertical. It means we consider the variation of potential energy of a particle in x direction.

We can see that, U is minimum at A and maximum at B.

Let $U_{min} = U_A$ and $U_{max} = U_B$.

Hence, the slope of U–x graph, that is dU/dx, must be zero at A and B. Furthermore, $dU/dx = 0$ at C because U remains constant near C. Since $F = -dU/dx = 0$ at the points A, B, and C, the net force acting on the particle (under consideration) at these points is zero. In other words, the particle is in equilibrium at A, B, and C.

Let us now enquire more about the nature of equilibrium by just looking at the U–x graph.

Stable Equilibrium

First of all consider point A. When we take two points (1) and (2) at both sides of A, we notice that slope of U–x graph.

At position (1): $\frac{dU}{dx} < 0$; but $F_1 = -\frac{dU}{dx}$

Hence F_1 is positive at position (1)

Similarly, at position (2), $\frac{dU}{dx} > 0$; but $F_2 = -\frac{dU}{dx}$

Hence, F_2 is negative at position (2).

The slope changes from –ve to +ve at A when we move along $+x$ direction. We can conclude that the direction of force changes from positive to negative at A.

As the force is always directed towards point A, the particle always accelerates towards A. This signifies that when a particle is displaced in either side of A and released, it will return to the point A. It means, the particle tries to attain a stable position at A. As the particle is in equilibrium at A, we call it "stable equilibrium." This is the sufficient and necessary condition for "oscillations" of an object.

At stable equilibrium, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} > 0$.

Unstable Equilibrium

Let us enquire to point B and take two points (2) and (3) at both sides of the point.

At position (2): $\frac{dU}{dx} > 0$; but $F_2 = -\frac{dU}{dx}$

Hence, F_2 is negative at position (2).

Similarly, at position (3): $\frac{dU}{dx} < 0$; but $F_3 = -\frac{dU}{dx}$

Hence, F_3 is positive at position (3). That means the force changes its direction from –ve to +ve when we pass through the point B while moving in x- direction. In other words, the force F is directed away from the point B. hence, when we displace a particle from B, it will never accelerate (return) to the point B. Hence, when we displace a particle from B, it will never accelerate (return) to the point B. That means, the position of particle is unstable even though it is in equilibrium at B. In other words, the particle is in "unstable equilibrium" at B.

At stable equilibrium, $\frac{dU}{dx} = 0$ and $\frac{d^2U}{dx^2} > 0$

Neutral Equilibrium

Finally, let us come the point C. At C, $F = 0$ as discussed earlier and the particle is in equilibrium at C. Let us displace the particle slowly in either side to (4) and (5). Since at both (4) and (5); $dU/dx = 0$, we can say that the particle experiences no force when it is displaced near C. In other words, the particle will remain at rest (or move with constant velocity) at the points (4) and (5). Then, we can say that the particle is simultaneously neutral and it is in equilibrium because no net force acts on the particle when it undergoes a displacement near C. Hence, at C, the particle is said to be in "neutral equilibrium."

ILLUSTRATION 8.46

The potential energy of a particle in a certain field has the form $U = (a/r^2) - (b/r)$, where a and b are positive constants and r is the distance from the center of the field. Find the value of r_0 corresponding to the equilibrium position of the particle; examine whether this position is stable.

Sol. $U(r) = \frac{a}{r^2} - \frac{b}{r}$

$$\text{Force} = -\frac{dU}{dr} = -\left(\frac{-2a}{r^3} + \frac{b}{r^2}\right) = -\frac{(br - 2a)}{r^3}$$

At equilibrium, $F = -\frac{dU}{dr} = 0$

Hence, $br - 2a = 0$, at equilibrium.

$r = r_0 = \frac{2a}{b}$ corresponds to equilibrium.

At stable equilibrium, the potential energy is minimum and at unstable equilibrium, it is maximum.

For minimum potential energy,

$\frac{du}{dr} = 0$ and $d^2U/dr^2 > 0$ at $r = r_0$.

Let us investigate the second derivative.

$$\frac{d^2U}{dr^2} = \frac{d}{dr}\left(\frac{dU}{dr}\right) = \frac{d}{dr}\left(\frac{-2a}{r^3} + \frac{b}{r^2}\right) = \frac{6a}{r^4} - \frac{2b}{r^3}$$

$$\text{At } r = r_0 = \frac{2a}{b} \Rightarrow \frac{d^2U}{dr^2} = \frac{6a - 2br_0}{r_0^4} = \frac{2a}{r_0^4} > 0$$

Hence, the potential energy function $U(r)$ has a minimum value at $r_0 = 2a/b$. The system has a stable equilibrium at minimum potential energy state.

Note: Conservative force can be defined in three ways as follows:

- If the work done by a force on a body depends only upon the initial and final positions of that body, then the force is conservative, e.g., gravitational, electrostatic,

  $W = k_f - k_i = U(x_i) - U(x_f)$
- A force is conservative if it can be derived from a scalar quantity $U(x)$ by the relation

  $F(x) = -\Delta U(x)/\Delta x$ or $F = -dU/dx$
- If the work done by a force on a body that has moved in closed path and has come back to its initial position is zero, the force is conservative.

ILLUSTRATION 8.47

A particle of mass $m = 2$ kg is free to move along x axis under influence of a conservative force. The potential energy function for the particle is given by

$$U(x) = a\left[\left(\frac{x}{b}\right)^4 - 5\left(\frac{x}{b}\right)^2\right] \text{J}$$

Where $a = 1.0$ J and $b = 1.0$ m. If the total mechanical energy of the particle is zero,

(a) Plot this potential energy function, identify the regions where particle may be found

(b) Calculate the maximum speed of the particle.

Sol.

(a) As $a = 1$ and $b = 1$

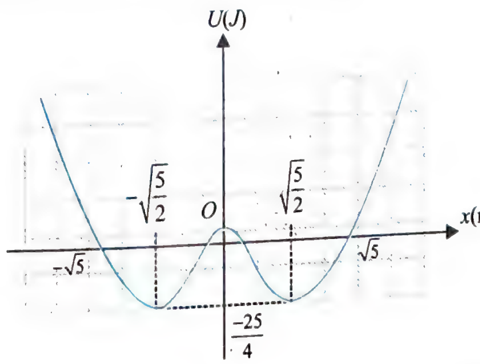
$$\therefore \quad U(x) = x^4 - 5x^2 \qquad \text{...(i)}$$

$$U = 0 \text{ at } x = 0 \text{ and } x = \pm\sqrt{5} \text{ m}$$

Differentiating (i) w.r.t x, $\dfrac{dU}{dx} = 4x^3 - 10x$ ...(ii)

$$\frac{dU}{dx} = 0 \text{ where } x = 0,\ x = \sqrt{\frac{5}{2}} \text{ m},\ x = -\sqrt{\frac{5}{2}} \text{ m}$$

x(m)	±1	$\pm\sqrt{\frac{5}{2}}$	±2	±3
U(J)	–4	$-\frac{25}{4}$	–4	+36



(b) As the total mechanical energy of the particle is zero, $K + U = 0$

$\therefore$ The particle will remain between $-\sqrt{5}$ m to $+\sqrt{5}$ m.

The kinetic energy of the particle will be maximum where the potential energy of the particle become minimum.

As $U_{min} = -\frac{25}{4}$ J, hence $K_{max} + \left(-\frac{25}{4}\right) = 0 \Rightarrow K_{max} = \frac{25}{4}$ J

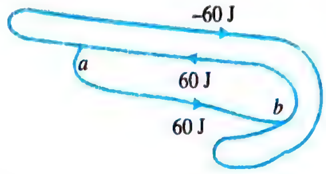
$$\Rightarrow K_{max} = \frac{1}{2}mv_{max}^2 = \frac{25}{4} \text{J}$$

$$\Rightarrow \frac{1}{2} \times 2 \times v_{max}^2 = \frac{25}{4} \text{J}$$

$$\therefore \quad v_{max} = \frac{5}{2} \text{ m/s}$$

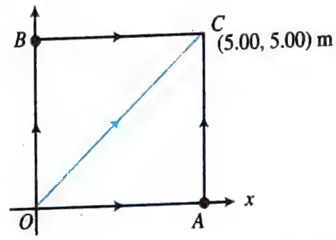
CONCEPT APPLICATION EXERCISE 8.3

1. Figure shows three paths connecting points a and b. A single force F does the indicated work on a particle moving along each path in the indicated direction. On the basis of this information, is force F conservative?



2. The potential energy of a conservative system is given by $U = ax^2 - bx$, where a and b are positive constants. Find the equilibrium position and discuss whether the equilibrium is stable, unstable, or neutral.

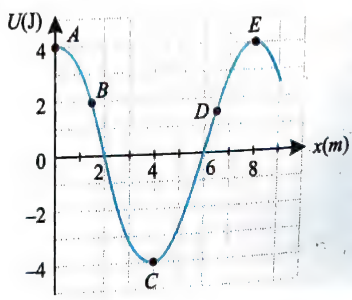
3. A particle moves in the x-y plane in Figure under the influence of a friction force with magnitude 3.00 N and acting in the direction opposite to the particle's displacement. Calculate the work done by the friction force on particle as it moves along the following closed paths: (a) path OA followed by the return path AO, (b) path OA followed by AC and the return path CO, (c) path OC followed by the return path CO, and (d) each of your three answers should be non-zero. What is the significance of this observation?



4. A single conservative force acting on a particle varies as $\vec{F} = (-Ax + Bx^2)\hat{i}$ N, where A and B are constants and x is in metres. (a) Calculate the potential energy function $U(x)$ associated with this force, taking $U = 0$ at $x = 0$. (b) Find the change in potential energy and the change in kinetic energy of the system as the particle moves from $x = 2.00$ m to $x = 3.00$ m.

5. A potential energy function for a two-dimensional force is of the form $U = 3x^3y - 7x$. Find the force that acts at the point (x, y).

6. For the potential energy curve shown in figure, (a) Determine whether the force F_x is positive, negative, or zero at the five points indicated; (b) Indicate points of stable, unstable, and neutral equilibrium; (c) Sketch the curve for F_x versus x from $x = 0$ to $x = 8$ m.



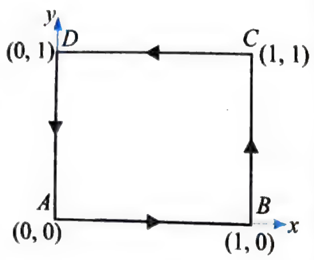
7. A particle is confined to move on x-axis only and its kinetic energy is 60 J. The potential energy of particle is given as $U = x^2 - 7x + 30$. Find how this particle will move.

8. A particle can move along x-axis under influence of a conservative force. The potential energy of the particle is given by $U = 5x^2 - 20x + 2$ joule where x is co-ordinate of the particle expressed in meter. The particle is released at $x = -3$ m.

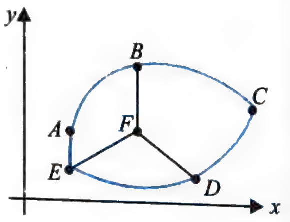
   (a) Find the maximum kinetic energy of the particle during subsequent motion.

   (b) Find the maximum x co-ordinate of the particle.

9. A particle moves in x-y plane under the action of a force $\vec{F} = 2Ax^2 y\hat{i} + Axy^2\hat{j}$. Calculate the work done in going in anticlockwise direction along the side of a square of vertices (0,0), (1,0), (1,1) and (0,1). Can this force be conservative?



10. A conservative force does 2 J of work in moving a particle from A to C via ABC The force does – 1 J work when moving from D to F, 3 J for the path EB, 1 J for EF and 1 J for BC. What is the work done by the force when the particle moves from C to A, A to E and D to C?



ANSWERS

1. No **2.** $b/2a$, stable equilibrium

3. (a) –30.0 J **(b)** –51.2 J **(c)** –42.4 J

(d) The force of friction is a non-conservative force.

4. (a) $\frac{Ax^2}{2} - \frac{Bx^3}{3}$

(b) $\Delta U = \frac{5.00}{2}A - \frac{19.0}{3}B$, $\Delta K = \left(-\frac{5.00}{2}A + \frac{19.0}{3}B\right)$

5. $(7 - 9x^2 y)\hat{i} - 3x^3\hat{j}$

6. (a) Zero **(b)** A and E are unstable, and C is stable.

8. (a) 125 J **(b)** 7 m

9. $W = \Sigma W_i = -\frac{A}{3} \neq 0$, non-conservative

10. $W_{CA} = -2$ J, $W_{AE} = -2$ J, $W_{DC} = 2$ J

MECHANICAL ENERGY AND ITS CONSERVATION

Mechanical energy, E, of an object or a system is defined as the sum of kinetic energy K and potential energy U, i.e.,

$$E = K + U$$

We know the work-energy theorem:

$$W_{\text{conservative}} + W_{\text{non-conservative}} + W_{\text{other}} = \Delta K \quad \text{...(i)}$$

But we know the definition of the potential energy

$$W_{\text{conservative}} = -\Delta U \quad \text{...(ii)}$$

From (i) and (ii),

$$-\Delta U + W_{\text{non-conservative}} + W_{\text{other}} = \Delta K \quad \text{...(iii)}$$

$$W_{\text{non-conservative}} + W_{\text{other}} = \Delta K + \Delta U$$

= Change in mechanical energy of the system

If only conservative forces act on the system, then we have

$$W_{\text{non-conservative}} = 0 \text{ and } W_{\text{other}} = 0.$$

Then we have, $\Delta K + \Delta U = 0$ ...(iv)

or we can say the mechanical energy of the system is constant.

Equation (iv) is a statement of conservation of mechanical energy for an isolated system with no non-conservative forces acting. The mechanical energy in such a system is conserved: the sum of the kinetic and potential energies remains constant.

If there are non-conservative forces acting within the system, mechanical energy is transformed to internal energy. If non-conservative forces act in an isolated system, the total energy of the system is conserved although the mechanical energy is not.

Let us summarize the concepts developed so far in this section:

- Work done on a particle is equal to the change in its kinetic energy.
- Work done on a system by all the (external and internal) forces is equal to the change in its kinetic energy.
- A force is called conservative if the work done by it during a round trip of a system is always zero. The force of gravitation, Coulomb force, force by a spring etc. are conservative. If the work done by it during a round trip is not zero, the force is non-conservative. Friction is an example of non-conservative force.
- The change in the potential energy of a system corresponding to conservative internal forces is equal to negative of the work done by this forces.
- If no external forces act (or the work done by them is zero) and the internal forces are conservative, the mechanical energy of the system remains constant. This is known as the principle of conservation of mechanical energy.
- If some of the internal forces are non-conservative and they do some work, the mechanical energy of the system is not constant.
- If the internal forces are conservative, the work done by the external forces is equal to the change in mechanical energy.

$$W_{\text{external}} = \Delta K + \Delta U$$

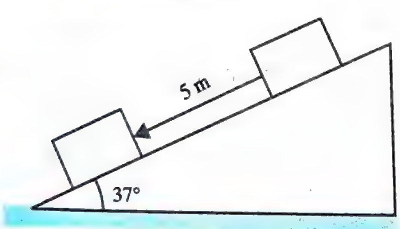
Important Points:

Mechanical energy

- It is a scalar quantity having dimensions $[ML^2T^{-2}]$ and SI units joule.
- It depends on the frame of reference.
- A body can have mechanical energy without having either kinetic energy or potential energy. However, if both kinetic and potential energies are zero, mechanical energy will be zero. The converse may or may not be true, i.e., if $E = 0$ either both PE and KE are zero or PE may be negative and KE may be positive such that KE + PE = 0.
- As mechanical energy $E = K + U$, i.e., $E - U = K$. Now as K is always positive, $E - U \geq 0$, i.e., for existence of a particle in the field, $E \geq U$.
- As mechanical energy $E = K + U$ and K is always positive, so, if 'U' is positive 'E' will be positive. However, if potential energy U is negative, 'E' will be positive if $K > |U|$ and E will be negative if $K < |U|$, i.e., mechanical energy of a body or system can be negative, and negative mechanical energy means that potential energy is negative and in magnitude it is more than kinetic energy. Such a state is called bound state, e.g., electron in an atom or a satellite moving around a planet are in bound state.

ILLUSTRATION 8.48

A block is placed on the top of a plane inclined at 37° with horizontal. The length of the plane is 5 m. The block slides down the plane and reaches the bottom.



(a) Find the speed of the block at the bottom if the inclined plane is smooth.

(b) Find the speed of the block at the bottom if the coefficient of friction is 0.25.

Sol. Let h be the height of the inclined plane.

$h = 5 \sin 37° = 3$ m

(a) As the block slides down the inclined plane, it loses gravitational potential energy and gains KE.

Loss in GPE = gain in KE

mg (loss in height) = $KE_f - KE_i$

$$\Rightarrow \quad mgh = \frac{1}{2}mv^2 - 0$$

$$\Rightarrow \quad v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 3} = 7.67 \text{ m s}^{-1}$$

Note:

- Loss in mechanical energy = Initial energy – Final energy
- Gain in mechanical energy = Final energy – Initial energy

(b) As the block comes down, it loses GPE. It gains KE and does work against friction.

Loss in GPE = gain in KE + work done against friction

$$\Rightarrow \quad mgh = \left(\frac{1}{2}mv^2 - 0\right) + (\mu mg \cos 37°)s$$

$$\Rightarrow \quad 3mg = \frac{1}{2}mv^2 + (0.25) \times mg \times \frac{4}{5} \times 5$$

$$\Rightarrow \quad v = \sqrt{4g} = 6.26 \text{ m s}^{-1}$$

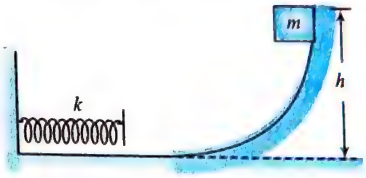
ILLUSTRATION 8.49

A smooth block of mass m is released from rest from a height h. It slides and compresses the spring of stiffness k. Find the maximum compression of the spring.

Taking block spring a system. No external force is doing work on system and no non-conservative force is present. The mechanical energy should be conserved.

$$\Delta K + \Delta U = 0$$

Sol. Let x = maximum compression of the spring.



Here $\Delta U = \Delta U_{sp} + \Delta U_{gr}$

$$\Delta U_{sp} = \frac{k}{2}x^2$$

because the spring is deformed from $x = 0$ to $x = x$ and $\Delta U_{gr} = -mgh$ because the block falls down through a vertical distance h. Hence,

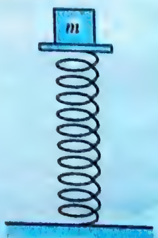
$$\Delta U = \frac{k}{2}x^2 - mgh$$

Since, the block will come to rest at the time of maximum compression $\Delta K = 0$. Substituting ΔU and ΔK in the equation $\Delta U + \Delta K = 0$, we have $\frac{k}{2}x^2 - mgh = 0$

Then $x = \sqrt{\frac{2mgh}{k}}$

ILLUSTRATION 8.50

A block of mass m is suddenly released from the top of a spring of stiffness constant k.



(a) Find the acceleration of block just after release.

(b) What is the compression in the spring at equilibrium.

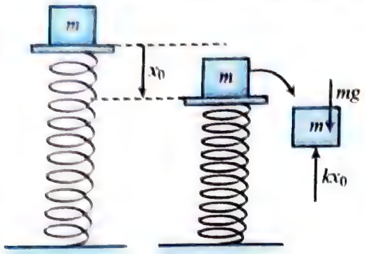
(c) What is the maximum compression in the spring.

(d) Find the acceleration of block at maximum compression in the spring.

Sol.

(a) Just after release the spring is relaxed. Net force on the block is only its weight. Hence, acceleration of the block is g downward.

(b) At equilibrium, net force on the block should be zero. Let x_0 be the compression in the spring at equilibrium.



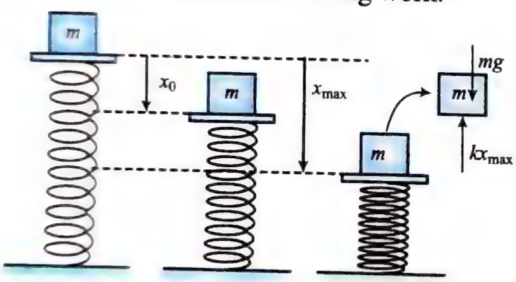
Applying Newton's second law,

$$mg = kx_0 \quad \text{or} \quad x_0 = \frac{mg}{k}$$

(c) At maximum compression, the velocity of block should be zero.

Let x_{max} be the maximum compression in the spring.

If we take block and spring as system, no external force and non-conservative force is doing work.



Applying mechanical energy conservation $\Delta K + \Delta U =$

As the block is released from rest net charge and finally comes to rest. Hence, net change in its kinetic energy will be zero.

Net change the potential energy of the system

$$\Delta U = \Delta U_{gr} + \Delta U_{sp} = (-mgx_{max}) + \left(\frac{1}{2}kx_{max}^2\right) = 0$$

or $\quad x_{max} = \dfrac{2mg}{k}$

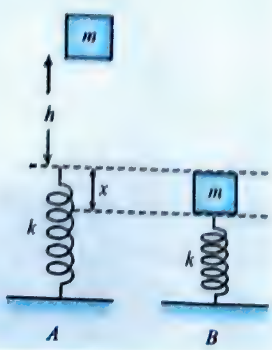
(d) At maximum compression, writing equation of motion of block

$$kx_{max} - mg = ma \text{ where } x_{max} = \frac{2mg}{k}$$

Hence, acceleration of the block is g upward.

ILLUSTRATION 8.51

A block of mass m strikes a light pan fitted with a vertical spring after falling through a distance h. If the stiffness of the spring is k, find the maximum compression of the spring.



Sol. Let the maximum compression in the spring be x. The reference level for potential energy is assumed at the position of maximum compression.

Applying mechanical energy conservation, $\Delta K + \Delta U = 0$

As block is released from rest and finally comes to rest. Hence, net change in kinetic energy, $\Delta K = 0$.

Net change in potential energy,

$$\Delta U = \Delta U_{gr} + \Delta U_{sp} = [-mg(x+h)] + \left(\frac{1}{2}kx^2\right)$$

$$0 + [-mg(h+x)] + \left(\frac{1}{2}kx^2 - 0\right) = 0$$

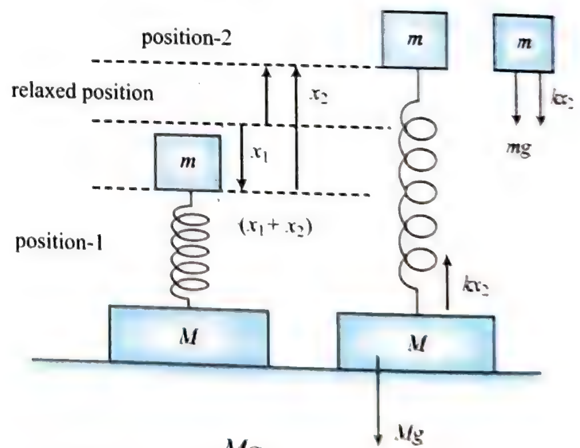
$$x^2 - 2\left(\frac{mg}{k}\right)x - 2\left(\frac{mg}{k}\right)h = 0$$

After solving, we get $x = \dfrac{mg}{k}\left[1 + \sqrt{1 + \dfrac{2kh}{mg}}\right]$

ILLUSTRATION 8.52

Two blocks of masses m and M connected by a light spring of stiffness k, are kept on a smooth horizontal surface as shown in figure. What should be the initial compression of the spring so that the system will be about to break off the surface, after releasing the block m_1?

Sol. In order to lift the block m_2 from the surface, the spring force F must be equal to the weight of the block M and acts in upward direction, at position 2 (say). Let the corresponding elongation of the spring be x_2.



Hence, $kx_2 = Mg \Rightarrow x_2 = \dfrac{Mg}{k}$ ...(i)

PE stored in the spring in position-2 $U_F = \dfrac{1}{2}kx_2^2$

Let, initially, the spring be compressed by x_1. Therefore, the potential energy stored in the spring at this position $= \dfrac{1}{2}kx_1^2$

$$\Delta U = (\Delta PE)_{spring}$$

$$= \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

$$= \frac{1}{2}k(x_2^2 - x_1^2) \quad \text{...(ii)}$$

As the block m_1 rises through a distance $(x_1 + x_2)$, the change in its gravitational potential energy is,

$$(\Delta PE)_{gr} = mg(x_1 + x_2) \quad \text{...(iii)}$$

Since the blocks are at rest at both the position 1 and 2

$(\Delta KE)_{system} = 0$.

Conserving the energy of the system between position 1 and 2

$\Delta PE + \Delta KE = 0$

$(\Delta PE)_{gr} + (\Delta PE)_{spring} = 0$

$$mg(x_1 + x_2) + \frac{1}{2}k(x_2^2 - x_1^2) = 0$$

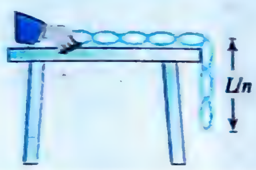
$$mg + \frac{\{(x_2 - x_1)\}}{2} = 0$$

Putting $x_2 = Mg/k$, we obtain $mg + \frac{k}{2}\left(\frac{Mg}{k} - x_1\right) = 0$

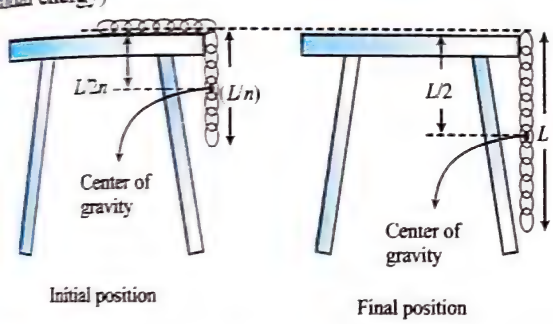
$$\Rightarrow (2m + M)g = kx_1 \Rightarrow x_1 = \frac{(2m+M)g}{k}$$

ILLUSTRATION 8.53

A chain of length L and mass M is held on a frictionless table with $(1/n)$th of its length hanging over the edge. When the chain is released, find the velocity of chain while leaving the table.



Sol. Taking surface of table as a reference level (zero potential energy)



Potential energy of chain when $1/n$th length hanging from the edge $U_{initial} = \frac{-MgL}{2n^2}$

Potential energy of chain when it leaves the table,

$$U_{final} = -\frac{MgL}{2}$$

Using conservation of mechanical energy

$$\Delta K + \Delta U = 0$$

or Kinetic energy of chain = Loss in potential energy

$$\Rightarrow \frac{1}{2}Mv^2 = \frac{MgL}{2} - \frac{MgL}{2n^2}$$

$$= \frac{MgL}{2}\left[1 - \frac{1}{n^2}\right]$$

Therefore, velocity of chain $v = \sqrt{gL\left(1 - \frac{1}{n^2}\right)}$

ILLUSTRATION 8.54

A body of mass m hangs by an inextensible string that passes over a smooth mass less pulley that is fitted with a light spring of stiffness k as shown in figure. If the body is released from rest and the spring is released, calculate the maximum elongation of the spring.

Sol. Let the spring be elongated by x. That means the pulley falls through a distance x.

Consequently the string of the pulley is slackened to the same extent x. Therefore, the total distance moved by the body is $2x$. Since the spring is elongated by x,

$$\Delta PE_{spring} = \frac{1}{2}kx^2$$

As the body falls through a distance $2x$,

$$(\Delta PE)_{gr} = -mg(2x)$$

Since the body is initially at rest and comes to rest just at the instant of maximum elongation of the spring, $\Delta KE = 0$

Applying the principle of conservation of energy, we get

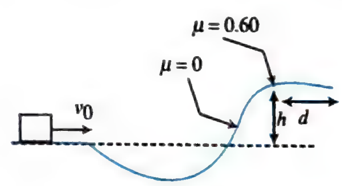
$$(\Delta PE) + (\Delta KE) = 0$$

$$\Rightarrow \Delta PE_{spring} + \Delta PE_{gr} + \Delta KE = 0$$

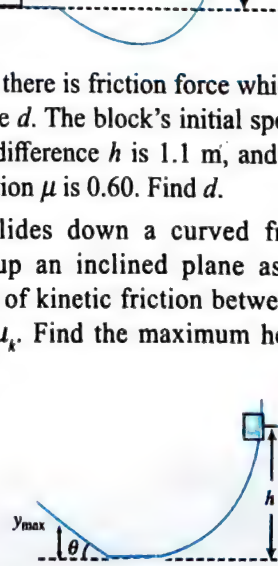
$$\frac{1}{2}kx^2 - mg(2x) + 0 = 0 \Rightarrow x = \frac{4mg}{k}$$

CONCEPT APPLICATION EXERCISE 8.4

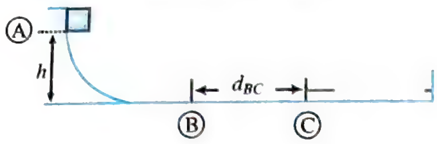
1. A uniform chain of length l and mass m overhangs a smooth table with its two-thirds part lying on the table. Find the kinetic energy of the chain as it completely slips off the table.
2. In figure, a block slides along a track from one level to a higher level by moving through an intermediate valley. The track is friction less until the block reaches the higher



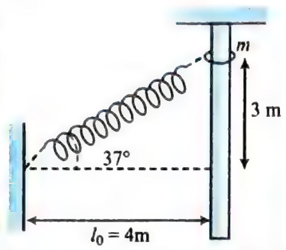
level. Then there is friction force which stops the block at a distance d. The block's initial speed v_0 is 6.0 m s^{-1}; the height difference h is 1.1 m, and the coefficient of kinetic friction μ is 0.60. Find d.
3. A block slides down a curved frictionless track and then up an inclined plane as in figure. The coefficient of kinetic friction between the block and incline is μ_k. Find the maximum height reached by the block.



4. A block of mass m is released from point (A) in Figure. The track is frictionless except for the portion between points (B) and (C), which has a length of d_{BC}. The block travels down the track, hits a spring of force constant k, and compresses the spring x from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between the block and the rough surface between (B) and (C).



5. A ring of mass m = 10 kg can slide through a vertical rod with friction. It is connected with a spring of force constant $k = 400$ N m^{-1}. The relaxed length of spring is 4 m. The ring is displaced 3 m as shown in the figure and released. Find velocity of ring when spring becomes horizontal.

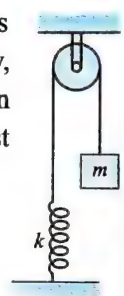


6. In figure, the stiffness of the spring is k and mass of the block is m. The pulley is fixed. Initially, the block m is held such that the elongation in the spring is zero and then released from rest Find:

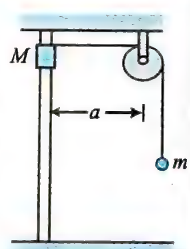
(a) the maximum elongation in the spring

(b) the maximum speed of the block m.

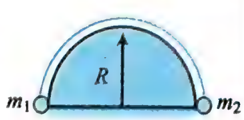
Neglect the mass of the spring and that of the string. Also neglect the friction.



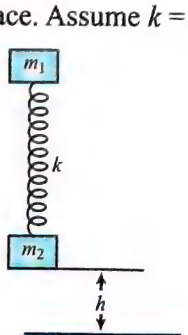
7. In figure, an inextensible string that connects two bodies of mass M and m, passing over a fixed smooth pulley. The body M slides along a smooth vertical rigid bar. If the body M is released from the given position, find the maximum distance raised by body m.



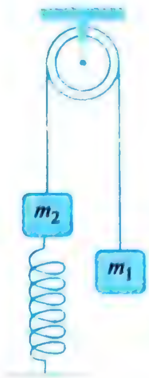
8. Two smooth balls of mass m_1 and m_2 connected by a light inextensible string are at the opposite points of horizontal diameter of a smooth semi cylindrical surface of radius R. If m_1 is released, find its speed at any angular distance θ moved by m_2.



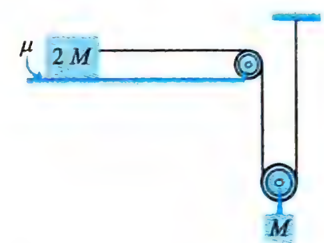
9. A spring–mass system (m_1 + massless spring + m_2) fall freely from a height h before m_2 colliding inelastically with the ground. Find the maximum value of h so that block m_2 will break off the surface. Assume k = stiffness of the spring.



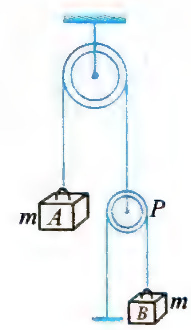
10. In an ideal pulley particle system, the mass m_2 is connected with a vertical spring of stiffness k. If the mass m_2 is released from rest, when the spring is undeformed, find the maximum compression of the spring.



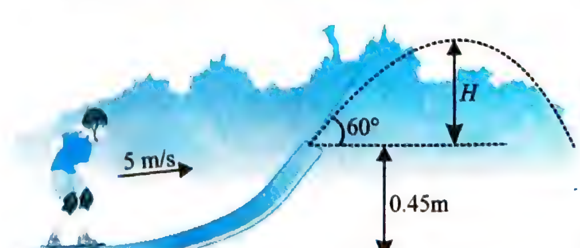
11. The figure below shows two blocks of masses $2M$ and M, respectively. The coefficient of friction between the block of mass $2M$ and the horizontal plane is μ. The system is released from rest. Find the velocity of the block of mass 'M' when the block of mass $2M$ has moved a distance 's' towards right.



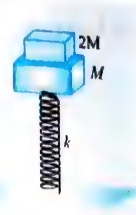
12. In the figure shown, the system is released from rest. Find the velocity of block A when block B has fallen a distance 'l'. Assume all pulleys to be massless and frictionless.



13. In the figure shown, a skateboarder is moving at 5.0 m/s along the horizontal section of a track that is slanted upward by 60° above the horizontal at its end, which is 0.45 m above the ground. When he leaves the track, he follows the characteristic path of projectile motion. Ignoring friction and air resistance, find the maximum height H to which he rises above the end of the track.



14. In the fig shown, a block of mass M is attached to the spring and another block of mass $2M$ has been placed over it. The system is in equilibrium. The block are pushed down so that the spring compresses further by $9Mg/K$. System is released.



(a) At what height above the position of release, the block of mass $2M$ will lose contact with the other block?

(b) What is maximum height attained by $2M$ above the point of release?

ANSWERS

1. $\frac{4}{9}mgl$ 2. 1.2 m 3. $\frac{h}{1+\mu_k \cot\theta}$ 4. $\frac{mgh - \frac{1}{2}kx^2}{mgd_{BC}}$

5. $10\ ms^{-1}$ 6. (a) $2mg/k$ (b) $\left(\sqrt{\frac{m}{k}}\right)g$ 7. $\frac{2a}{\left(\frac{m^2}{M^2}-1\right)}$

8. $\sqrt{\frac{2gR(m_1\theta - m_2\sin\theta)}{(m_1+m_2)}}$ 9. $\frac{m_2(2m_1+m_2)g}{2m_1k}$

10. $\frac{2(m_2-m_1)g}{k}$ 11. $\frac{1}{3}\sqrt{gs(1-4\mu)}$ 12. $\sqrt{\frac{gl}{5}}$

13. 0.60 m 14. (a) $\frac{12Mg}{k}$ (b) $\frac{24\ Mg}{k}$

MECHANICAL POWER

In many situations, when work is being done, we are not only interested in the amount of work done but also in the time in which it is done. Here comes the concept of mechanical power, the idea of power incorporates both the concepts of work and time. The time rate at which work is done by a force is said to be the power due to the force. Power is also called **activity**. We shall define two types of power.

(a) **Average Power:** Average power $\bar{P}$ is the average rate at which work W is done, and it is obtained by dividing W by the time t required to perform the work: $\bar{P} = \frac{\text{Work}}{\text{Time}} = \frac{W}{t}$ ...(1)

We can also define average power as the rate at which the energy is changing, or as the change in energy divided by the time during which the change occurs:

$$\bar{P} = \frac{\text{Change in energy}}{\text{Time}}$$

Since work, energy, and time are scalars, power is also a scalar.

(b) **Instantaneous Power:** P. It is the power at any instant t and is the limit of the ratio $\Delta W/\Delta t$ as At tends to zero, i.e.

$$P = \lim_{\Delta t \to \infty} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} \quad ...(2)$$

In case, the force applied $(\vec{F})$ to deliver power is constant,

$$P = \frac{dW}{dt} = \frac{d}{dt}\left(\vec{F}\cdot\vec{s}\right) = \vec{F}\cdot\frac{d\vec{s}}{dt} = \vec{F}\cdot\vec{v} \quad ...(3)$$

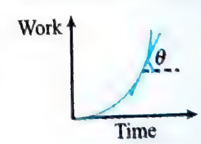
Thus, instantaneous power is the dot product of the force $(\vec{F})$ and the velocity $(\vec{v})$.

When $\vec{F}$ and $\vec{v}$ and are in the same direction,

$$P = Fv \quad ...(4)$$

Important Points:

- Dimension: $[P] = [F]\,[v] = [MLT^{-2}]\,[LT^{-1}]$
  $\therefore [P] = [ML^2T^{-3}]$
- Units: Watt or $J s^{-1}$ [SI]
  Erg/s [CGS]
  Practical units: Kilowatt (KW), Megawatt (MW) and Horse power (hp)
  Relations between different units:
  1 Watt = 1 $Js^{-1} = 10^7$ erg s^{-1}
  1 hp = 746 W
  1 MW = 10^6 W
  1 KW = 10^3 W
- If work done by the two bodies is same, then
- Power $\propto \frac{1}{\text{time}}$
  i.e., the body which performs the given work in lesser time possesses more power and vice-versa.
- As power = work/time, any unit of power multiplied by a unit of time gives unit of work (or energy) and not power, i.e., kilowatt-hour or watt-day are units of work or energy.
  $$1\ \text{kWh} = 10^3\ \frac{\text{J}}{\text{s}} \times (60 \times 60\ \text{s}) = 3.6 \times 10^6\ \text{J}$$
- The slope of work time curve gives the instantaneous power. As $P = dW/dt = \tan\theta$
- Area under power–time curve gives the work done as



- $P = \frac{dW}{dt}$
  $\therefore \quad W = \int P\,dt$
  = Area under P-t curve

ILLUSTRATION 8.55

An insect of mass m moves up along a hanging stationary thread, with acceleration a. Find the power delivered by the gravity after a time t.

Sol. Power delivered by gravity at any instant is

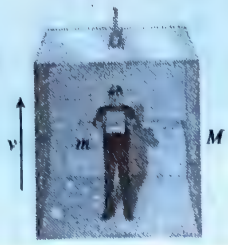
$$P_g = m\vec{g}.\vec{v}$$

where v is the instantaneous velocity of the insect.
Since the insect moves with an upward acceleration a, $v = at$.
Substituting $v = at$ in the above expression, we have

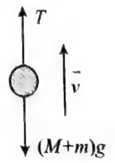
$$P_g = -mgat$$

ILLUSTRATION 8.56

An elevator of mass M with a person of mass m is moving upward with uniform velocity $\vec{v}$. What is the power delivered by the elevator?



Sol. Since the system moves up with constant velocity, the change in KE of the system is zero. Let T be the tension in the rope of the elevator acceleration.



Therefore, $F_{net} = T - (M+m)g = (M+m)a$

Putting $a = 0$, we obtain $T = (M+m)g$.

Therefore, power delivered by the elevator is

$$P = \vec{T}\cdot\vec{v} = T\,v\cos 0 = T\,v = (M+m)gv$$

ILLUSTRATION 8.57

A body is thrown with a velocity v_0 at an angle θ_0 with horizontal. Find the (a) instantaneous power delivered by gravity after a time t measured from the instant of projection and (b) average power delivered by gravity during the time t.

Sol.

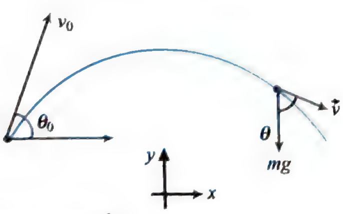
(a) Power delivered by gravity after a time t when the particle has velocity v is,

$$P_g = m\vec{g}.\vec{v} \qquad \text{...(i)}$$

Substituting $\vec{g} = -g\hat{j}$

and $\vec{v} = v_0\cos\theta_0\hat{i} + (v_0\sin\theta_0 - gt)\hat{j}$ in Eq. (i), we have

$$P_g = -mg(v_0\sin\theta_0 - gt)$$



(b) The average power is

$$P_{av} = \frac{\int_0^t P dt}{t}$$

Substituting $P = P_g = -mg\,(v_0\sin\theta_0 - gt)$,

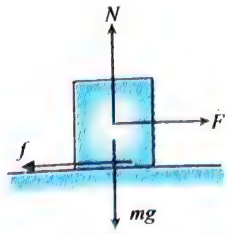
we have $P_{av} = -\dfrac{mg}{t}\displaystyle\int_0^t (v_0\sin\theta_0 - gt)dt$

This gives $P_{av} = -mg\left(v_0\sin\theta_0 - \dfrac{gt}{2}\right)$

ILLUSTRATION 8.58

A horizontal constant force F pulls a block of mass m placed on a horizontal surface. If the coefficient of kinetic friction between the block and ground is μ, find the power delivered by the external agent after a time t measured from the beginning of action of the force.

Sol. The forces acting on the block are $F\rightarrow$, kinetic friction $f\leftarrow$, $mg\downarrow$, and $N\uparrow$.



Power delivered by the external agent (force F) is

$$P_{ext} = \vec{F}.\vec{v} = Fv\cos 0° = Fv \qquad \text{...(i)}$$

we have $v = \int a\,dt$ ...(ii)

Since a is decided by the net force

$$F_{net} = (F - f) \text{ (but not } F \text{ or } f) \qquad \text{...(iii)}$$

Force equation:

Using eqs. (i), (ii) and (iii), we have

$$P_{ext} = F\left(\int \frac{F-f}{m}dt\right) = F\left(\frac{F-f}{m}\right)t \qquad \text{...(iv)}$$

Law of kinetic friction $f = \mu mg$

Using eqs. (iv) and (v), we have $P_{ext} = \dfrac{F(F-\mu mg)t}{m}$ ...(v)

ILLUSTRATION 8.59

A small body of mass m is located on a horizontal plane at the point O. The body acquires a horizontal velocity v_0. Find the mean power developed by the friction force during the whole time of motion, if the frictional coefficient $\mu = 0.27$, $m = 1.0$ kg and $v_0 = 1.5\ \mathrm{m\,s^{-1}}$.

Sol. $\bar{P} = \dfrac{1}{T}\displaystyle\int_0^T P.dt$

$$= \frac{1}{T}\int_0^T \mu mg(v_0 - \mu gt)dt \text{ (numerically)}$$

$$= \frac{\mu mg}{T}\left(v_0T - \frac{\mu gT^2}{2}\right) = \mu mg\left(v_0 - \frac{\mu gT}{2}\right)$$

where $T = \dfrac{v_0}{\mu g} \Rightarrow \bar{P} = \dfrac{\mu mgv_0}{2} = \dfrac{0.27\times1\times9.8\times1.5}{2}$

$\Rightarrow \quad \bar{P} = 2.025$ W

ILLUSTRATION 8.60

A car of mass 500 kg moving with a speed 36 kmh in a straight road unidirectionally doubles its speed in 1 min. Find the power delivered by the engine.

Sol. Its initial speed $v_1 = \dfrac{36000}{3600} = 10\,\mathrm{m\,s^{-1}}$.

If the car doubles its speed, finally its speed becomes $v_2 = 20\,\mathrm{m\,s^{-1}}$.

Change in KE of the car = ΔKE = $(1/2)\, mv_2^2 - (1/2)\, mv_1^2$

During $\Delta t = 1$ min = 60 s

the power delivered by the engine,

$$P = \frac{\text{Work done}}{\text{Time taken}} = \frac{|\Delta KE|}{\Delta t}$$

$$= \frac{\frac{1}{2} \times m\left[v_2^2 - v_1^2\right]}{\Delta t}$$

$$= \frac{1/2 \times 500\left[20^2 - 10^2\right]}{60} = 1250 \text{ W}$$

ILLUSTRATION 8.61

An automobile of mass m accelerates, starting from rest, while the engine supplies constant power P. Find its position and instantaneous velocity changes w.r.t. time.

Sol. Velocity: As $Fv = P$ = constant, i.e.,

$$m\frac{dv}{dt}v = P \quad \left[\text{as } F = \frac{m\,dv}{dt}\right]$$

or $$\int v\,dv = \int \frac{P}{m}dt$$

By integrating both sides, we get $\frac{v^2}{2} = \frac{P}{m}t + C_1$

As initially the body is at rest, i.e., $v = 0$ at $t = 0$, so $C_1 = 0$.

$$v = \left(\frac{2Pt}{m}\right)^{1/2}$$

Position: From the above expression, $v = \left(\frac{2Pt}{m}\right)^{1/2}$

or $$\frac{ds}{dt} = \left(\frac{2Pt}{m}\right)^{1/2} \quad \left[\text{As } v = \frac{ds}{dt}\right]$$

i.e. $$\int ds = \int \left(\frac{2Pt}{m}\right)^{1/2} dt$$

By integrating both sides, we get

$$s = \left(\frac{2P}{m}\right)^{1/2} \cdot \frac{2}{3}t^{3/2} + C_2$$

Now as at $t = 0$, $s = 0$, so $C_2 = 0$.

$$s = \left(\frac{8P}{9m}\right)^{1/2} t^{3/2}$$

ILLUSTRATION 8.62

A car of mass 1.0×10^3 kg, starts from rest and maintains an acceleration of 5.0 m/s² for 6.0 s in the $+x$ direction as shown in figure. Assume that a single horizontal force (not shown) accelerates the car. Determine the average power generated by this force.

Sol. The force can be obtained from Newton's second law as the product of the car's mass and acceleration. The average speed can be determined by using the equations of kinematics.

The average power P_{av} is the product of the magnitude F of the horizontal force acting on the car and the car's average speed v_{av},

$$P_{av} = Fv_{av} \qquad \text{...(i)}$$

We can find the force acting on the car can be obtained from Newton's second law, the net force is equal to the mass m of the car times its acceleration a. Since there is only one horizontal force acting on the car, it is the net force. Thus,

$$F = ma \qquad \text{...(ii)}$$

The final velocity is related to the initial velocity, the acceleration a, and the time t (all of which are known) by: $v = v_0 + at$

Now we can find the average speed by using the equations of kinematics. Since the acceleration is constant, the equations of kinematics apply, and the car's average velocity $\bar{v}$ is one-half the sum of its initial velocity v_0 and its final velocity v:

$$\bar{v} = \frac{1}{2}(v_0 + v) \qquad \text{...(iii)}$$

Substituting this expression for v into equation (iii) above for $\bar{v}$ yields

$$\bar{v} = \frac{1}{2}(v_0 + v) = \frac{1}{2}\left[v_0 + (v_0 + at)\right] = v_0 + \frac{1}{2}at \qquad \text{...(iv)}$$

The average power generated by the net force that accelerates the car is

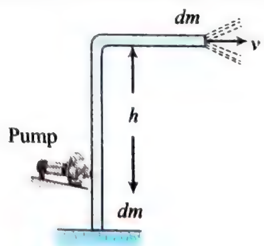
$$P_{av} = Fv_{av} = (ma)\left(v_0 + \frac{1}{2}at\right)$$

$$= (1.0\times10^3\text{ kg})(5.0\text{ m/s}^2)\left[0\text{ m/s} + \frac{1}{2}(5.0\text{ m/s}^2)(6.0\text{ s})\right]$$

$$= 7.5 \times 10^4 \text{ W}$$

POWER OF A WATER-DRAWING PUMP

Let a water pump draws water through a height h from deep well and delivering at the rate of (dm/dt) with a velocity of v.



Suppose dm amount of water is delivered in time dt. The work done,

$$dW = (dm)gh + \frac{1}{2}(dm)v^2$$

Therefore, power delivered, $P = \frac{dW}{dt} = \left(\frac{dm}{dt}\right)\left[gh + \frac{v^2}{2}\right]$

Important Points:

- Power of pump required to just lift the water, $v = 0$,

  $$P = gh\left(\frac{dm}{dt}\right) \quad [\text{if } h \text{ is constant}]$$

- If η be the efficiency of the pump, then

  $$\eta = \frac{\text{Output power}}{\text{Input power}}$$

ILLUSTRATION 8.63

A water pump, rated 400 W, has an efficiency of 75%. If it is employed to raise water through a height of 40 *m*, find the volume of water drawn in 10 min.

Sol. Output power = η × Input power

$= 0.75 \times 400 = 300$ W

Now,

$$P = \frac{dW}{dt} = \frac{d\,(mgh)}{dt}$$

$$\frac{dm}{dt} = \frac{P}{gh} = \frac{300}{10 \times 40} = \frac{3}{4}\ \text{kg} = \frac{3}{4} \times 60\ \text{kg min}^{-1} = 45\ \text{kg min}^{-1}$$

Hence, water drawn in 10 min = $45 \times 10 = 450$ kg min^{-1}

Since density of water is 1000 kg m^{-3}, therefore, from

$$m = \rho V$$

$$V = \frac{m}{\rho} = \frac{450}{1000} = 0.45\ \text{m}^3$$

ILLUSTRATION 8.64

A pump is required to lift 1000 kg of water per minute from a well 20 m deep and eject it at a rate of 20 m s^{-1}.

(a) How much work is done in lifting water?

(b) How much work is done in giving it KE?

(c) What HP (horsepower) engine is required for the purpose of lifting water?

Sol. Work done in lifting water = Gain in PE (potential energy)

Work = $1000 \times g \times 20 = 1.96 \times 10^5$ J min^{-1}

Work done (per minute) in giving it KE

$$= \frac{1}{2}mv^2 = \frac{1}{2}(1000)(20)^2 = 2 \times 10^5\ \text{J min}^{-1}$$

Power of the engine = Work done per second

$$= \frac{1}{60}(1.96 + 2) \times 10^5\ \text{J} = 6.6 \times 10^3\ \text{W}$$

Since 1 HP = 746 W, HP required = 8.85.

CONCEPT APPLICATION EXERCISE 8.5

1. A block moves in uniform circular motion because a cord tied to the block is anchored at the center of a circle. Is the power of the force exerted on the block by the cord positive, negative, or zero?
2. What is the power of an engine which can lift 20 metric tonne of coal per hour from a 20-m deep mine?
3. A 1-kW motor pumps out water from a well 10 m deep. Calculate the quantity of water pumped out per second.
4. One coolie takes 1 min to raise a box through a height of 2 m. Another one takes 30 s for the same job and does the same amount of work. Which one of the two has greater power and which one uses greater energy?
5. A large family uses 8 kW of power. Direct solar energy is incident on the horizontal surface at an average rate of 200 W m^{-2}. If 20% of this energy can be converted to useful electrical energy, how large an area is needed to supply 8 kW?
6. An elevator of total mass (elevator + passenger) 1800 kg is moving up with a constant speed of 2 m s^{-1}. A frictional force of 4000 N opposes its motion. Determine the minimum power delivered by the motor to the elevator. Take $g = 10$ m s^{-2}.
7. Two persons of equal weight are running at speeds of 4 m s^{-1} and 5 m s^{-1}, respectively. Both increase their speeds by 1 m s^{-1} in a time span of 10 s. Who does more work? Who develops more power?
8. A pump is required to lift 1000 kg of water per minute from a well 12 m deep and eject it with a speed of 20 m s^{-1}. How much work is done per second in lifting the water?
9. A helicopter lifts a body of mass 100 kg to a height of 500 m at a constant speed. It takes 5 min to lift the body. Find the work done by the helicopter and the power required.
10. A 700 N marine in basic training climbs a 10.0 m vertical rope at a constant speed in 8.00 s. What is his power output?
11. The electric motor of a model train accelerates the train from rest to 0.620 m s^{-1} in 21.0 m s. The total mass of the train is 875 g. Find the average power delivered to the train during the acceleration.
12. Water falling from a 50 m high fall is to be used for generating electric energy. If 1.8×10^5 kg of water falls per hour and half the gravitational potential energy can be converted into electric energy, how many 100 W lamps can be lit?
13. A particle of mass m is moving in a circular path of constant radius r, such that its centripetal force F_r varies with time t as $F_r = k^2rt^2$, where k is a constant. What is the power delivered to the particle by the forces acting on it?

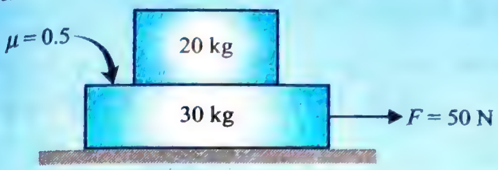
ANSWERS

1. Zero	**2.** 1.1×10^3 W
3. 10 kg	**5.** 200 m^2
6. 58.98 hp	**8.** 2000 J s^{-1}
9. 4900 kJ, 16.333 kW	**10.** 875 W
11. 8.0 W	**12.** 125
13. k^2r^2t	

Solved Examples

EXAMPLE 8.1

A small block of mass 20 kg rests on a bigger block of mass 30 kg, which lies on a smooth horizontal plane. Initially the whole system is at rest. The coefficient of friction between the blocks is 0.5. The horizontal force $F = 50$ N is applied on the lower block.

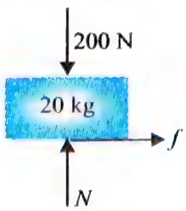


Find the work done by frictional force on upper block and on the lower block in $t = 2$ sec.

Sol. Acceleration of the system. Assume 20 kg and 30 kg blocks move together. Then

$$a = \frac{50}{50} = 1\,\text{m/s}^2$$

Frictional force on 20 kg block is $f = 20 \times 1 = 20$ N



The maximum value of frictional force is

$$f_{max} = \mu N = \frac{1}{2} \times 200 = 100\,\text{N}$$

As $f < f_{max}$, Hence no slipping is occurring. The value of frictional force is $f = 20$ N.

Distance travelled by system in $t = 2$, $s = \frac{1}{2} \times 1 \times 4 = 2\,\text{m}$

Work done by frictional force on upper block is

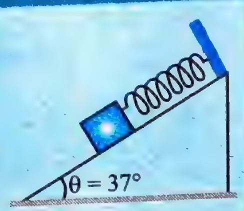
$$\left(W_f\right)_{upper} = f \cdot s = 20 \times 2 = 40\,\text{J}$$

Work done by frictional force on lower block,

$$\left(W_f\right)_{lower} = -f \cdot s = -20 \times 2 = -40\,\text{J}$$

EXAMPLE 8.2

A small block of mass $m = 1$ kg is attached with one end of the spring of force constant $K = 110\,\text{N m}^{-1}$. Other end of the spring is fixed to a rough plane having coefficient of friction $\mu = 0.2$. The spring is kept in its natural length by an inextensible thread tied between its ends as shown in figure. If the thread is burnt, calculate elongation of spring when the block attains static equilibrium position.

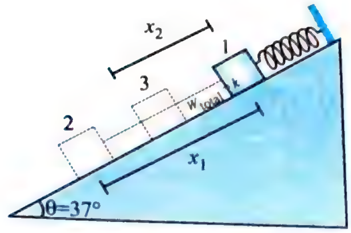


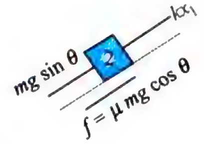
Sol. When the string burns, the block starts moving down. Due to friction and string force, the block stops but at the bottom most position, the block is the static position but it is not in equilibrium position.

Applying work energy theorem between the points when the block is released and bottom most position. At position (1) and (2),

$$W_{total} = \Delta K \text{ (for block)}$$

$$W_{(gravity)} + W_{(friction)} + W_{(spring)} = K_{(final)} - K_{(initial)}$$





$$mg\sin\theta \cdot x_1 - \mu mg\cos\theta \cdot x_1 - \frac{1}{2}kx_1^2 = 0 - 0$$

$$1 \times 10 \times \frac{3}{5}x_1 - 0 \times 2 \times 1 \times 10 \times \frac{4}{5}x_1 - \frac{1}{2} \times 110 \times x_1^2 = 0$$

$$x_1 = 0.08 \text{ m or } 8 \text{ cm}$$

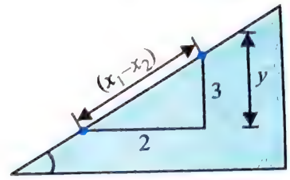
As at position (2), there is net force on the block in upward direction

$$kx_1 > \mu mg\cos\theta + mg\sin\theta$$

Now the block will start moving up. Let it stop at position (3). Now consider position (2) and (3), Consider the (block + spring) system.

Now work done by external force = Change in total energy of the system.

Let at this position, there will be a stretch x_2 in the spring.



$$W_{(external)} = \Delta E$$

$$W_{friction} = \Delta E_{(gravitational)} + \Delta E_{(spring)}$$

$$-\mu mg\cos\theta(x_1 - x_2) = mgy + \left[\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right]$$

$$-\mu mg\cos\theta(x_1 - x_2) = mg(x_1 - x_2)\sin\theta - \frac{1}{2}k(x_1 - x_2)(x_1 + x_2)$$

$$-\mu mg\cos\theta = mg\sin\theta - \frac{1}{2}k(x_1 + x_2)$$

$$\frac{1}{2}k(x_1 + x_2) = mg(\sin\theta + \mu\cos\theta)$$

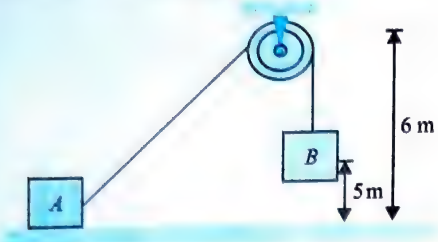
$$(x_1 + x_2) = \frac{2mg(\sin\theta + \mu\cos\theta)}{k}$$

$$x_2 = \frac{2mg(\sin\theta + \mu\cos\theta)}{k} - x_1$$

$$= \frac{2 \times 1 \times 10\left(\frac{3}{5} + 0.2 \times \frac{4}{5}\right)}{110} - 0.08 = \frac{64}{11}\text{ cm}$$

EXAMPLE 8.3

A block A of mass m is held at rest on a smooth horizontal floor. A light frictionless, small pulley is fixed at a height of 6 m from the floor. A light inextensible string of length 16 m, connected with A passes over the pulley and another identical block B is hung from the string. The initial height of B is 5 m from the floor as shown in figure.

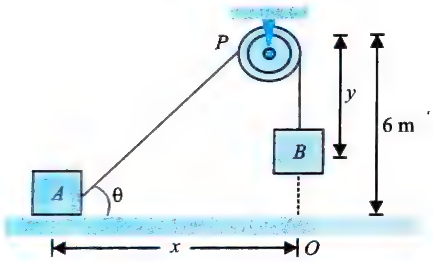


When the system is released frtom rest, B starts to move vertically downwards and A slides on the floor towards right.

(i) If at an instant string makes an angle θ with horizontal, calculate relation between velocity u of A and v of B.

(ii) Calculate v when B strikes the floor ($g = 10\ \text{m s}^{-2}$).

Sol. Let distance of block A from foot O of pulley be x and let the depth of block B from pulley by y when string makes angle θ with horizontal as shown in figure.



Then length, $AP = \sqrt{x^2 + 6^2} = \sqrt{36 + x^2}$

But $AP + BP = 16 \Rightarrow \sqrt{36 + x^2} + y = 16$

Differentiating above equation w.r.t. time,

$$\frac{x}{\sqrt{36 + x^2}} \cdot \frac{dx}{dt} + \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = v \text{ and } \frac{dx}{dt} = -u$$

$$-u.\cos\theta + v = 0 \Rightarrow u = v\sec\theta$$

When B strikes the floor, $y = 16$ m

$$AP = 16 - 6 = 10 \text{ m}$$

Hence, $\sin\theta = \dfrac{6}{16} \Rightarrow \theta = 37°$

Now applying conservation of mechanical energy, $\Delta K + \Delta U = 0$

Or gain in kinetic energy of system = loss in potential energy of the system

At that instant, kinetic energy of the two blocks = loss of potential energy to B

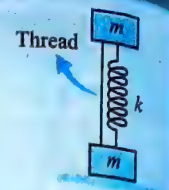
$$\frac{1}{2}mu^2 + \frac{1}{2}mv^2 = mg \times 5$$

where $u = v \sec 37°$

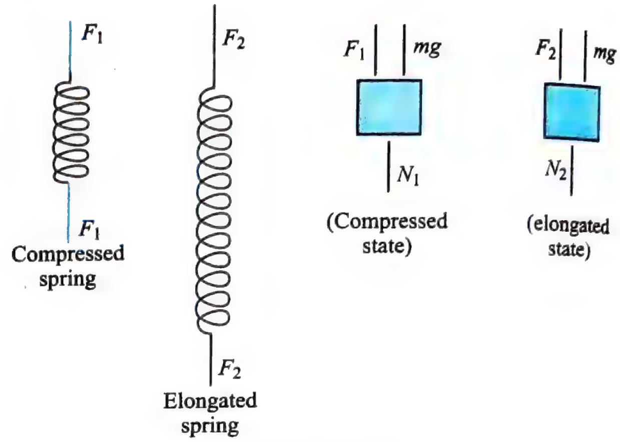
Hence, $v = \dfrac{40}{\sqrt{41}}\ \text{m s}^{-1}$

EXAMPLE 8.4

The figure shows two identical blocks, each of mass m, connected together by a weightless spring of stiffness constant k. The two cubes are also connected by a thread which is burned at a certain moment. Find the value of Δl, the initial compression of the spring, for which the lower block would bounce after the thread has been burnt.



Sol. In the given figures, the direction of restoring forces generated in the spring during compressed and elongated states, respectively, are shown.



F.B.D. of lower block

When the spring is in compressed state, the restoring forces press the lower block down. However, when in elongated state the spring lifts (or has a tendency to lift) the block.

After burning the thread, let F_2 be the minimum restoring force generated in the elongated spring. Then for the lower block to bounce.

$$F_2 = mg \quad \text{...(i)}$$

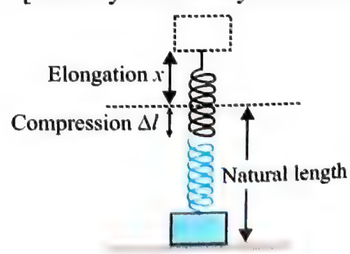
Let x be the elongation.

Then, $kx = F_2 \Rightarrow x = \dfrac{mg}{k}$ [From (i)]

Now, from conservation of mechanical energy,

$$\frac{1}{2}k(\Delta l)^2 = \frac{1}{2}kx^2 + mg(\Delta + x)$$

[Initially and finally the two blocks are at rest]



Putting, $x = \dfrac{mg}{k}$, we have $\dfrac{k}{2}(\Delta l)^2 = \dfrac{k}{2}\left(\dfrac{m^2g^2}{k^2}\right) + mg\left(\Delta l + \dfrac{mg}{k}\right)$

$$\Rightarrow k^2(\Delta l)^2 = m^2g^2 + 2mg\,k\,\Delta l\,k + 2m^2g^2$$

$$\Rightarrow k^2(\Delta l)^2 - (2mgk)\Delta l - 3m^2g^2 = 0$$

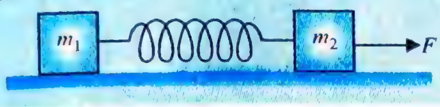
$$\Rightarrow \Delta l = \frac{2mgk \pm \sqrt{4m^2g^2k^2 + 12m^2g^2k^2}}{2k^2} = \frac{2mgk \pm 4mgk}{2k^2}$$

Taking positive sign, we get $\Delta l = \dfrac{3mg}{k}$

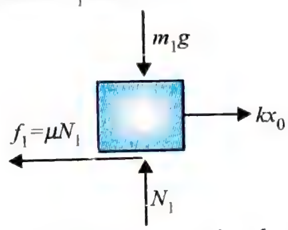
Thus, the initial compression Δl should be greater than or equal to $3mg/k$, for the lower block to bounce.

EXAMPLE 8.5

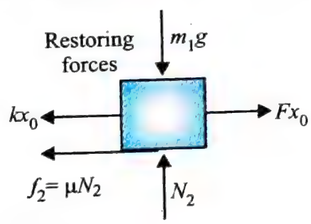
Two blocks of masses m_1 and m_2 connected by an undeformed massless spring rest on a horizontal plane. Find the minimum constant force F that has to be applied in the horizontal direction to the block of mass m_2, so that the other block gets shifted, if μ be the coefficient of friction between the blocks and the surface.



Sol. As the force F is applied on the block of mass m_2, it shifts the block towards right (if F exceeds the friction force acting on the block). This process elongates the spring. Consequently the restoring forces generated in the spring tends to move the block of mass m_1. If this restoring force exceeds the frictional (limiting) force, the block of mass m_1 moves.



Let x_0 be the minimum elongation in the spring so that the restoring forces generated in the spring is just able to overcome the frictional forces acting on block of mass m_1 and shifts it. Figure above shows the F.B.D. for block of mass m_1.



For vertical equilibrium, $m_1g = N_1$

And for horizontal motion to impend:

If k be the spring constant, then $kx_0 \geq f_1$

$\because \quad f_1 = \mu N_1 \quad \Rightarrow \quad kx_0 \geq \mu m_1 g$

But x_0 is the minimum elongation

$$\therefore \quad kx_0 = \mu m_1 g \qquad \text{...(i)}$$

For the block of mass m_2, the force F does the work in shifting by a distance x_0 against the frictional forces.

Besides, it also does work, in elongation the spring by a length x_0.

$$\therefore \quad Fx_0 = f_2 x_0 + \frac{1}{2}kx_0^2 = \mu(m_2 g)x_0 + \frac{1}{2}kx_0^2$$

$$\Rightarrow \quad F = \mu m_2 g + \frac{kx_0}{2} = \mu m_2 g + \frac{\mu m_2 g}{2} \quad \text{[From equation (i)]}$$

$$= \mu g\left(m_2 + \frac{m_1}{2}\right)$$

EXAMPLE 8.6

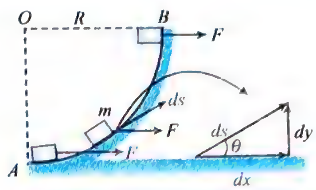
The figure below shows a smooth circular path of radius R on the horizontal plane which is quarter of a circle A block of mass m is taken from position A to B under the action of a constant force F. Calculate the work done by force F.

(a) If it is always directed horizontally.

(b) If the block is pulled by a force F which is always tangential to the surface.

(c) Block is pulled with a constant force F which is always directed towards the point B.

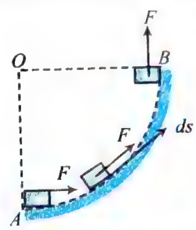
Sol.



(a) $W_R = \int \vec{F} \cdot d\vec{S} = \int F \, ds \cos\theta$

or $\quad W = \int_0^R F \, dx = FR$

As the block moves from A to B, the displacement of the block in the direction of force is equal to radius R.



Therefore, the work done by the constant force F is $W = FR$.

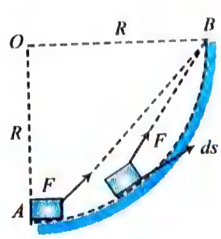
(b) If the block is pulled by a force F which is always tangential to the surface, in this case, force and displacement are always parallel to each other. The displacement of the block in the direction of force is $\dfrac{\pi}{2}R$.

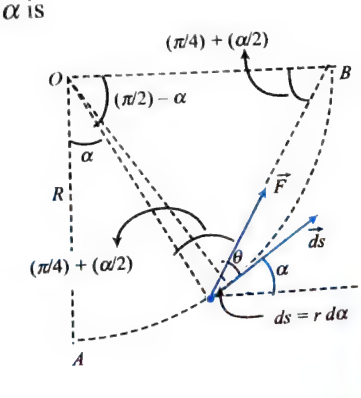
Thus, the work done by the force is

$$W = F\left(\frac{\pi R}{2}\right) = \frac{\pi}{2}FR$$

(c) Block is pulled with a constant force F which is always directed towards the point B. In this case, angle between force vector and displacement vector is varying.

In figure the angle between $\vec{F}$ and $d\vec{S}$ is θ. Block is at angle α from vertical The magnitude of ds is $R. d\alpha$ The relation between θ and α is





$$\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+\theta=\frac{\pi}{2}$$

$$\therefore \quad \theta=\frac{\pi}{4}-\frac{\alpha}{2}$$

Thus, $dW=\vec{F}\,d\vec{s}=Fds\cos\theta=F(R\,d\alpha)\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)$

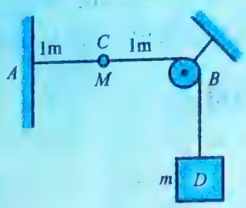
or $dW=\frac{FR}{\sqrt{2}}\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)d\alpha$

$$=\frac{FR}{\sqrt{2}}\left(\int_0^{\pi/2}\cos\frac{\alpha}{2}d\alpha+\int_0^{\pi/2}\sin\frac{\alpha}{2}d\alpha\right)$$

or $W=FR\sqrt{2}$

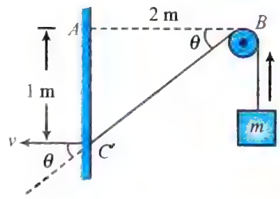
EXAMPLE 8.7

A string with one end fixed on a rigid wall, passing over a fixed frictionless pulley at a distance of 2 m from the wall, has a point mass M of 2 kg attached to it at a distance of 1 m from the wall. A mass m of 0.5 kg is attached to the free end. The system is initially held at rest so that the string is horizontal between wall and pulley and vertical beyond the pulley as shown in figure.



What will be the speed with which point mass M will hit the wall when the system is released? ($g = 10\ \text{m s}^{-2}$)

Sol. When M strikes the wall, the vertically downward component of its displacement from initial position is 1 m and its distance from pulley B is, $C'B=\sqrt{1^2+2^2}=\sqrt{5}\,\text{m}$.



while its initial distance from the pulley was $CB = 1$ m. It means vertically upward displacement of mass m is$(\sqrt{5}-1)$ m,

Let M strike the wall with velocity v, Since, string between two blocks always remains taut, therefore at any instant speed of m is equal to that component of velocity of M which is along the string $C'B$. Hence, the velocity of m when M strikes the wall is $v = \cos\theta$, where

$$\cos\theta=\frac{2}{\sqrt{5}}$$

According to the law of conservation of energy the loss of potential energy of M

= Increase in PE of m + KE of M = KE of m

$$Mg\times 1=mg(\sqrt{5}-1)+\frac{1}{2}Mv^2+\frac{1}{2}m(v\cos\theta)^2$$

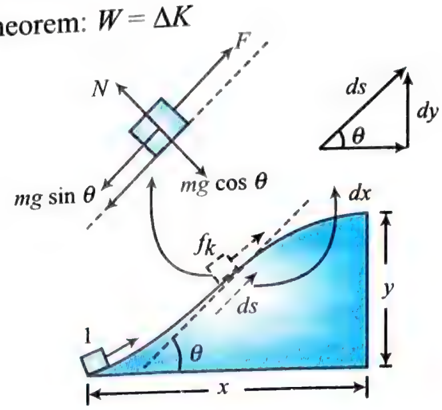
$$v=5\sqrt{\frac{5-\sqrt{5}}{6}}\ \text{m s}^{-1}=3.39\ \text{m s}^{-1}$$

EXAMPLE 8.8

A block of mass m is slowly pulled along a curved surface from position 1 and 2. If the coefficient of kinetic friction between the block and surface is m, find the work done by the applied force.

Sol. The forces acting on the block are gravit
$mg\downarrow, N\nwarrow, f_k\swarrow$, and $F\nearrow$.

Using W-E theorem: $W = \Delta K$



$$W_N+W_{gr}+W_N+W_f=\Delta K \quad ...($$

For slowly shifting the body,

$$\Delta K=0 \quad ...(i$$

Work done normal reaction

$$W_N=\vec{N}\cdot d\vec{s}=0 \qquad (\because N\perp ds) \quad ...(iii$$

Work done by the gravity

$$W_{gr}=-mgy\ (\because \text{ the block moves up}) \quad ...(iv$$

Using equations (i), (ii), (iii), and (iv),

$$W_F+W_f=mgy \quad ...(v$$

$$W_f=\int\vec{f}_k\cdot d\vec{s}$$

$$=\int f_k ds\cos 180^\circ$$

$$=-\int f_k ds=-\int\mu N\,ds$$

$$=-\int\mu\,(mg\cos\theta)\,ds$$

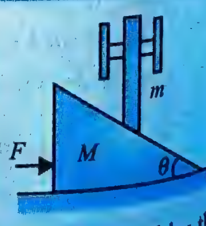
$$=-\mu mg\int ds\cos\theta \quad (ds\cos\theta=dx)$$

$$=-\mu mg\int_0^x dx=-\mu mgx \quad ...(vi)$$

Using equations (v) and (vi), $W_F = mg\,(\mu x + y)$

EXAMPLE 8.9

A vertical rod of mass m is kept on a wedge of mass M. If a horizontal force F acts on the wedge and the rod is constrained to move vertically, after releasing the rod-wedge system, (a) find their speeds when the wedge moves through a distance x. (b) What is the power delivered by the rod on the wedge after a time t measured from the instant of release?



(a) Let the displacement of rod is y when wedge moves a distance x,

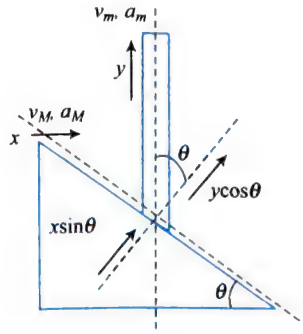
$x\sin\theta = y\cos\theta \Rightarrow x = y\tan\theta$

Also $v_m = v_M\tan\theta$ and $a_m = a_M\tan\theta$ ...(i)

Apply work–energy theorem: $W = \Delta KE$

$$\Rightarrow \quad Fx - mgy = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_m^2$$

$$\Rightarrow \quad (F - mg\tan\theta)x = \frac{1}{2}Mv_M^2 + \frac{1}{2}mv_M^2\tan^2\theta$$

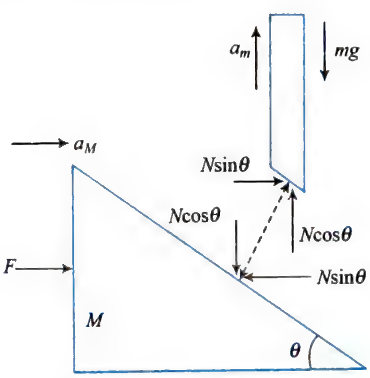


$$\Rightarrow \quad v_M = \sqrt{\frac{2[F - mg\tan\theta]x}{M + m\tan^2\theta}}$$

and

$$v_m = v_M\tan\theta = \sqrt{\frac{2[F - mg\tan\theta]x}{M\cot^2\theta + m}}$$

(b) From FBD of wedge and rod,



$F - N\sin\theta = Ma_M$ ...(ii)

$N\cos\theta - mg = ma_m$ ...(ii)

From (i), and (ii)

$$a_M = \frac{F - mg\tan\theta}{M + m\tan^2\theta}$$

$$N = \frac{(F\tan\theta + Mg)m}{(M + m\tan^2\theta)\cos\theta}$$

Velocity of M after time t,

$$v_M = a_M t$$

Power given by rod by wedge

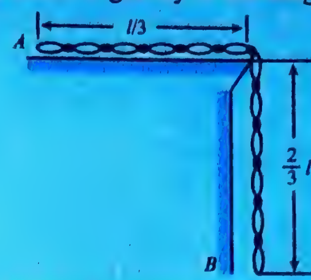
$$P = -(N\sin\theta)\, v_M$$

$$= -Na_M \sin\theta t$$

$$= \frac{-m\tan\theta[F\tan\theta + Mg][F - mg\tan\theta]}{[M + m\tan^2\theta]^2}$$

EXAMPLE 8.10

A uniform chain of mass m and length l is at the verge of sliding under the effect of gravity of the hanging part. Find the



(a) coefficient of friction, between chain and table.

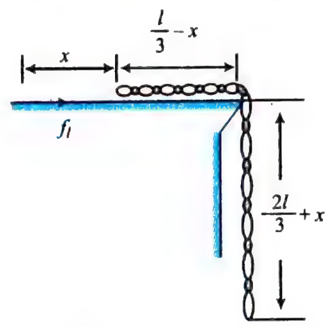
(b) work done by friction and gravity till the chain leaves the table if the hanging part is pulled gently and released.

(c) speed of the chain at the time of leaving the table in part (b).

(d) work done by the external force acting at the end A of the chain in slowly pulling the chain completely onto the table.

Sol.

(a) Limiting friction acting on the upper part of chain will balance the weight of overhanging part.



$$\mu\frac{m}{3}g = \frac{2m}{3}g \quad \Rightarrow \quad \mu = 2$$

(b) Let at any time, chain slides down by a distance x. Then

$$f_l = \mu\left[\left(\frac{l}{2} - x\right)\frac{m}{l}\right]g$$

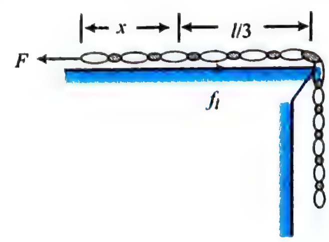
$$W_f = -\int f_l dx = -\frac{\mu mg}{l}\int_0^{l/3}\left(\frac{l}{3} - x\right)dx$$

$$\Rightarrow \quad W_f = -\frac{\mu mgl}{18} - \frac{2mgl}{18} = -\frac{mgl}{9}$$

$$\Rightarrow \quad W_{gravity} = \frac{2m}{3}g\frac{l}{3} + \frac{m}{3}g\frac{l}{6} = \frac{5mgl}{18}$$

(c) $W_{gravity} + W_f = \frac{1}{2}mv^2$

$$\Rightarrow \quad \frac{5mgl}{18} - \frac{mgl}{9} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{gl}{3}}$$



(d) $f_l = \mu\left(\dfrac{l}{3}+x\right)\dfrac{m}{l}g$

$$W_f = -\int f_l dx$$

$$= -\frac{\mu mg}{l}\int_0^{2l/3}\left(\frac{l}{3}+x\right)dx$$

$$= -\frac{4\mu mgl}{9} = -\frac{8mgl}{9}$$

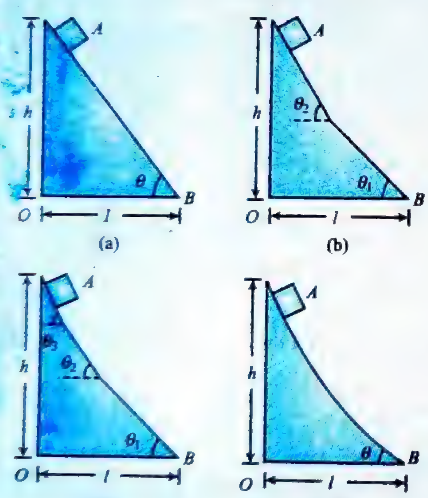
$$W_{\text{gravity}} = -\frac{2m}{3}g\frac{l}{3} = -\frac{2mgl}{9}$$

$$W_{\text{ext}} + W_f + W_{\text{gravity}} = 0$$

$$\Rightarrow \quad W_{\text{ext}} = \frac{8mgl}{9}+\frac{2mgl}{9} = \frac{10mgl}{9}$$

EXAMPLE 8.11

Block of mass m are released from rest and they slide down the inclined surfaces as shown in figure (a)–(d). Show that the work done by the frictional forces on the blocks are same in all the cases. Also calculate the speeds with which the blocks reach point B. The coefficient of friction for all the surfaces is μ_k.



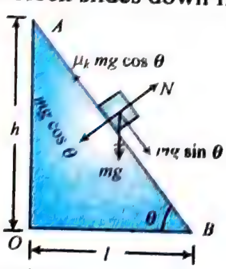
(a) (b) (c) (d)

Sol.

For Fig. (a):

Frictional force on the block $= \mu_k N = \mu_k\, mg \cos\theta$

If S be the length of the incline, work done by the frictional force on the block when the block slides down from A to B,



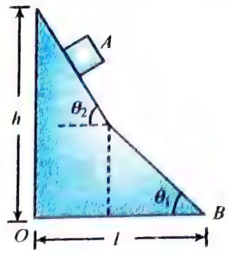
$W_{fl} = (\mu_k\, mg \cos\theta)\,(s)$

{Force and displacement are antiparallel}

$= -\mu_k\, mg\,(s \cos\theta) = -\mu_k\, mg$

For Fig. (b):

Frictional forces on the block at the planes with the inclination θ_1 and θ_2 are $\mu_k\, mg \cos\theta_1$ and $\mu_k \cos\theta_2$. If the lengths of the planes be S_1 and S_2, work done by the frictional force on the block,



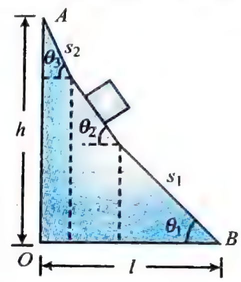
$l = s_1 \cos\theta_1 + s_2 \cos\theta_2$

$$W_{fk} = -(\mu_k\, mg \cos\theta_1)s_1 - (\mu_k\, mg \cos\theta_2)s_2$$

$$= -\mu_k\, mg(s_1 \cos\theta_1 + s_2 \cos\theta_2) = -\mu_k\, mgl$$

For Fig. (c)

Let the length of the inclines with inclination θ_1, θ_2, and θ_3 be s_1, s_2, and s_3, respectively. At these planes, the frictional force on the block are $\mu_k\, mg\cos\theta_1$, $\mu_k\, mg\cos\theta_2$, and $\mu_k\, mg\cos\theta_3$, respectively. Work done by the frictional force on the block,



$$W_{fk} = -(\mu_k\, mg \cos\theta_1)s_1 - (\mu_k\, mg \cos\theta_2)s_2 - (\mu_k\, mg \cos\theta_3)s_3$$

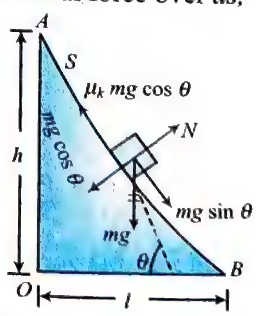
$$= -\mu_k\, mg\,(s_1 \cos\theta_1 + s_2 \cos\theta_2 + s_3 \cos\theta_3) = -\mu_k\, mgl.$$

For Fig. (d):

Let the block be at point C where the tangent to the parabola makes an angle θ with the horizontal as shown in the figure below. Frictional force on the block at

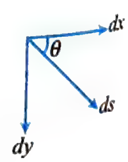
$C = \mu_k mg \cos\theta$, as shown. Consider an infinitesimal displacement ds at C.

Work done by the frictional force over ds,



$$dW = \mu_1\, mg \cos\theta\, ds$$

Work done from A to B,



$$W_{fk} = \int_A^B dW \int_A^B -\mu_k mg\cos\theta\, ds$$

$$= -\int_A^B \mu_k mg\, dx = -\mu_k mgl$$

Calculations show that the work done by the frictional force in each case are same.

Speed at B:

Using work-energy theorem.

$$W_{mg} = W_{f_k} = K_2 - K_1$$

$$\Rightarrow \quad mgh - m_k\, mgl = \frac{1}{2}mv^2 = 0$$

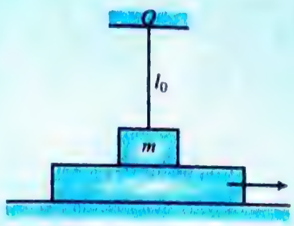
(Block is at rest at A, $K_1 = 0$)

$$\Rightarrow \quad 2g(h - m_k l) = v^2 \quad \Rightarrow \quad v = \sqrt{2g(h - \mu_k l)}$$

In each case, the block will reach B with same speed, which equals $\sqrt{2g(h - \mu_k l)}$.

EXAMPLE 8.12

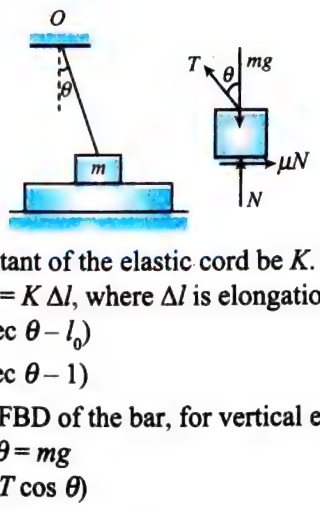
A horizontal plane supports a plank with a bar of mass m placed on it and attached by a light elastic non-deformed cord of length l_0, to a point O. The coefficient of friction between the bar and the plank is equal to m. The plank is slowly shifted to the right until the bar starts sliding over it. It occurs at the moment when the cord deviates from the vertical by an angle θ.



Find the work that has been performed by the moment by the frictional force acting on the bar in the reference frame fixed to the plane.

Sol. When the plank is pulled to the right, bar also moves to the right with the plank, and elastic cord elongates. Hence, a tension is developed in it.

Since the bar starts sliding over the plank when inclination of the elastic cord becomes equal to θ with the vertical, therefore at this moment, friction acting between the bar and the plank is at its limiting value.



Let the force constant of the elastic cord be K. Then tension in it will be equal to $T = K\,\Delta l$, where Δl is elongation of the cord. It is equal to $\Delta l = (l_0 \sec\theta - l_0)$

$$T = K.l_0(\sec\theta - 1)$$

Now considering FBD of the bar, for vertical equilibrium,

$$N + T\cos\theta = mg$$

$$N = (mg - T\cos\theta)$$

For horizontal equilibrium, $\mu N = T\sin\theta$

$$\mu(mg - T\cos\theta) = T\sin\theta$$

Substituting, $T = K.l_0(\sec\theta - 1)$

$$K = \frac{\mu mg}{l_0(\sec\theta - 1)(\sin\theta + \mu\cos\theta)}$$

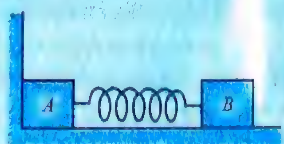
Since there is no slipping between the bar and plank up to this moment, therefore no energy is lost against friction.

Since the only force responsible for rightward displacement of the bar and hence for elongation of the cord is friction, therefore whole of the work done by the friction is used to elongate the cord. In fact it is stored in the cord in the form of its deformation energy. Hence, the work that has been performed by the moment by the frictional force acting on the bar in the reference frame fixed to the plane is equal to $\frac{1}{2}K(\Delta l)^2$.

$$\Rightarrow \quad W = \frac{1}{2}\frac{\mu mgl_0(\sec\theta - 1)}{(\sin\theta + \mu\cos\theta)}$$

EXAMPLE 8.13

Two identical blocks A and B, each of mass m = 2 kg are connected to the ends of an ideal spring having force constant K = 1000 N m^{-1}. System of these blocks and spring is placed on a rough floor. Coefficient of friction between blocks and floor is m = 0.5. Block B is passed towards left so that spring gets compressed.



Calculate initial minimum compression x_0 of spring such that block A leaves contact with the wall when system is released.

Sol. Since, spring is initially compressed, therefore, on releasing the system, block B experiences a resultant force (to the right) and starts moving to the right.

Block A leaves contact with the wall when rightward force on it applied by spring becomes just equal to the force of friction between this block and floor. Let at that instant, elongation of spring (from its natural length) be x'; then tension in spring is $T = Kx' = 1000x'$ newton. Considering free body diagram of A at that instant.

$$V_1 = mg = 20 \text{ N}$$

$$1000.x' = \mu.\ V_1 = 0.5 \times 20$$

$$x' = 0.01 \text{ m}$$

Since spring was initially compressed by x_0 and it has an elongation of x', therefore, displacement of block B (upto this instant) $= x_0 = x'$

According to the law of conservation of energy,

Decrease in energy stored in the spring = Work done by block B against friction

$$\left[\frac{1}{2}Kx_0^2 - \frac{1}{2}K(x')^2\right] = \mu mg(x_0 + x')$$

Substituting x' = 0.01 m, K = 1000 N m^{-1}, μ = 0.5, n = 2 kg and x_0 = 0.03 cm.

If initial compression of spring is $x = 2\,x_0$ = 6 cm, initial energy stored in spring is

$$U_0 = \frac{1}{2}K(2x_0)^2 = 1.80 \text{ J}$$

When block A loses contact with the wall, at that instant, elongation of spring (from natural length) is $x' = 1$ cm.

At that instant energy stored in the spring is,

$$U = \frac{1}{2}K(x')^2 = 0.05 \text{ J}$$

Decrease in energy stored in the spring is,

$$(U_0 - U) = 1.75 \text{ J}$$

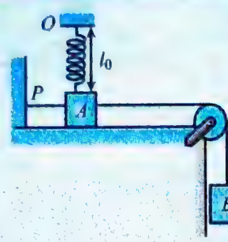
This energy is used in two ways.

To over come frictional loss during motion of block. But displacement of block B is $(2x_0 + x') = 7$cm. Hence, energy lost against friction is

$$\mu mg = (2x_0 + x') = 0.7 \text{ J}$$

EXAMPLE 8.14

A small bar A resting on a smooth horizontal plane is attached by threads to a point P and by means of weightless pulley, to a weight B possessing the same mass as the bar itself. The bar is also attached to a point O by means of a light non-deformed spring of length $l_0 = 50$ cm and stiffness $k = mg/l_0$, where m is the mass of the bar. The thread PA having been burned, the bar starts moving to the right. Find its velocity at the moment when it is breaking off the plane.



Sol. Let θ be the angle between spring and vertical at the instant when block A breaks off the plane ($N = 0$).

$$\Rightarrow \quad \cos\theta = \frac{l_0}{l_o + x} \qquad \text{...(i)}$$

$$N + kx \cos\theta = mg$$

$$\Rightarrow \quad kx\cos\theta = mg \qquad (\text{as } N = 0) \qquad \text{...(ii)}$$

Let d is the distance covered by A and B till this instant and V is the speed acquired by A and B.

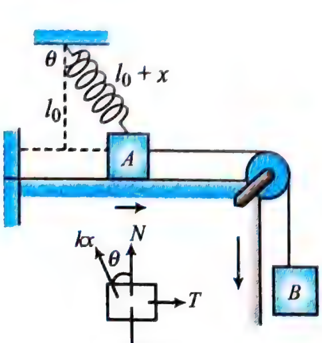
(same because they are connected)

From (i) and (ii), using $k = \dfrac{5mg}{l_0}$

$$\Rightarrow \quad \frac{5mg}{l_0} \times \frac{l_0}{l_0 + x} = mg$$

$$\Rightarrow \quad x = \frac{1}{4}l_0 \quad \Rightarrow \quad d = \sqrt{(l_o + x)^2 - l_0^2} = \frac{3l_o}{4}$$

Using energy conservation:



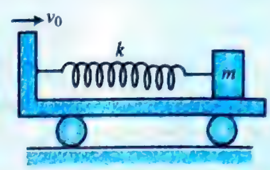
$$mgd = 2\left(\frac{1}{2}mv^2\right) + \frac{1}{2}kx^2$$

$$\Rightarrow \quad mg\frac{3l_0}{4} = mv^2 + \frac{1}{2}\frac{5mg}{l_0}\frac{l_0^2}{16}$$

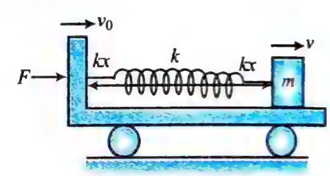
$$\Rightarrow \quad v^2 = \frac{3\,gl_0}{4} - \frac{5}{32}gl_0 \quad \Rightarrow \quad v = \sqrt{\frac{19gl_0}{32}}$$

EXAMPLE 8.15

A block of mass m is connected rigidly with a smooth wedge (plank) by a light spring of stiffness k. If the wedge is moved with constant velocity v_0, find the work done by the external agent till the maximum compression of the spring.



Sol. Let us take wedge + spring + block as a system. The forces responsible for performing work are spring force $kx(\leftarrow)$ and the external force $F(\rightarrow)$.



Applying W–E theorem for block + spring + plank relative to ground:

We have $W_{ext} + W_{sp} = \Delta k$,

where W_{sp} is the total work done by the spring on wedge and block $= -\dfrac{1}{2}kx^2$ and ΔK is the change in KE of the block (because the plank does not change its kinetic energy). Then

$$W_{ext} = \frac{1}{2}kx^2 + \Delta k$$

As the block was initially stationary, it will acquire a velocity v_0 equal to that of the plank at the time of maximum compression of the spring. The change in kinetic energy of the block relative to ground is

$$\Delta K = \frac{1}{2}mv_0^2$$

Substituting ΔK in the above equation, we have

$$W_{ext} = \frac{1}{2}kx^2 + \frac{1}{2}mv_0^2$$

Applying W–E theorem for block + spring + plank relative to the plank:

Now we need to find x. For this, let us climb onto the plank. Since the plank moves with constant velocity, there is no pseudo force acting on the block. Then the net work done on the system (block + plank) due to the spring can be given as

$$W_{sp} = -\frac{1}{2}kx^2$$

As the relative velocity between the observer (plank) and block decrease from v_0 to zero at the time of maximum compression of the spring, the change in kinetic energy of the block is

$\Delta K = -\frac{1}{2}mv_0^2$.

Applying work–energy theorem, we have

$$W = \Delta K, \text{ where } W = -\frac{1}{2}kx^2 \text{ and } \Delta K = -\frac{1}{2}mv_0^2$$

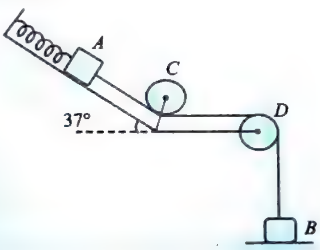
This gives $\frac{1}{2}kx^2 = \frac{1}{2}mv_0^2$

Substituting $\frac{1}{2}kx^2 = \frac{1}{2}mv_0^2$ from Eq. (ii) in Eq. (i), we have

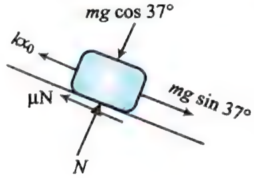
$$W_{ext} = mv_0^2$$

EXAMPLE 8.16

A block A of mass $m = 5$ kg is attached with a spring having force constant $K = 2000$ N m^{-1}. The other end of the spring is fixed to a rough plane, inclined at 37° with horizontal and having coefficient of friction $m = 0.25$. Block A is gently placed on the plane such that the spring has no tension. Then block A is released slowly. Calculate elongation of the spring when equilibrium is achieved. Now an inextensible thread is connected with block A and passes below pulley C and over pulley D, as shown in figure. Other end of the thread is connected with another block B of mass 3 kg. Block B is resting over a table and thread is loose. If the table collapses suddenly and B falls freely through 80/9 cm, the thread becomes taut, calculate combined speed of blocks at that instant, and maximum elongation of spring in the process of motion. ($g = 10$ m s^{-2}).



Sol. Since the block is released slowly, therefore, it starts to slide down the plane till equilibrium of forces is achieved.



Let at that instant elongation of spring be x_0, then tension in it is

$$T_0 = Kx_0 = 2000x_0$$

Considering free body diagram of block,

$$N = mg \cos 37° - 40 \text{ N} \quad \text{...(i)}$$

$$Kx_0 + \mu N = mg \sin 37° \quad \text{...(ii)}$$

From Eq. (i) and (ii),

$$x_0 = 0.01\ m = 1 \text{ cm}$$

When table collapses, first block B falls freely under gravity through height 80/9. Therefore, its speed just before the string becomes taut is

$$v_0 = \sqrt{2g \times \left(\frac{0.80}{9}\right)} = \frac{4}{3} \text{ m s}^{-1}$$

Now block A is jerked into motion and a large tension (for a very small time interval) is developed in string due to which both the blocks A and B experience numerically equal impulses.

Let its magnitud be J and let the combined speed of blocks be v.

Then for block A, $J = 5v$ ...(iii)

for block B, $3v_0 - J = 3v$ ...(iv)

From equations (iii) and (iv) $v = 0.5$ m s^{-1}

At the instant of maximum elongation of spring, blocks are momentarily at rest.

Let distance moved by the blocks be x from the instant when blocks A was jerked into motion to the instant of maximum elongation of the spring.

According to law of conservation of energy,

Loss of potential energy of A + Loss of potential energy of B + Loss of kinetic energy of blocks

= Increase in energy stored in spring + Work done by the block A against friction.

$$= 5.g(x \sin 37°) + 3gx + \left\{\frac{1}{2} \times (5v^2) + \frac{1}{2} \times (3v^2)\right\}$$

$$= \left\{\frac{1}{2}K(x_0 + x)^2 - \frac{1}{2}Kx_0^2\right\} + \mu Nx$$

$$x = 0.5 \text{ mm or } 5 \text{ cm}$$

Maximum elongation of spring = Its initial elongation (x_0) + Further elongation (x) = 6 cm

Exercises

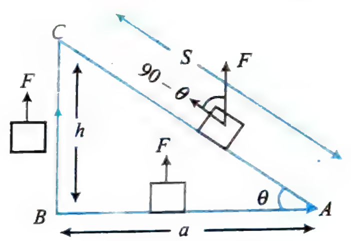
Single Correct Answer Type

1. In which of the following cases can the work done increase the potential energy?
   (1) Both conservative and non-conservative forces
   (2) Conservative force only
   (3) Non-conservative force only
   (4) Neither conservative nor non-conservative forces.

2. Work done by a conservative force on a system is equal to
   (1) The change in kinetic energy of the system
   (2) The change in potential energy of the system
   (3) The change in total mechanical energy of the system
   (4) None of the above

3. Which of the following statements is correct?
   (1) Kinetic energy of a system can be changed without changing its momentum.
   (2) Kinetic energy of a system cannot be changed without changing its momentum.
   (3) Momentum of a system cannot be changed without changing its kinetic energy.
   (4) A system cannot have energy without having momentum.

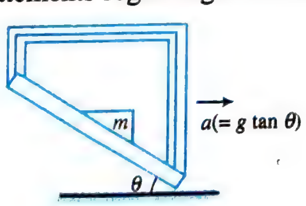
4. If we shift a body in equilibrium from A to C in a gravitational field via path AC or ABC,



   (1) The work done by the force $\vec{F}$ for both paths will be same
   (2) $W_{AC} > W_{ABC}$
   (3) $W_{AC} < W_{ABC}$
   (4) None of the above

5. A heavy weight is suspended from a spring. A person raises the weight till the spring becomes slack. The work done by him is W. The energy stored in the stretched spring was E. What will be the gain in gravitational potential energy?
   (1) W (2) E
   (3) $W + E$ (4) $W - E$

6. A block m is kept stationary on the surface of an accelerating cage as shown in figure. At the given instant, study the following statements regarding the block.

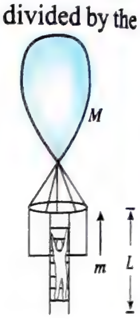


   (i) Normal reaction performs positive work on the block.
   (ii) Frictional work done on the block is negative.
   (iii) No net work is done by normal reaction and friction on the block.

   Now mark the correct answer.
   (1) Only statement (i) is correct.
   (2) Only statement (ii) is correct.
   (3) Only statement (iii) is correct.
   (4) All the statements are correct.

7. One end of an unstretched vertical spring is attached to the ceiling and an object attached to the other end is slowly lowered to its equilibrium position. If S is the gain in spring energy and G is the loss in gravitational potential energy in the process, then
   (1) $S = G$ (2) $S = 2G$
   (3) $G = 2S$ (4) None of these

8. A rope ladder of length L is attached to a balloon of mass M. As the man of mass m climbs the ladder into the balloon basket, the balloon comes down by a vertical distance s. Then the increase in potential energy of man divided by the increase in potential energy of balloon is

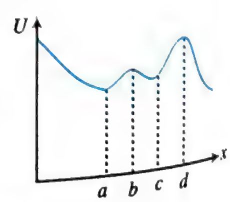


   (1) $\dfrac{L-s}{s}$
   (2) $\dfrac{L}{s}$
   (3) $\dfrac{s}{L-s}$
   (4) $L - s$

9. Two springs P and Q having stiffness constants k_1 and $k_2 (< k_1)$, respectively are stretched equally. Then
   (1) More work is done on Q
   (2) More work is done on P
   (3) Their force constants will become equal
   (4) Equal work is done on both the springs

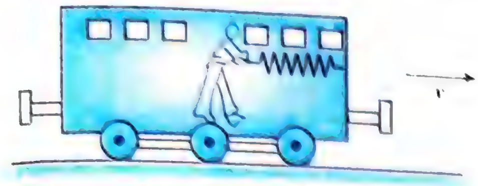
10. In the above question, if equal forces are applied on two springs, then
   (1) More work is done on Q
   (2) More work is done on P
   (3) Heir force constants will become equal
   (4) Equal work is done on both the springs

11. Figure below shows a plot of the potential energy as a function of x for a particle moving along the x-axis. Which of the following statement(s) is/are true?



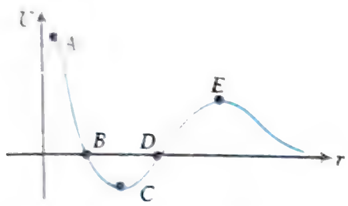
   (1) a, c, and d are points of equilibrium
   (2) a is a point of stable equilibrium
   (3) b is a stable equilibrium point
   (4) All of the above

12. A moving railway compartment has a spring of constant k fixed to its front wall. A boy stretches this spring by distance x and in the mean time the compartment moves by a distance s. The work done by boy w.r.t. earth is



(1) $\frac{1}{2}kx^2$
(2) $\frac{1}{2}(kx)(s+x)$
(3) $\frac{1}{2}kxs$
(4) $\frac{1}{2}kx(s+x+s)$

13. The given plot shows the variation of U, the potential energy of interaction between two particles, with the distance separating them, r:

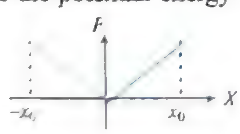


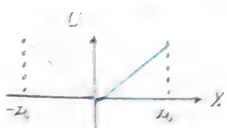
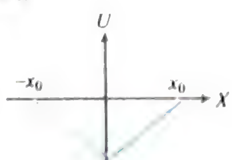
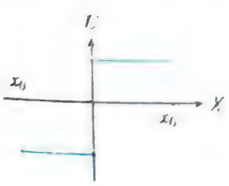
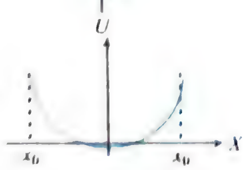
(i) B and D are equilibrium points.
(ii) C is a point of stable equilibrium.
(iii) The force of interaction between the two particles is attractive between points C and B, and repulsive between points D and E on the curve.
(iv) The force of interaction between the particles is repulsive between points C and A.

Which of the above statements are correct?

(1) (i) and (iii)
(2) (i) and (iv)
(3) (ii) and (iv)
(4) (ii) and (iii)

14. The figure below shows a plot of the conservative force F in a unidimensional field. The plot representing the function corresponding to the potential energy (U) in the field is



(1) 
(2) 
(3) 
(4) 

15. A spring is compressed between two toy carts of masses m_1 and m_2. When the toy carts are released, the spring exerts on each toy cart equal and opposite forces for the same small time t. If the coefficients of friction μ between the ground and the toy carts are equal, then the magnitude of displacements of the toy carts are in the ratio

(1) $\frac{s_1}{s_2}=\frac{m_2}{m_1}$
(2) $\frac{s_1}{s_2}=\frac{m_1}{m_2}$
(3) $\frac{s_1}{s_2}=\left(\frac{m_2}{m_1}\right)^2$
(4) $\frac{s_1}{s_2}=\left(\frac{m_1}{m_2}\right)^2$

16. The displacement x in meter of a particle of mass m kg moving in one dimension under the action of a force is related to the time t in second by the equation $x=(t-3)^2$. The work done by the force (in joules) in first six seconds is

(1) $18m$
(2) Zero
(3) $9m/2$
(4) $36m$

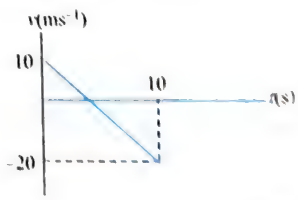
17. A mass M is lowered with the help of a string by a distance h at a constant acceleration $g/2$. The work done by the string will be

(1) $\frac{Mgh}{2}$
(2) $-\frac{Mgh}{2}$
(3) $\frac{3Mgh}{2}$
(4) $-\frac{3Mgh}{2}$

18. A bus can be stopped by applying a retarding force F when it is moving with speed v on a level road. The distance covered by it before coming to rest is s. If the load of the bus increases by 50% because of passengers, for the same speed and same retarding force, the distance covered by the bus to come to rest shall be

(1) $1.5s$
(2) $2s$
(3) $1s$
(4) $2.5s$

19. The velocity–time graph of a particle moving in a straight line is shown in figure. The mass of the particle is 2 kg. Work done by all the forces acting on the particle in time interval between $t = 0$ to $t = 10$ s is



(1) 300 J
(2) −300 J
(3) 400 J
(4) −400 J

20. A pump motor is used to deliver water at a certain rate from a given pipe. To obtain n times water from the same pipe in the same time, by what amount should the power of the motor be increased?

(1) n^2 times
(2) n^3 times
(3) n times
(4) $n^{1/2}$ times

21. The speed v reached by a car of mass m in travelling a distance x, driven with constant power P, is given by

(1) $v=\frac{3xP}{m}$
(2) $v=\left(\frac{3xP}{m}\right)^{1/2}$
(3) $v=\left(\frac{3xP}{m}\right)^{1/3}$
(4) $v=\left(\frac{3xP}{m}\right)^2$

22. Power supplied to a particle of mass 2 kg varies with t[illegible] as $P = 3t^2/2$ W. Here t is in second. If the velocity of partic[illegible] $t = 0$ is $v = 0$, the velocity of particle at time $t = 2$ s will

(1) $1\ \text{ms}^{-1}$ (2) $4\ \text{ms}^{-1}$
(3) $2\ \text{ms}^{-1}$ (4) $2\sqrt{2}\ \text{ms}^{-1}$

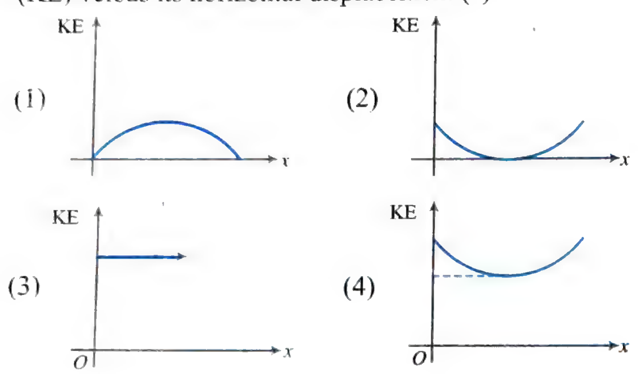
23. A body is moved along a straight line by a machine delivering a constant power. The distance moved by the body in time t is proportional to

(1) $\sqrt{t}$ (2) $t^{3/4}$
(3) $t^{3/2}$ (4) t^2

24. A car drives along a straight level frictionless road by an engine delivering constant power. Then velocity is directly proportional to

(1) t (2) $\dfrac{1}{\sqrt{t}}$
(3) $\sqrt{t}$ (4) None of these

25. A projectile is fired with some velocity making certain angle with the horizontal. Which of the following graphs is the best representation for the kinetic energy of a projectile (KE) versus its horizontal displacement (x)?



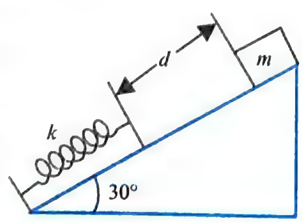
26. An object of mass m slides down a hill of arbitrary shape and after travelling a certain horizontal path stops because of friction. The total vertical height descended is h. The friction coefficient is different for different segments for the entire path but is independent of the velocity and direction of motion. The work that a tangential force must perform to return the object to its initial position along the same path is

(1) mgh (2) $-mgh$
(3) $-2mgh$ (4) $2mgh$

27. A toy gun uses a spring of force constant K. Before being triggered in the upward direction, the spring is compressed by a distance x. If the mass of the shot is m, on being triggered, it will go up to a maximum height of

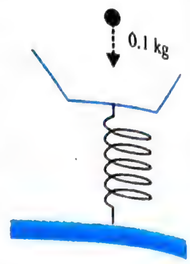
(1) $\dfrac{Kx^2}{mg}$ (2) $\dfrac{x^2}{Kmg}$
(3) $\dfrac{Kx^2}{2mg}$ (4) $\dfrac{K^2x^2}{mg}$

28. A block of 4 kg mass starts at rest and slides a distance d down a friction less incline (angle 30°) where it runs into a spring of negligible mass. The block slides an additional 25 cm before it is brought to rest momentarily by compressing the spring. The force constant of the spring is $400\ \text{N m}^{-1}$. The value of d is (take $g = 10\ \text{m s}^{-2}$)



(1) 25 cm (2) 37.5 cm
(3) 62.5 cm (4) None of the above

29. A massless platform is kept on a light elastic spring as shown in figure. When a particle of mass 0.1 kg is dropped on the pan from a height of 0.24 m, the particle strikes the pan, and the spring is compressed by 0.01 m. From what height should the particle be dropped to cause a compression of 0.04 m?

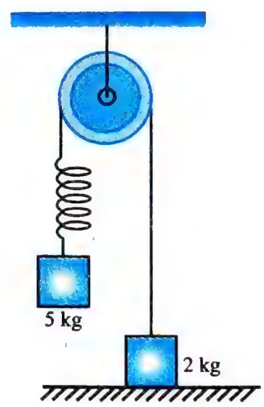


(1) 0.96 m (2) 2.96 m
(3) 3.96 m (4) 0.48 m

30. A particle is released one by one from the top of two inclined rough surfaces of height h each. The angles of inclination of the two planes are 30° and 60°, respectively. All other factors (e.g., coefficient of friction, mass of block, etc.) are same in both the cases. Let K_1 and K_2 be the kinetic energies of the particle at the bottom of the plane in the two cases. Then

(1) $K_1 = K_2$ (2) $K_1 > K_2$
(3) $K_1 < K_2$ (4) Data insufficient

31. The system shown in figure is released from rest with mass 2 kg in contact with the ground. Pulley and spring are massless, and friction is absent everywhere. The speed of 5 kg block when 2 kg block leaves the contact with the ground is (force constant of the spring $k = 40\ \text{N m}^{-1}$ and $g = 10\ \text{m s}^{-2}$)



(1) $\sqrt{2}\ \text{ms}^{-1}$ (2) $2\sqrt{2}\ \text{ms}^{-1}$
(3) $2\ \text{ms}^{-1}$ (4) $\sqrt{2}\ \text{ms}^{-1}$

32. A particle of mass m is projected at an angle a to the horizontal with an initial velocity u. The work done by gravity during the time it reaches its highest point is

(1) $u^2 \sin^2\alpha$ (2) $\dfrac{mu^2\cos^2\alpha}{2}$
(3) $\dfrac{mu^2\sin^2\alpha}{2}$ (4) $-\dfrac{mu^2\sin^2\alpha}{2}$

33. In the above question, the average power delivered by gravity is

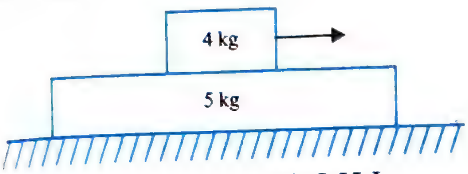
(1) $-mg\,u\cos\alpha$ (2) $-mgu\sin\alpha$
(3) $-\dfrac{mgu\cos\alpha}{2}$ (4) $-\dfrac{mgu\sin\alpha}{2}$

34. A particle located in a one-dimensional potential field has its potential energy function as $U(x) = \dfrac{a}{x^4} - \dfrac{b}{x^2}$, where a and b are positive constants. The position of equilibrium x corresponds to

(1) $\dfrac{b}{2a}$ (2) $\sqrt{\dfrac{2a}{b}}$

(3) $\sqrt{\dfrac{2b}{a}}$ (4) $\dfrac{a}{2a}$

35. A large slab of mass 5 kg lies on a smooth horizontal surface, with a block of mass 4 kg lying on the top of it. The coefficient of friction between the block and the slab is 0.25. If the block is pulled horizontally by a force of F = 6 N, the work done by the force of friction on the slab, between the instants t = 2 s and t = 3 s, is ($g = 10\ \text{m s}^{-2}$)



(1) 2.4 J (2) 5.55 J

(3) 4.44 J (4) 10 J

36. A particle of mass m slides along a curved-flat-curved track. The curved portions of the track are smooth. If the particle is released at the top of one of the curved portions, the particle comes to rest at flat portion of length l and of $\mu = \mu_{\text{kinetic}}$ after covering a distance of

(1) $\dfrac{l}{3\mu}$ (2) $\dfrac{H}{2\mu_{\text{kinetic}}}$

(3) $\dfrac{l}{6}$ (4) $\dfrac{H}{\mu_{\text{kinetic}}}$

37. A small block of mass 2 kg is kept on a rough inclined surface of inclination $\theta = 30°$ fixed in a lift. The lift goes up with a uniform speed of $1\ \text{m s}^{-1}$ and the block does not slide relative to the inclined surface. The work done by the force of friction on the block in a time interval of 2 s is

(1) Zero (2) 9.8 J

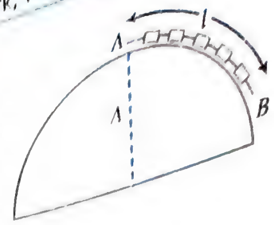
(3) 29.4 J (4) 16.9 J

38. A particle of mass m moves with a variable velocity v, which changes with distance covered x along a straight line as $v = k\sqrt{x}$, where k is a positive constant. The work done by all the forces acting on the particle, during the first t seconds is

(1) $\dfrac{mk^4}{t^2}$ (2) $\dfrac{mk^4t^2}{4}$

(3) $\dfrac{mk^4t^2}{8}$ (4) $\dfrac{mk^4t^2}{16}$

39. A uniform chain AB of mass m and length l is placed with one end A at the highest point of a hemisphere of radius R. Referring to the top of the hemisphere as the datum level, the potential energy of the chain is (given that $\left(\text{given that } l < \dfrac{\pi R}{2}\right)$



(1) $\dfrac{mR^2g}{l}\left(\dfrac{l}{R} - \sin\dfrac{l}{R}\right)$ (2) $\dfrac{mR^2g}{2l}\left(\dfrac{l}{R} - \sin\dfrac{l}{R}\right)$

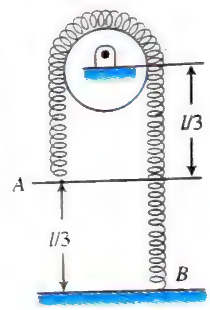
(3) $\dfrac{mR^2g}{2l}\left(\sin\dfrac{l}{R} - \dfrac{l}{R}\right)$ (4) $\dfrac{mR^2g}{l}\left(\sin\dfrac{l}{R} - \dfrac{l}{R}\right)$

40. Two discs, each having mass m, are attached rigidly to the ends of a vertical spring. One of the discs rests on a horizontal surface and the other produces a compression x_0 on the spring when it is in equilibrium. How much further must the spring be compressed so that when the force causing compression is removed, the extension of the spring will be able to lift the lower disc off the table?

(1) x_0 (2) $2x_0$

(3) $3x_0$ (4) $1.5x_0$

41. Two ends A and B of a smooth chain of mass m and length l are situated as shown in figure. If an external agent pulls A till it comes to same level of B, work done by external agent is



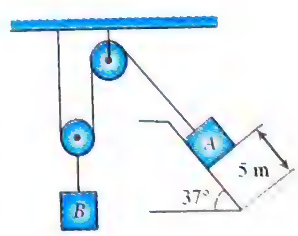
(1) $\dfrac{mgl}{36}$

(2) $\dfrac{mgl}{15}$

(3) $\dfrac{mgl}{9}$

(4) None of the above

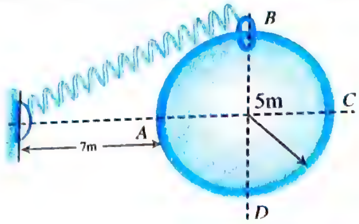
42. The blocks A and B shown in figure have masses $M_A = 5$ kg and $M_B = 4$ kg. The system is released from rest. The speed of B after A has travelled a distance 1 m along the incline is



(1) $\dfrac{\sqrt{3}}{2}\sqrt{g}$ (2) $\dfrac{\sqrt{3}}{4}\sqrt{g}$

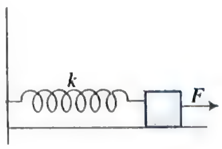
(3) $\dfrac{\sqrt{g}}{2\sqrt{3}}$ (4) $\dfrac{\sqrt{g}}{2}$

43. A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m. The spring lying in the plane of the circular track and having spring constant $200\ \text{N m}^{-1}$ is undeformed when the collar is at A. If the collar starts from rest at B, the normal reaction exerted by the track on the collar when it passes through A is



(1) 360 N (2) 720 N
(3) 1440 N (4) 2880 N

44. A block attached to a spring, pulled by a constant horizontal force, is kept on a smooth surface as shown in figure. Initially, the spring is in the natural length state. Then the maximum positive work that the applied force F can do is (given that string does not break)



(1) $\frac{F^2}{k}$ (2) $\frac{2F^2}{k}$

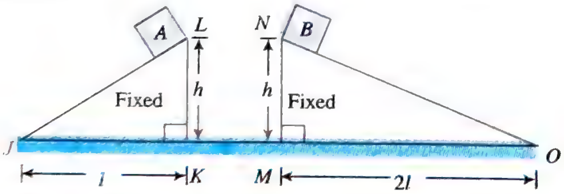
(3) ∞ (4) $\frac{F^2}{2k}$

45. A particle is projected along a horizontal field whose coefficient of friction varies as $\mu = A/r^2$, where r is the distance from the origin in metres and A is a positive constant. The initial distance of the particle is 1 m from the origin and its velocity is radially outwards. The minimum initial velocity at this point so the particle never stops is

(1) ∞ (2) $2\sqrt{gA}$

(3) $\sqrt{2gA}$ (4) $4\sqrt{gA}$

46. Two identical blocks A and B are placed on two inclined planes as shown in figure. Neglect resistance and other friction.



Read the following statements and choose options.

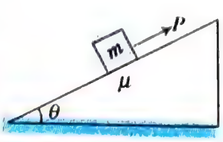
Statement I: The kinetic energy of A on sliding to J will be greater than the kinetic energy of B on sliding to O.

Statement II: The acceleration of A will be greater than acceleration of B when both are released on the inclined plane.

Statement III: The work done by external agent to move the block slowly from position B to O is negative.

(1) Only statement I is true
(2) Only statement II is true
(3) Only I and III are true
(4) Only II and III are true

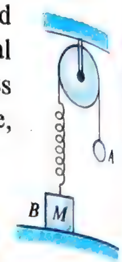
47. A block of mass m is being pulled up a rough incline by an agent delivering constant power P. The coefficient of friction between the block and the incline is μ. The maximum speed of the block during the course of ascent is



(1) $\frac{P}{mg\sin\theta + \mu\, mg\cos\theta}$ (2) $\frac{P}{mg\sin\theta - \mu\, mg\cos\theta}$

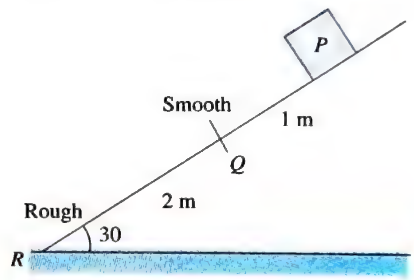
(3) $\frac{2P}{mg\sin\theta - \mu\, mg\cos\theta}$ (4) $\frac{3P}{mg\sin\theta - \mu\, mg\cos\theta}$

48. In the figure shown, the ball A is released from rest, when the spring is at its natural (unstretched) length. For the block B of mass M to leave contact with ground at some stage, the minimum mass of A must be



(1) $2M$ (2) M
(3) $M/2$ (4) $M/4$

49. A block of mass 5.0 kg slides down from the top of an inclined plane of length 3 m. The first 1 m of the plane is smooth and the next 2 m is rough. The block is released from rest and again comes to rest at the bottom of the plane. If the plane is inclined at 30° with the horizontal, find the coefficient of friction on the rough portion.



(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{\sqrt{3}}{2}$

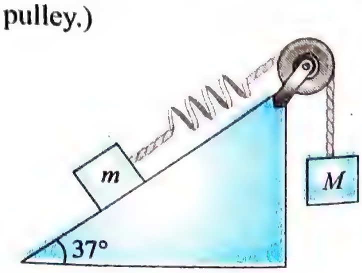
(3) $\frac{\sqrt{3}}{4}$ (4) $\frac{\sqrt{3}}{5}$

50. The potential energy for a force field $\vec{F}$ is given by $U(x, y) = \cos(x + y)$. The force acting on a particle at position given by coordinates $(0, \pi/4)$ is

(1) $-\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (2) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

(3) $\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$ (4) $\left(\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right)$

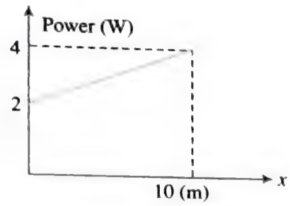
51. A block of mass m is attached with a massless spring of force constant k. The block is placed over a fixed rough inclined surface for which the coefficient of friction is $\mu = 3/4$. The block of mass m is initially at rest. The block of mass M is released from rest with spring in unstretched state. The minimum value of M required to move the block up the plane is (neglect mass of string and pulley and friction in pulley.)



(1) $\frac{3}{5}m$ (2) $\frac{4}{5}m$

(3) $\frac{6}{5}m$ (4) $\frac{3}{2}m$

52. A particle A of mass 10/7 kg is moving in the positive direction of x-axis. At initial position $x = 0$, its velocity is 1 m s^{-1}, then its velocity at $x = 10$ m is (use the graph given)



(1) 4 m s^{-1} (2) 2 m s^{-1}

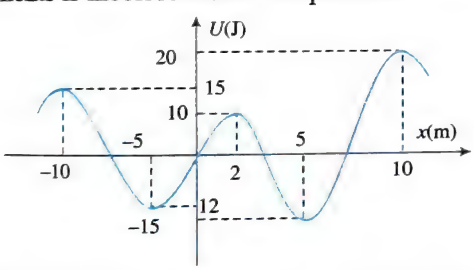
(3) $3\sqrt{2}$ m s^{-1} (4) $\frac{100}{3}$ m s^{-1}

53. A particle is projected vertically upwards with a speed of 16 m s^{-1}. After some time, when it again passes through the point of projection, its speed is found to be 8 m s^{-1}. It is known that the work done by air resistance is same during upward and downward motion. Then the maximum height attained by the particle is (take $g = 10$ m s^{-2})

(1) 8 m (2) 4.8 m

(3) 17.6 m (4) 12.8 m

54. In the figure below, the variation of potential energy of a particle of mass $m = 2$ kg is represented w.r.t its x-coordinate. The particle moves under the effect of the conservative force along the x-axis. Which of the following statements is incorrect about the particle?



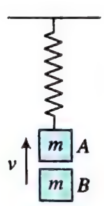
(1) If it is released at the origin, it will move in negative x-axis.

(2) If it is released at $x = 2 + \Delta$, where $\Delta \to 0$, then its maximum speed will be 5 m s^{-1} and it will perform oscillatory motion.

(3) If initially $x = -10$ and $\vec{u} = \sqrt{6}\,\hat{i}$, then it will cross $x = 10$.

(4) $x = -5$ and $x = +5$ are unstable equilibrium positions of the particle.

55. Block A is hanging from a vertical spring and is at rest. Block B strikes block A with velocity v and sticks to it. Then the value of v for which the spring just attains natural length is



(1) $\sqrt{\frac{60mg^2}{k}}$ (2) $\sqrt{\frac{6mg^2}{k}}$

(3) $\sqrt{\frac{10mg^2}{k}}$ (4) None of these

56. A machine delivers power to a body which is proportional to velocity of the body. If the body starts with a velocity which is almost negligible, then the distance covered by the body is proportional to

(1) $\sqrt{v}$ (2) $\sqrt[3]{\frac{v}{2}}$

(3) $v^{5/3}$ (4) v^2

57. The kinetic energy acquired by a mass m in travelling a certain distance d, starting from rest, under the action of a force F such that the force F is directly proportional to t is

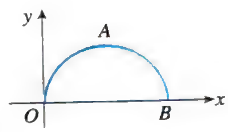
(1) Directly proportional to t^2

(2) Independent of t

(3) Directly proportional to t^4

(4) Directly proportional to t

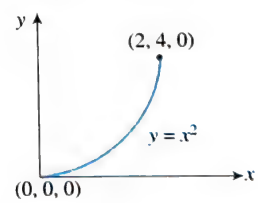
58. Given $\vec{F} = (xy^2)\hat{i} + (x^2y)\hat{j}$ N. The work done by $\vec{F}$ when a particle is taken along the semicircular path OAB where the coordinates of B are (4, 0) is



(1) $\frac{65}{3}$ J (2) $\frac{75}{2}$ J

(3) $\frac{73}{4}$ J (4) Zero

59. A force $\vec{F} = (3xy - 5z)\hat{j} + 4z\hat{k}$ is applied on a particle. The work done by the force when the particle moves from point (0, 0, 0) to point (2, 4, 0) as shown in figure is



(1) $\frac{280}{5}$ units (2) $\frac{140}{5}$ units

(3) $\frac{232}{5}$ units (4) $\frac{192}{5}$ units

60. The potential energy of a particle is determined by the expression $U = \alpha(x^2 + y^2)$, where α is a positive constant. The particle begins to move from a point with coordinates (3, 3), only under the action of potential field force. Then its kinetic energy T at the instant when the particle is at a point with the coordinates (1, 1) is

(1) 8α (2) 24α

(3) 16α (4) Zero

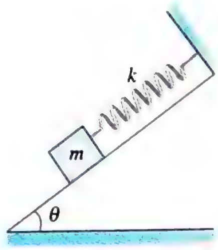
61. In the position shown in the figure, the spring is at its natural length. The block of mass m is given a velocity v_0 towards the vertical support at $t = 0$. The coefficient

of friction between the block and the surface is given by $\mu = \alpha x$, where α is a positive constant and x is the position of the block from its starting position. The block comes to rest for the first time at x, which is

(1) $v_0\sqrt{\dfrac{m}{k+\alpha mg}}$ (2) $v_0\sqrt{\dfrac{m}{k}}$

(3) $v_0\sqrt{\dfrac{m}{\alpha g}}$ (4) None of these

62. In the figure shown, a spring of spring constant k is fixed at one end and the other end is attached to the mass 'm'. The coefficient of friction between block and the inclined plane is 'μ'. The block is released when the spring is in its natural length. Assuming that $\tan\theta > \mu$, find the maximum speed of the block during the motion.



(1) $(\sin\theta + \mu\cos\theta)g\sqrt{\dfrac{m}{k}}$

(2) $(\sin\theta - \mu\cos\theta)g\sqrt{\dfrac{m}{k}}$

(3) $(\cos\theta - \mu\sin\theta)g\sqrt{\dfrac{m}{k}}$

(4) $(\cos\theta + \mu\sin\theta)g\sqrt{\dfrac{m}{k}}$

63. Let r be the distance of a particle from a fixed point to which it is attracted by an inverse square law force given by $F = k/r^2$ (k = constant). Let m be the mass of the particle and L be its angular momentum with respect to the fixed point. Which of the following formulae is correct about the total energy of the system?

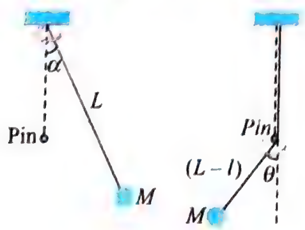
(1) $\dfrac{1}{2}m\left(\dfrac{dr}{dt}\right)^2 - \dfrac{k}{r} + \dfrac{L}{2mr^2} = \text{Constant}$

(2) $\dfrac{1}{2}m\left(\dfrac{dr}{dt}\right)^2 - \dfrac{k}{r} = \text{Constant}$

(3) $\dfrac{1}{2}m\left(\dfrac{dr}{dt}\right)^2 + \dfrac{k}{r} + \dfrac{L^2}{2mr^2} = \text{Constant}$

(4) None

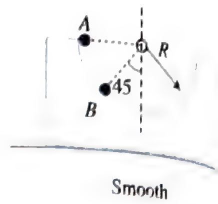
64. A simple pendulum consisting of a mass M attached to a string of length L is released from rest at an angle α. A pin is located at a distance l below the pivot point. When the pendulum swings down, the string hits the pin as shown in figure. The maximum angle θ which the string makes with the vertical after hitting the pin is



(1) $\cos^{-1}\left[\dfrac{L\cos\alpha + l}{L+l}\right]$ (2) $\cos^{-1}\left[\dfrac{L\cos\alpha + l}{L-l}\right]$

(3) $\cos^{-1}\left[\dfrac{L\cos\alpha - l}{L-l}\right]$ (4) $\cos^{-1}\left[\dfrac{L\cos\alpha - l}{L+l}\right]$

65. A ball of mass m is released from A inside a smooth wedge of mass m as shown in figure. What is the speed of the wedge when the ball reaches point B?



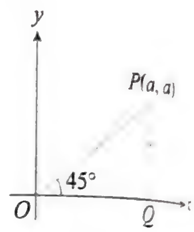
(1) $\left(\dfrac{gR}{3\sqrt{2}}\right)^{1/2}$

(2) $\sqrt{2gR}$

(3) $\left(\dfrac{5gR}{2\sqrt{3}}\right)^{1/2}$

(4) $\sqrt{\dfrac{3}{2}gR}$

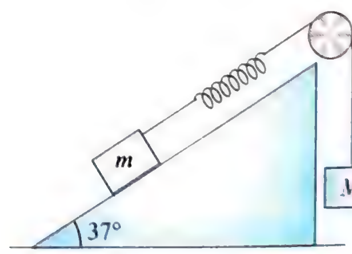
66. A particle is moved from (0, 0) to (a, a) under a force $\vec{F} = (3\hat{i} + 4\hat{j})$ from two paths. Path 1 is OP and path 2 is OQP. Let W_1 and W_2 be the work done by this force in these two paths. Then



(1) $W_1 = W_2$ (2) $W_1 = 2W_2$

(3) $W_2 = 2W_1$ (4) $W_2 = 4W_1$

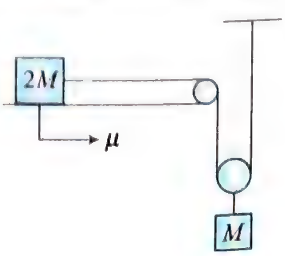
67. A block of mass m is attached with a massless spring of force constant k. The block is placed over a rough inclined surface for which the coefficient of friction is $\mu = 3/4$. The minimum value of M required to move the block up the plane is (neglect mass of string and pulley and friction is pulley).



(1) $\dfrac{3}{5}m$ (2) $\dfrac{4}{5}m$

(3) $2m$ (4) $\dfrac{3}{2}m$

68. The figure shows two blocks of masses $2M$ and M respectively. The coefficient of friction between the block of mass $2M$ and the horizontal plane is μ. The system is released from rest. Find the velocity of the block of mass M when the block of mass $2M$ has moved a distance s towards right.



(1) $\frac{1}{3}\sqrt{gs(1-4\mu)}$ (2) $\frac{1}{3}\sqrt{gs(1+4\mu)}$

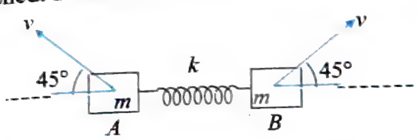
(3) $\frac{2}{3}\sqrt{gs(1-4\mu)}$ (4) None of these

69. A block attached with a spring is kept on a smooth horizontal surface. Now the free end of the spring is pulled with a constant velocity u horizontally. Then the maximum energy stored in the spring and block system during subsequent motion is

(1) (1/2) mu^2 (2) mu^2

(3) $2mu^2$ (4) $4mu^2$

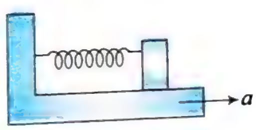
70. Blocks A and B of mass m each are connected with spring of constant k. Both blocks lie on frictionless ground and are imparted horizontal velocity v as shown when spring is unstretched. Find the maximum stretch of spring.



(1) $v\sqrt{\frac{m}{k}}$ (2) $v\sqrt{\frac{m}{2k}}$

(3) $v\sqrt{\frac{2m}{k}}$ (4) None of the above

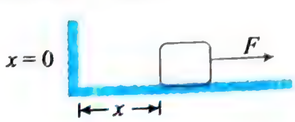
71. Find the maximum compression in the spring, if the lower block is shifted to rightwards with acceleration 'a'. All the surfaces are smooth:



(1) $\frac{ma}{2k}$ (2) $\frac{2ma}{k}$

(3) $\frac{ma}{k}$ (4) $\frac{4ma}{k}$

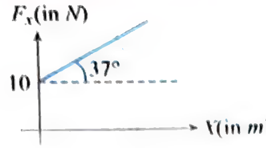
72. The block of mass m initially at $x = 0$ is acted upon by a horizontal force $F = a - bx^2$ (where $a > \mu mg$), as shown in the figure. The co-efficient of friction between the surfaces of contact is μ. The net work done on the block is zero, if the block travels a distance of

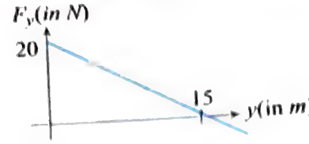


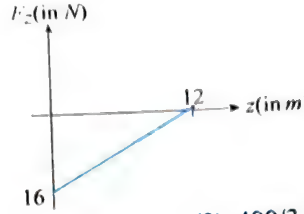
(1) $x=\sqrt{\frac{3(a-\mu mg)}{b}}$ (2) $x=\sqrt{\frac{3(a+\mu mg)}{b}}$

(3) $x=\sqrt{\frac{2(a-\mu mg)}{b}}$ (4) $x=\sqrt{\frac{2(a+\mu mg)}{b}}$

73. The components of a force acting on a particle are varying according to the graphs shown. When the particle moves from (0, 5, 12) to (4, 20, 0) then the work done by this force is



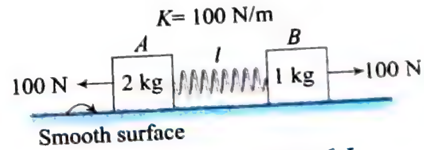




(1) 192 J (2) 400/3 J

(3) 0 (4) None of these

74. In the figure shown, initially spring is in unstretched state and blocks are at rest. Now 100 N force is applied on block A and B as shown in figure. After some time velocity of A becomes 2 m/s and that of B 4 m/s and block A displaced by amount 10 cm and spring is stretched by amount 30 cm. Then work done by spring force on A will be:



(1) 9/3 J (2) −6 J

(3) 6 J (4) None of these

75. In the figure shown, a spring of spring constant K is fixed at one end and the other end is attached to the mass 'm'. The coefficient of friction between block and the inclined plane is 'μ'. The block is released when the spring is in its natural length. Assuming that $\tan\theta > \mu$, the maximum speed of the block during the motion is

(1) $(\cos\theta+\mu\sin\theta)g\sqrt{\frac{m}{k}}$ (2) $(\cos\theta-\mu\sin\theta)g\sqrt{\frac{m}{k}}$

(3) $(\sin\theta+\mu\cos\theta)g\sqrt{\frac{m}{k}}$ (4) $(\sin\theta-\mu\cos\theta)g\sqrt{\frac{m}{k}}$

Multiple Correct Answers Type

1. Choose the correct statement(s) from the following.
   (1) Force acting on a particle for equal time intervals can produce the same change in momentum but different change in kinetic energy.
   (2) Force acting on a particle for equal displacements can produce same change in kinetic energy but different change in momentum.
   (3) Force acting on a particle for equal time intervals can produce different change in momentum but same change in kinetic energy.
   (4) Force acting on a particle for equal displacements can produce different change in kinetic energy but same change in momentum.

2. Mark the correct statement(s).
   (1) The work–energy theorem is valid only for particles
   (2) The work–energy theorem is an invariant law of physics.
   (3) The work–energy theorem is valid only in inertial frames of reference.
   (4) The work–energy theorem can be applied in non-inertial frames of reference too.

3. Mark the correct statement(s).
   (1) Total work done by internal forces of a system on the system is always zero.
   (2) Total work done by internal forces of a system on the system is sometimes zero.
   (3) Total work done by internal forces acting between the particles of a rigid body is always zero.
   (4) Total work done by internal forces acting between the particles of a rigid body is sometimes zero.

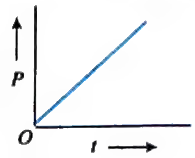
4. Select the correct option(s).
   (1) A single external force acting on a particle necessarily changes its momentum and kinetic energy.
   (2) A single external force acting on a particle necessarily changes its momentum.
   (3) The work–energy theorem is valid for all types of forces: internal, external, conservative as well as non-conservative.
   (4) The kinetic energy of the system can be increased without applying any external force on the system.

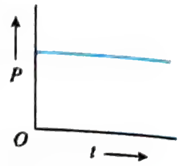
5. A block hangs freely from the end of a spring. A boy then slowly pushes the block upwards so that the spring becomes strain free. The gain in gravitational potential energy of the block during this process is not equal to
   (1) The work done by the boy against the gravitational force acting on the block
   (2) The loss of energy stored in the spring minus the work done by the tension in the spring
   (3) The work done on the block by the boy plus the loss of energy stored in the spring
   (4) The work done on the block by the boy minus the work done by the tension in the spring plus the loss of energy stored in the spring
   (5) The work done on the block by the boy minus the work done by the tension in the spring

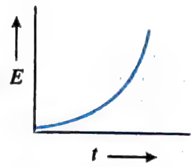
6. In which of the following case(s), no work is done by the force?
   (1) A man carrying a bucket of water, walking on a level road with a uniform velocity.
   (2) A drop of rain falling vertically with a constant velocity.
   (3) A man whirling a stone tied to a string in a circle with a constant speed.
   (4) A man walking upon a staircase.

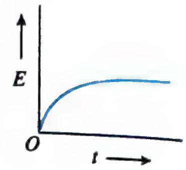
7. A particle is taken from point A to point B under the influence of a force field. Now it is taken back from B to A and it is observed that the work done in taking the particle from A to B is not equal to the work done in taking it from B to A. If W_{nc} and W_c are the work done by non-conservative and conservative forces present in the system, respectively, ΔU is the change in potential energy and Δk is the change in kinetic energy, then
   (1) $W_{nc} - \Delta U = \Delta k$
   (2) $W_c = -\Delta U$
   (3) $W_{nc} + W_c = \Delta k$
   (4) $W_{nc} - \Delta U = -\Delta k$

8. A vehicle is driven along a straight horizontal track by a motor which exerts a constant driving force. The vehicle starts from rest at $t = 0$ and the effects of friction and air resistance are negligible. If the kinetic energy of the vehicle at time t is E and power developed by the motor is P, which of the following graphs is/are correct?

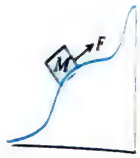
(1) 

(2) 

(3) 

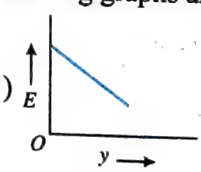
(4) 

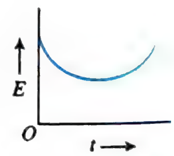
9. A body of mass M was slowly hauled up a rough hill by a force F which at each point was directed along a tangent to the hill. Work done by the force

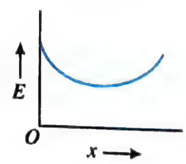


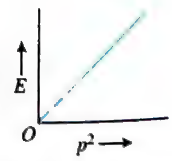
   (1) is independent of the shape of trajectory
   (2) depends upon the vertical component of displacement but is independent of horizontal component
   (3) depends upon both the components
   (4) does not depend upon the coefficient of friction

10. A particle is projected from a point at an angle with the horizontal at $t = 0$. At an instant t, if p is linear momentum, x is horizontal displacement, y is vertical displacement and E is kinetic energy of the particle, then which of the following graphs are correct?

(1) 

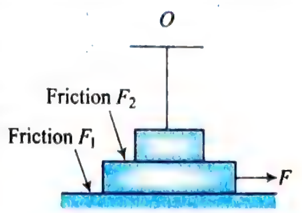
(2) 

(3) 

(4) 

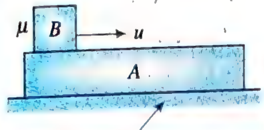
11. A block is suspended by an ideal spring of force constant k. If the block is pulled down by applying a constant force F and if maximum displacement of the block from its initial position of rest is δ, then
   (1) $\dfrac{F}{k} < \delta > \dfrac{2F}{k}$
   (2) $\delta = \dfrac{2F}{k}$
   (3) Work done by force F is equal to $F\delta$
   (4) Increase in energy stored in the spring is $\dfrac{1}{2}k\delta^2$

12. A horizontal plane supports a plank with a block placed on it. A light elastic string is attached to the block, which is attached to a fixed point O. Initially, the cord is unstretched and vertical. The plank is slowly shifted to right until the block starts sliding over it. It occurs at the moment when the cord deviates from vertical by an angle $\theta = 0°$. Work done by the force F equals



(1) Energy lost against friction F_1 plus strain energy in cord
(2) Work done against total friction acting on the plank alone
(3) Work done against total friction acting on the plank plus strain energy in the cord
(4) Work done against total friction acting on the plank plus strain energy in the cord minus work done by friction acting on the block

13. A long block A is at rest on a smooth horizontal surface. A small block B whose mass is half of mass of A is placed on A at one end and is given an initial velocity u as shown in the figure. The coefficient of friction between the blocks is μ. Choose the correct statement(s).

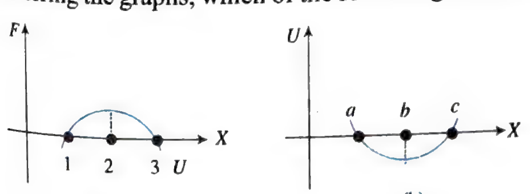


Smooth

(1) Finally both move with a common velocity $2u/3$.
(2) Acceleration of B relative to A initially is $3\mu g/2$ towards left.
(3) Magnitude of total work done by friction is equal to the final kinetic energy of the system.
(4) The ratio of initial to final momentum of the system is 1.

14. A body of mass 1 kg is taken from infinity to a point P. When the body reaches that point, it has a speed of 2 m s^{-1}. The work done by the conservative force is – 5 J. Which of the following is/are true (assuming non-conservative and pseudo-forces to be absent)?
(1) Work done by the applied force is +7 J.
(2) The total energy possessed by the body at P is +7 J.
(3) The potential energy possessed by the body at P is + 5 J.
(4) Work done by all forces together is equal to the change in kinetic energy.

15. Referring the graphs, which of the following is/are correct?

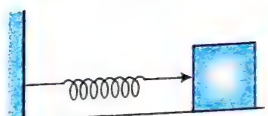


(a) (b)

(1) The particle has stable equilibrium at points 3 and b.
(2) The particle is in neutral equilibrium at points b and 2.
(3) No power is delivered by the force on the particle at points 1, 3, and b.
(4) The particle has least kinetic energy at position 1.

16. The potential energy ϕ, in joule, of a particle of mass 1 kg, moving in the x-y plane, obeys the law $\phi = 3x + 4y$, where (x, y) are the coordinates of the particle in metre. If the particle is at rest at (6, 4) at time $t = 0$, then
(1) The particle has constant acceleration.
(2) The work done by the external forces, the position of rest of the particle and the instant of the particle crossing the x-axis is 25 J.
(3) The speed of the particle when it crosses the y-axis is 10 m s^{-1}.
(4) The coordinates of the particle at time $t = 4$ s are (−18, −28).

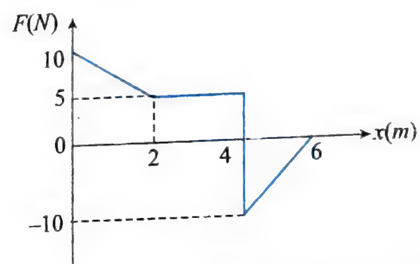
17. A block of mass 1 kg is pressed against a spring of force constant 400 N/m. The spring is compressed by 10 cm and block is released. Which of the following is a possible velocity of the block during subsequent motion?



Smooth level ground

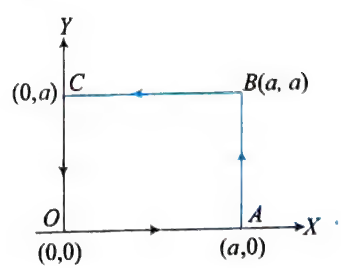
(1) 2 m/s
(2) 1 m/s
(3) 3 m/s
(4) 4 m/s

18. A particle is acted upon by a force, which varies with position x as shown in the figure. Then:



(1) work done by force F between $X = 0$ to $X = 2$ m is + 15 J
(2) work done by force F between $X = 4$ m to $X = 6$ m is −10 J
(3) work done by force F between $X = 4$ m to $X = 6$ m is +10 J
(4) Change in KE between $X = 0$ to $X = 6$ m is 50 J

19. Regarding the work done by the force $\vec{F} = x^2\hat{i} + y^2\hat{j}$ around the closed path $OABCO$ shown in the figure which is/are correct?



(1) zero on the entire closed path
(2) same along OA and AB
(3) same along BC and CO
(4) same along AB and BC

20. A moving particle is acted by several forces $\vec{F}_1, \vec{F}_2 \ldots$ etc. One of the force is chosen say $\vec{F}_2$, then which of the following statement about work done by $\vec{F}_2$ will be true.

(1) Work done by $\vec{F}_2$ will be negative if speed of particle is decreasing

(2) Work done by $\vec{F}_2$ will be positive if speed of particle is increasing

(3) Work done by $\vec{F}_2$ will be equal in magnitude to the sum of work done by all other forces if speed of particle remains constant

(4) If $\vec{F}_2$ is conservative force, then work done by all other forces will be equal to 'change in P.E. due to $\vec{F}_2$, if speed remains constant

21. A block of mass 1 kg kept on a rough horizontal surface (μ = 0.4) is attached to a light spring (force constant = 200 N/m) whose other end is attached to a vertical wall. The block is pushed to compress the spring by a distance d and released. Find the value(s) of 'd' for which (spring + block) system loses its entire mechanical energy in form of heat.

(1) 4 cm (2) 6 cm

(3) 8 cm (4) 10 cm

22. A man is standing on a plank which is placed on smooth horizontal surface. There is sufficient friction between feet of man and plank. Now man starts running over plank. The correct statements is/are

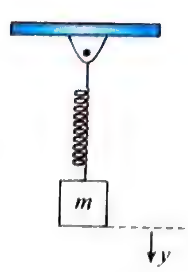
(1) Work done by friction on man with respect to ground is negative

(2) Work done by friction on man with respect to ground is positive

(3) Work done by friction on plank with respect to ground is positive

(4) Work done by friction on man with respect to plank is zero

23. A block suspended from a spring at natural length and is free to move vertically in the y-direction. Then



(1) Mass is released when $y = 0$; the maximum value of y reached by the mass m is $2mg/k$

(2) Mass is released when $y = 0$; the maximum value of y reached by the mass m is mg/k

(3) Due to air resistance the mass settles down into an equilibrium position y_{eq}, the mechanical energy lost is $\frac{1}{2}(m^2g^2/k)$

(4) Due to air resistance the mass settles down into an equilibrium position y_{eq}, the mechanical energy lost is (m^2g^2/k)

24. A ball of mass m is attached to the lower end of light vertical spring of force constant k. The upper end of the spring is fixed. The ball is released from rest with the spring at its normal (unstretched) length and comes to rest again after descending through a distance x.

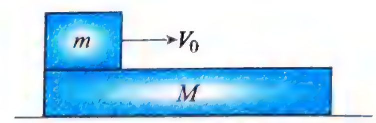
(1) $x = mg/k$

(2) $x = 2mg/k$

(3) The ball will have no acceleration at the position where it has descended through $x/2$.

(4) The ball will have an upward acceleration equal to g at its lowermost position

25. The coefficient of friction between the block and plank is μ and its value is such that the block becomes stationary with respect to plank before it reaches the other end. Then:



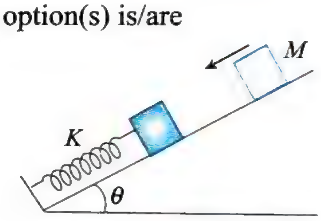
(1) The work done by friction on the block is negative.

(2) The work done by friction on the plank is positive.

(3) The net work done by friction is negative.

(4) Net work done by the friction is zero.

26. The correct option(s) is/are



[where x = compression]

(1) Gravitational potential energy is decreased and its magnitude of $mg\,x \sin\theta$.

(2) Elastic potential energy stored in spring = $1/2\; kx^2$

(3) Total potential energy of the system = $- mgx \sin\theta + 1/2\; kx^2$

(4) None of these

27. A block of mass 2 kg is hanging over a smooth and light pulley through a light string. The other end of the string is pulled by a constant force F = 40 N. The kinetic energy of the particle increase 40 J in a given interval of time. Then: (g = 10 m/s^2)

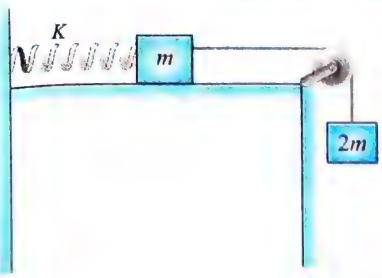
(1) tension in the string is 40 N

(2) displacement of the block in the given interval of time is 2 m

(3) work done by gravity is –20 J

(4) work done by tension is 80 J

28. Two blocks, of masses M and $2M$, are connected to a light spring of spring constant K that has one end fixed, as shown in figure. The horizontal surface and the pulley are frictionless. The blocks are released from rest when the spring is non-deformed. The string is light. Select the correct statement(s).



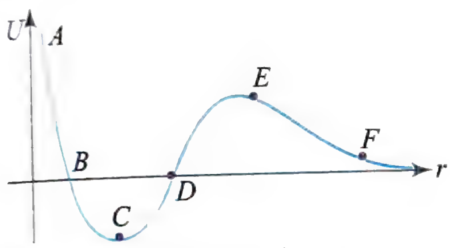
(1) Maximum extension in the spring is $\dfrac{4Mg}{K}$

(2) Maximum kinetic energy of the system is $\dfrac{2M^2g^2}{K}$

(3) Maximum energy stored in the spring is four times that of maximum kinetic energy of the system

(4) When kinetic energy of the system is maximum energy stored in the spring is $\dfrac{4M^2g^2}{K}$

The given plot shows the variation of U, the potential energy of interaction between two particles with the distance separating them, r. Then which of the following statement(s) is/are correct?



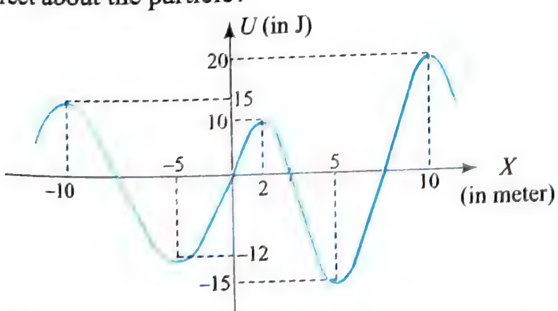
(1) B and D are equilibrium points

(2) C is a point of stable equilibrium

(3) The force of interaction between the two particles is attractive between points C and D and repulsive between points D and E on the curve

(4) The force of interaction between the particles is repulsive between points E and F on the curve

In the figure the variation of potential energy of a particle of mass $m = 2$ kg is represented w.r.t. its x-coordinate. The particle moves under the effect of this conservative force along the x-axis. Which of the following statement(s) is/are correct about the particle?



(1) If it is released at the origin, it will move in negative x-axis.

(2) If it is released at $x = 2 + \Delta$ where $\Delta \to 0$, then its maximum speed will be 5 m/s and it will perform oscillatory motion.

(3) If initially $x = -10$ m and $\vec{u} = \sqrt{6}\,\hat{i}$ m/s, then it will cross $x = 10$ m.

(4) $x = -5$ m and $x = +5$ m are unstable equilibrium positions of the particle.

Linked Comprehension Type

For Problems 1–4

A single conservative force $F(x)$ acts on a 1.0-kg particle that moves along the x-axis. The potential energy $U(x)$ is given by $U(x) = 20 + (x-2)^2$ where x is in meters. At $x = 5.0$ m, the particle has a kinetic energy of 20 J.

1. What is the mechanical energy of the system?
   (1) 35 J (2) 64 J
   (3) 86 J (4) 49 J

2. The maximum and minimum values of x, respectively, are
   (1) 7.38 m, –3.38 m (2) 6.38 m, –4.38 m
   (3) 7.38 m, –2.38 m (4) 6.38 m, –2.38 m

3. The maximum kinetic energy of the particle and the value of x at which maximum kinetic energy occurs are
   (1) 29 J, 0 m (2) 49 J, 0 m
   (3) 49 J, 2 m (4) 29 J, 2 m

4. Determine the equation of $F(x)$ as a function of x.
   (1) $F = 2 + x$ (2) $F = 2 + 3x$
   (3) $F = 2(2 - x)$ (4) $F = 3 + 2x$

For Problems 5 and 6

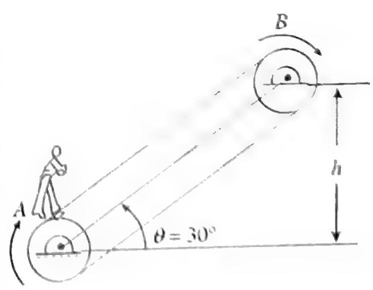
A 1.5-kg block is initially at rest on a horizontal frictionless surface when a horizontal force in the positive direction of x-axis is applied to the block. The force is given by $\vec{F} = (4 - x^2)\vec{i}$ N, where x is in meter and the initial position of the block is $x = 0$.

5. The maximum kinetic energy of the block between $x = 0$ and $x = 2.0$ m is
   (1) 2.33 J (2) 8.67 J
   (3) 5.33 J (4) 6.67 J

6. The maximum positive displacement x is
   (1) $2\sqrt{3}$ m (2) 2 m
   (3) 4 m (4) $\sqrt{2}$ m

For Problems 7–10

A boy of mass m climbs up a conveyor belt with a constant acceleration. The speed of the belt is $v = \sqrt{gh/6}$ and the coefficient of friction between the boy and conveyor belt is $\mu = \dfrac{5}{3\sqrt{3}}$. The boy starts from A and moves with the maximum possible acceleration till he reaches the highest point B.



7. The time taken by the boy to reach the height h is
   (1) $\sqrt{\dfrac{2h}{g}}$ (2) $\sqrt{\dfrac{6h}{g}}$
   (3) $2\sqrt{\dfrac{h}{g}}$ (4) None of above

8. Work done by gravity to w.r.t. the conveyor belt is

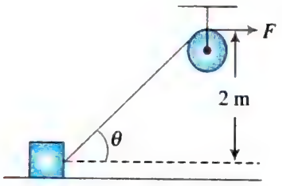
(1) $-mgh$
(2) $-\frac{1}{2}mgh$
(3) $\frac{1}{3}mgh$
(4) None of above

9. Work done by friction on the boy is
(1) Equal to work done by boy
(2) Equal to work done by the motor in running the conveyor belt
(3) Zero
(4) None of above

10. Work done by the boy is
(1) $\frac{5}{6}mgh$
(2) $\frac{1}{4}mgh$
(3) $\frac{4}{3}mgh$
(4) None of above

For Problems 11–13

A force $F = 50$ N is applied at one end of a string, the other end of which is tied to a block of mass 10 kg. The block is free to move on a frictionless horizontal surface. Take initial instant as $\theta = 30°$ and final instant as $\theta = 37°$. For the time between these two instants, answer the following questions?



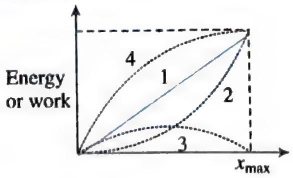
11. Net work done by the force F on the block is
(1) $\frac{50}{3}$ J
(2) $\frac{100}{3}$ J
(3) 75 J
(4) None of these

12. What is the final velocity of the block if initially it was at rest?
(1) $\sqrt{\frac{25}{3}}$ m s^{-1}
(2) 5 m s^{-1}
(3) $\sqrt{\frac{20}{3}}$ m s^{-1}
(4) None of these

13. Find the ratio of initial acceleration to final acceleration of the block.
(1) $\frac{3\sqrt{5}}{8}$
(2) $\frac{8\sqrt{3}}{5}$
(3) $\frac{8\sqrt{5}}{3}$
(4) $\frac{5\sqrt{3}}{8}$

For Problems 14–16

A spring lies along the x-axis attached to a wall at one end and a block at the other end. The block rests on a friction less surface at $x = 0$. A force of constant magnitude F is applied to the block that begins to compress the spring, until the block comes to a maximum displacement x_{max}.



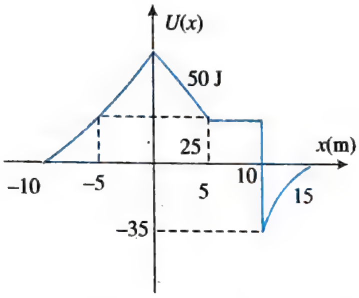
14. During the displacement, which of the curves shown in the graph best represents the kinetic energy of the block?
(1) 1
(2) 2
(3) 3
(4) 4

15. During the displacement, which of the curves shown in the graph best represents the work done on the spring block system by the applied force?
(1) 1
(2) 2
(3) 3
(4) 4

16. During the first half of the motion, applied force transfers more energy to the
(1) Kinetic energy
(2) Potential energy
(3) Equal to both
(4) Depends upon mass of the block

For Problems 17–19

The figure below shows the variation of potential energy of a particle as a function of x, the x-coordinate of the region. It has been assumed that potential energy depends only on x. For all other values of x, U is zero, i.e., $x < -10$ and $x > 15$, $U = 0$.



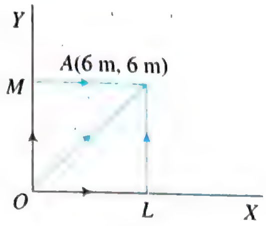
17. If the total mechanical energy of the particle is 25 J, then it can be found in region
(1) $-10 < x < -5$ and $6 < x < 15$
(2) $-10 < x < 0$ and $6 < x < 10$
(3) $-5 < x < 6$
(4) $-10 < x < 10$

18. If the total mechanical energy of the particle is -40 J, then it can be found in region
(1) $x < -10$ and $x > 15$
(2) $-10 < x < -5$ and $6 < x < 15$
(3) $10 < x < 15$
(4) It is not possible.

19. If the particle is isolated and its total mechanical energy is 60 J, then
(1) The particle can be found anywhere from $-\infty < x < \infty$.
(2) The particle's maximum kinetic energy is 95 J.
(3) The particle's kinetic energy is not getting zero anywhere on the x-axis.
(4) All of the above

For Problems 20–25

Force acting on a particle moving in the x-y plane is $\vec{F} = (y^2\hat{i} + x\hat{j})$ N, x and y are in metre. As shown in figure, the particle moves from the origin O to point A (6 m, 6 m). The figure shows three paths, OLA, OMA, and OA for the motion of the particle from O to A.



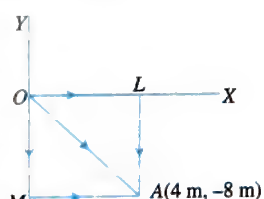
(A)

20. Which of the following is correct?
   (1) There is equal probability for the force being conservative or non-conservative.
   (2) Conservative or non-conservative nature of force cannot be predicted on the basis of given information.
   (3) The given force is non-conservative.
   (4) The given force is conservative.

21. Along which of the three paths is the work done maximum?
   (1) OA
   (2) OMA
   (3) OLA
   (4) Work done has the same value for all the three paths.

22. Work done for motion along path OA is nearly
   (1) 383 J (2) 90 J
   (3) 180 J (4) None of these

Now consider another situation. A force $\vec{F} = (4\hat{i} + 3\hat{j})$ N acts on a particle of mass 2 kg. The particle under the action of this force moves from the origin to a point A (4 m, –8 m). Initial speed of the particle, i.e., its speed at the origin is $2\sqrt{6}$ m s^{-1}.



(B)

Figure shows three paths for the motion of the particle from O to A.

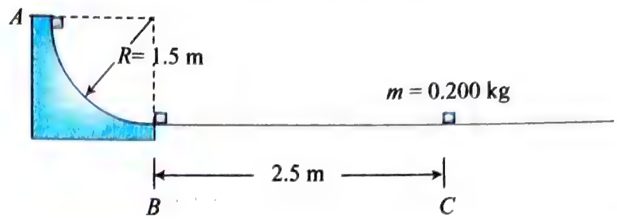
23. Which of the following is correct?
   (1) There is equal probability for the force being conservative or non-conservative.
   (2) Conservative or non-conservative nature of the force cannot be predicted on the basis of the given information.
   (3) The force is non-conservative.
   (4) The force is conservative.

24. Speed of the particle at A will be nearly
   (1) 4.0 m s^{-1} (2) 2.8 m s^{-1}
   (3) 3.6 m s^{-1} (4) 5.6 m s^{-1}

25. If the potential energy at O is 16 J, the potential energy at A will be
   (1) 14.5 J (2) 32 J
   (3) 24 J (4) 40 J

For Problems 26 and 27

In a truck-loading station at a post office, a small 0.200 kg package is released from rest at point A on a track that is one quarter of a circle with radius 1.5 m (figure). The size of the package is much less than 1.5 m, so the package can be treated as a particle. It slides down the track and reaches point B with a speed of 5 m/s. From point B, it slides on a level surface a distance of 2.5 m to point C, where it comes to rest.

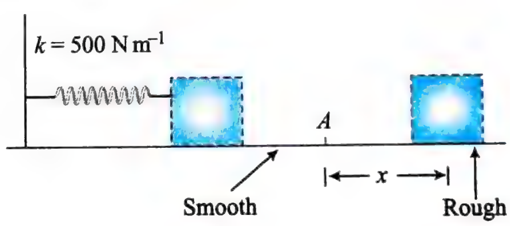


26. What is the coefficient of kinetic friction on the horizontal surface?
   (1) 0.40 (2) 0.25
   (3) 0.50 (4) None of these

27. How much work is done on the package by friction as it slides down the circular arc from A to B?
   (1) –10 J (2) –1 J
   (3) –2 J (4) None of these

For Problems 28–32

A small block of mass 50 g, compressing a spring (of spring constant k = 500 N m^{-1}) by 10 cm is released on a horizontal surface. The surface to the right of A is rough (μ = 0.2). Find the following:



28. The velocity with which the block separates from the spring is:
   (1) 0.5 ms^{-1} (2) 1 ms^{-1}
   (3) 1.5 ms^{-1} (4) None of these

29. The velocity (v) of the block as a function of the distance 'x' covered on the rough horizontal surface.
   (1) $v = \sqrt{0.25 + 4x}$ (2) $v = \sqrt{0.25 - 4x}$
   (3) $v = \sqrt{0.5 - 4x}$ (4) None of these

30. Find the maximum value of x (when the block stops)
   (1) 1.25 cm (2) 25 cm
   (3) 6.25 cm (4) 2.5 m

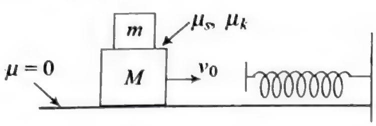
31. The velocity (v) of the block as a function of time (t), when it moves on to the rough surface.
   (1) $v = 0.5 + 2t$ (2) $v = 0.5 - 2t$
   (3) $v = 10 - 2t$ (4) None of these

32. Find the time elapsed, before which the block is at rest.

(1) 0.75 s (2) 1 s
(3) 0.25 s (4) 5 s

For Problems 33 and 34

A block of mass M slides on a frictionless surface with an initial speed of v_0. On top of block is a small box of mass m. The coefficients of friction between box and block are μ_s and μ_k. The sliding block encounters an ideal spring with force constant k. Answer following questions.



33. Assuming no relative motion between box and block what is the maximum possible acceleration of block and box at the instant of maximum compression?

(1) $\mu_s g$ (2) $\dfrac{\mu_s Mg}{m}$

(3) $\dfrac{\mu_s (M+m)g}{m}$ (4) $\dfrac{\mu_s mg}{M}$

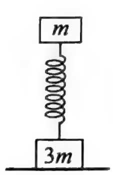
34. What is maximum value of k for which it remains true that box does not slide?

(1) $\left(\dfrac{\mu_s g}{v_0}\right)^2 \dfrac{M}{(M+m)}$ (2) $\left(\dfrac{\mu_s g}{v_0}\right)^2 M$

(3) $\left(\dfrac{\mu_s g}{2v_0}\right)^2 \dfrac{(M+m)^2}{M}$ (4) $\left(\dfrac{\mu_s g}{v_0}\right)^2 (M+m)$

For Problems 35–37

In the figure shown, the spring constant is K. The mass of the upper disc is m and that of the lower disc is $3m$. The upper block is depressed down from its equilibrium position by a distance y and released at $t = 0$.



35. The minimum value of y for which the lower disc loses contact with the ground is

(1) $3mg/K$ (2) $2mg/K$
(3) $4mg/K$ (4) mg/K

36. The value of y for which the minimum normal reaction on $3m$ from ground is mg is

(1) $3mg/K$ (2) $2mg/K$
(3) $mg/2K$ (4) mg/K

37. Suppose $y = 5mg/K$. Then, the velocity of m when normal reaction on $3m$ is mg is

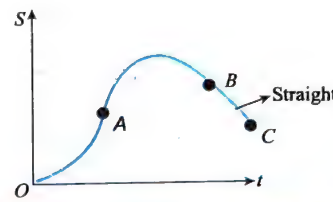
(1) zero (2) $g[m/K]^{1/2}$
(3) $2g[m/K]^{1/2}$ (4) $4g[m/K]^{1/2}$

Matrix Match Type

1. When a body is moving vertically up with constant velocity, then match the following.

Column I	Column II
i. Work done by lifting force is	**a.** negative
ii. Total work done by all the forces is	**b.** positive
iii. Work done by gravity	**c.** zero
iv. Work done by lifting force + work done by gravity force	**d.** higher positive values

2. The displacement–time graph of a body acted upon by some forces is shown in figure. For this situation, match the entries of Column I with the entries of Column II.

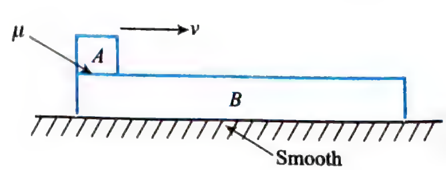


Column I	Column II
i. For OA, the total work done by all forces together is	**a.** always positive
ii. For OA, the work done by few of the acting forces is	**b.** can be positive
iii. For AB, the work done by few of the acting forces is	**c.** zero or can be zero
iv. For BC, the work done by all forces together is	**d.** can be negative

3. A man pushes a block of 30 kg along a level floor at a constant speed with a force directed at 45° below the horizontal. If the coefficient of friction is 0.20, then match the following.

Column I	Column II
i. Work done by all forces exerted by the surface on the block in 20 m	**a.** zero
ii. Work done by the force of gravity	**b.** –1500 J
iii. Work done by the man on the block in pushing it through 10 m	**c.** 750 J
iv. Net force on the block	**d.** 30 J

4. In the figure below, block A is kept on a larger block B. Both are initially at rest. Friction exists between the blocks but there is no friction between B and floor. An impulse gives block A a velocity v as shown. For some displacement after this, match the entries of Column I with that of Column II.

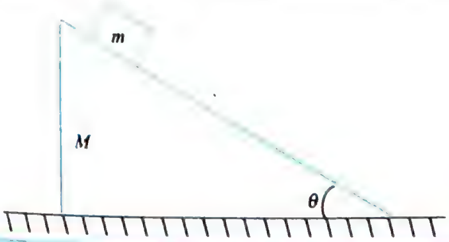


Column I	Column II
i. Work done by friction on B	a. positive
ii. Work done by friction on A	b. negative
iii. Net work done by friction on A and B	c. zero
iv. Work done by friction on B in the frame of A	d. positive, negative or zero

5. A small object of mass 0.5 kg is attached to an end of a massless 2 m long rope. It is rotated under gravity in a vertical circle with the other end of the rope being at the centre of the circle. The motion is started from the lowest point. Match columns I and II.

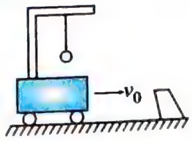
Column I	Column II
i. If the speed of the object at lowest point is 3.5 ms^{-1}	a. there will be some point on the circle at which speed of the object is zero but tension in the rope is not zero.
ii. If the speed of the object at lowest point is 8 ms^{-1}	b. there will be some point on the circle at which tension in the rope is zero but speed of the object is not zero.
iii. If the maximum tension in the rope is 15 N	c. the object will not be able to reach the highest point
iv. If the maximum tension in the rope is 30 N	d. the object will be able to reach the highest point

6. A block of mass m lies on a wedge of mass M. The wedge in turn lies on a smooth horizontal surface. Friction is absent everywhere. The wedge–block system is released from rest. All situations given in Column I are to be estimated in duration the block undergoes a vertical displacement h starting from rest. Match the statements in Column I with the results in Column II. (g is acceleration due to gravity.)



Column I	Column II
i. Work done by normal reaction acting on the block is	a. positive
ii. Work done by normal reaction (exerted by block) acting on the wedge is	b. negative
iii. The sum of work done by normal reaction on the block and work done by normal on wedge	c. zero
iv. Net work done by all forces on the block is	d. less than mgh in magnitude

7. A bob of mass 2 kg is suspended from a vehicle by a rope of length l = 5 m. The vehicle and the bob are moving at a constant speed v_0. The vehicle is suddenly stopped by a bumper and the bob on the rope swings out a maximum angle of 60°. Match the following.

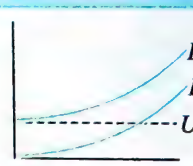
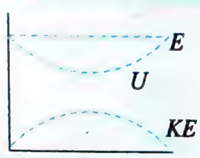
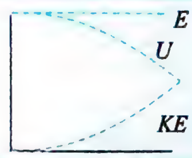
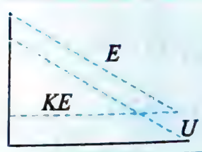


Column I	Column II (all values are in SI units)
i. Net force acting on the bob at lowest point just after the vehicle is stopped	a. $5\sqrt{3}$
ii. Acceleration of the bob at lowest point	b. 10
iii. Net force acting on the bob at its highest point	c. 20
iv. Acceleration of the bob at its highest point	d. $10\sqrt{3}$

8. A chain of length l and mass m lies on the surface of a smooth sphere of radius $R > l$ with one end tied to the top of the sphere.

Column I	Column II
i. Gravitational potential energy w.r.t. centre of the sphere	a. $\frac{Rg}{l}\left[1-\cos\left(\frac{l}{R}\right)\right]$
ii. The chain is released and slides down, its KE when it has slid by θ	b. $\frac{2Rg}{l}\left[\sin\left(\frac{l}{R}\right)+\sin\theta-\sin\left(\theta+\frac{l}{R}\right)\right]$
iii. The initial tangential acceleration	c. $\frac{MR^2g}{l}\sin\left(\frac{l}{R}\right)$
iv. The radial acceleration a_r	d. $\frac{MR^2g}{l}\left[\sin\left(\frac{l}{R}\right)+\sin\theta-\sin\left(\theta+\frac{l}{R}\right)\right]$

9. Match the physical situation on the left with the graph on the right. The graphs depict the variation of total energy (E), potential energy (U) and kinetic energy (KE) with time.

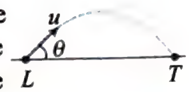
Column I	Column II
i. A mass attached to an unstretched ideal spring. Released in vertical plane from rest until it reaches its maximum extension.	a. 
ii. An object undergoing free fall.	b. 
iii. An object being pulled on a level, frictionless surface by a constant force in the horizontal direction.	c. 
iv. A block is coming down on a rough inclined plane with constant velocity.	d. 

Now match the given columns and select the correct option from the codes given below.

Codes:

	i	ii	iii	iv
(1)	a	b	c	d
(2)	b	c	a	d
(3)	d	a	b	c
(4)	a	c	b	d

10. A projectile is launched at angle θ to the horizontal from point L and it hits the target T on level ground. During the entire motion:



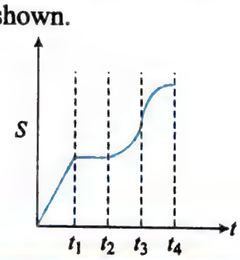
Column I	Column II
i. Magnitude of radial acceleration	**a.** remains constant
ii. Magnitude of tangential acceleration	**b.** either always decreases or always increases
iii. Power delivered by gravity	**c.** First increases, then decreases
iv. Total acceleration	**d.** First decreases, then increases

Now match the given columns and select the correct option from the codes given below.

Codes:

	i	ii	iii	iv
(1)	a	b	c	d
(2)	c	d	c	a
(3)	d	a	b	c
(4)	a	c	b	d

11. A particle is moving along a straight line. Its displacement–time graph is shown.



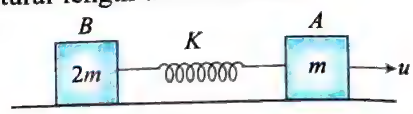
Column I	Column II
i. Work done on particle from 0 to t_1	**a.** Positive
ii. Work done on particle from t_1 to t_2	**b.** Negative
iii. Work done on particle from t_2 to t_3	**c.** Zero
iv Work done on particle from t_3 to t_4	**d.** Unpredictable

Now match the given columns and select the correct option from the codes given below.

Codes:

	i	ii	iii	iv
(1)	a	b	c	d
(2)	b	c	a	d
(3)	c	c	a	b
(4)	a	c	b	d

12. Two blocks A and B of mass m and $2m$ respectively are connected by a massless spring of spring constant K. This system lies over a smooth horizontal surface. At $t = 0$ the block. A has velocity u towards right as shown while the speed of block B is zero, and the length of spring is equal to its natural length at that instant.



Smooth horizontal surface

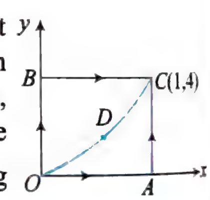
Column I	Column II
i. The velocity of block A	**a.** can never be zero
ii. The velocity of block B	**b.** may be zero at certain instants of time
iii. The kinetic energy of system of two blocks	**c.** is maximum at maximum compression of spring
iv. The potential energy of spring	**d.** is maximum at maximum extension of spring

Now match the given columns and select the correct option from the codes given below.

Codes:

	i	ii	iii	iv
(1)	b	b	a,c	b,d
(2)	b	a,c	a	d
(3)	d	a,c	b	b,c
(4)	a,d	c	b	d

13. A particle is moved along the different paths OAC, OBC and ODC as shown in the figure. Path ODC is a parabola, $y = 4x^2$. Find the work done by a force $\vec{F} = xy\hat{i} + x^2y\hat{j}$ on the particle along these paths.



Column I	Column II
i. W_{OAC}	**a.** 15/3 J
ii. W_{OBC}	**b.** 19/3 J
iii. W_{ODC}	**c.** 2 J
	d. 8 J

Now match the given columns and select the correct option from the codes given below.

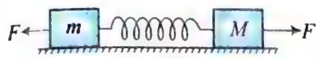
Codes:

	i	ii	iii
(1)	d	c	b
(2)	a	b	c
(3)	b	c	d
(4)	c	d	a

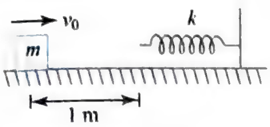
Numerical Value Type

1. The PE of a certain spring when stretched from natural length through a distance 0.3 m is 5.6 J. Find the amount of work in joule that must be done on this spring to stretch it through an additional distance 0.15 m.

2. In the situation shown in figure all contact surfaces are smooth. The force constant of the spring is K. Two forces F are applied as shown. The maximum elongation produced in the spring is how many times of F/K (initially the spring is relaxed)?



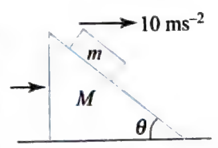
3. A block of mass $m = 0.14$ kg is moving with velocity v_0 towards a mass less unstretched spring of force constant $K = 10\ \text{Nm}^{-1}$. Coefficient of friction between the block and the ground is $\mu = 1/2$.



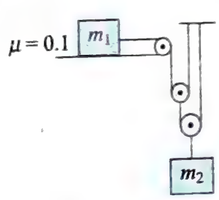
Find the maximum value of compression in the spring (in cm), so that after pressing the spring the block does not return back but stops there permanently.

4. A man slowly pulls a bucket of water from a well of depth $h = 20$ m. The mass of the uniform rope and bucket full of water are $m = 200$ g and M 19.9 kg, respectively. Find the work done (in kJ) by the man.

5. In the figure below, shown all the surfaces are frictionless, and mass of the block is $m = 100$ g. The block and the wedge are held initially at rest. Now the wedge is given a horizontal acceleration of $10\ \text{m s}^{-2}$ by applying a force on the wedge, so that the block does not slip on the wedge. Then find the work done in joules by the normal force in ground frame on the block in 1 s.

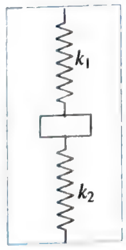


6. In the figure below, find the velocity of m_1 in m s^{-1} when m_2 falls by 9 m.

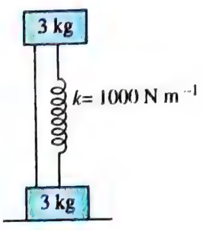


Given $m_1 = m$; $m_2 = 2m$ (take $g = 10\ \text{ms}^{-2}$).

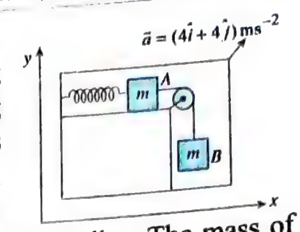
7. One end of a spring of force constant k_1 is attached to the ceiling of an elevator. A block of mass 1.5 kg is attached to the other end. Another spring of force constant k_2 is attached to the bottom of the mass and to the floor of the elevator as shown in the figure. At equilibrium, the deformation in both the springs is equal and is 40 cm. If the elevator moves with constant acceleration upward, the additional deformation in both the springs is 8 cm. Find the elevator's acceleration ($g = 10\ \text{m s}^{-2}$).



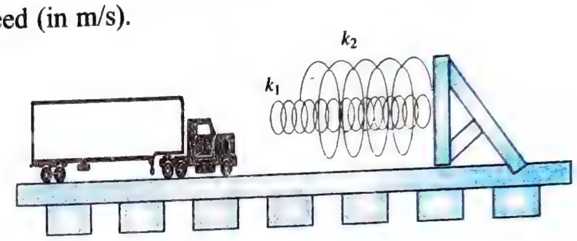
8. A system consists of two identical cubes, each of mass 3 kg, linked together by a compressed weightless spring of force constant $1000\ \text{N m}^{-1}$. The cubes are also connected by a thread which is burnt at a certain moment. At what minimum value of initial compression x_0 (in cm) of the spring will the lower cube bounce up after the thread is burnt through?

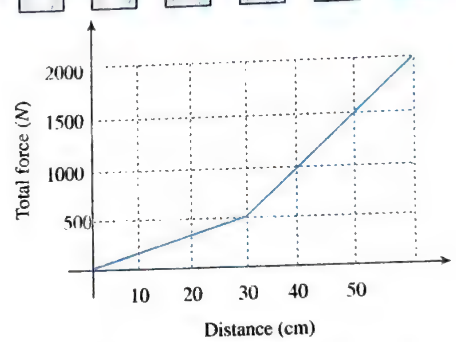


9. The arrangement shown in figure is at rest. An ideal spring of natural length l_0 having spring constant $k = 220\ \text{Nm}^{-1}$, is connected to block A. Blocks A and B are connected by an ideal string passing through a friction less pulley. The mass of each block A and B is equal to $m = 2$ kg when the spring was in natural length, the whole system is given an acceleration $\vec{a}$ as shown. If coefficient of friction of both surfaces is $\mu = 0.25$, then find the maximum extension (in cm) of the spring.($g = 10\ \text{m s}^{-2}$).

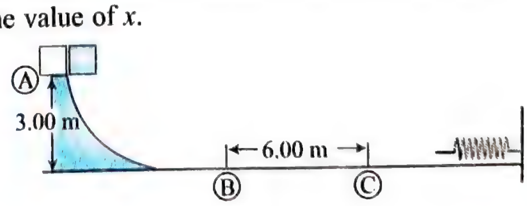


10. A 2144-kg freight car roles along rails with negligible friction. The car is brought to rest by a combination of two coiled springs as illustrated in fig. Both springs are described by Hooke's law with $k_1 = 1600\ \text{Nm}^{-1}$ and $k_2 = 3400\ \text{N m}^{-1}$. After the first spring compresses a distance of 30.0 cm, the second spring acts with the first to increase the force as additional compression occurs as shown in the graph in figure. The car comes to rest 50.0 cm after first contracting the two-spring system. Find the car's initial speed (in m/s).

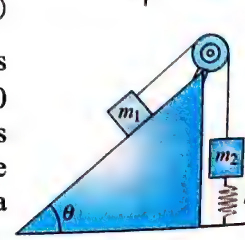




11. A 10.0-kg block is released from rest at point Ⓐ in figure. The track is frictionless except for the portion between points Ⓑ and Ⓒ, which has a length of 6.00 m. The block travels down the track, hits a spring of force constant 3000 N/m and compresses the spring 0.20 m from its equilibrium position before coming to rest momentarily. If the coefficient of kinetic friction between the block and the rough surface between points Ⓑ and Ⓒ is $10\,x$. Find the value of x.



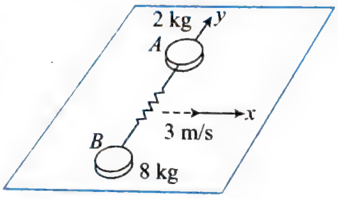
12. A block of mass $m_1 = 20.0$ kg is connected to a block of mass $m_2 = 30.0$ kg by a massless string that passes over a light, frictionless pulley. The 30.0-kg block is connected to a spring that has negligible mass



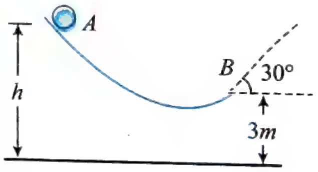
and a force constant of $k = 250$ N/m as shown in figure. The spring is unstretched when the system is as shown in the figure, and the incline is frictionless. The 20.0-kg block is pulled a distance $h = 20.0$ cm down the incline of angle $\theta = 30°$ and released from rest. If the speed of each block when the spring is again unstretched is $\frac{x}{\sqrt{5}}$ m/s. Find the value of x.

13. A block of mass M rests on a table. It is fastened to the lower end of a light, vertical spring. The upper end of the spring is fastened to a block of mass m. The upper block is pushed down by an additional force $3mg$, so the spring compression is 4 mg/k. In this configuration, the upper block is released from rest. The spring lifts the lower block off the table. What is the greatest possible value for $\frac{M}{m}$?

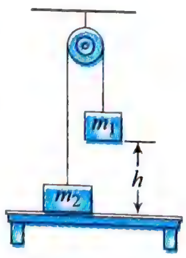
14. A massless spring of force constant 1000 N/m is compressed a distance of 20 cm between discs of 8 kg and 2 kg, spring is not attached to discs. The system is given an initial velocity 3 m/s perpendicular to length of spring as shown in figure. What is ground frame velocity of 2 kg block (in m/s) when spring regains its natural length.



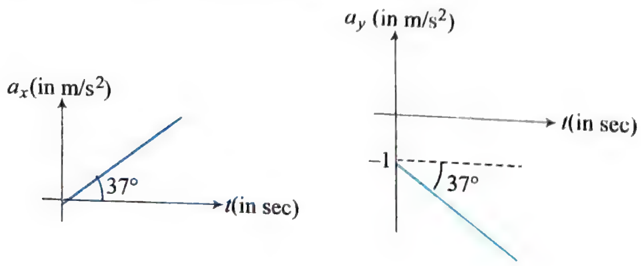
15. A ball released at A leaves the frictionless track at B, which is at a height of 3 m from the ground. The ball further rises maximum up to 4 m above the ground before falling down. Find h (in m) if the track at B makes an angle of 30° with the horizontal.



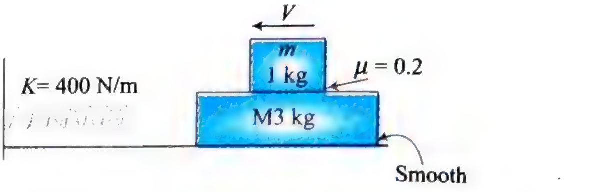
16. Two objects are connected by a light string passing over a light, frictionless pulley as shown in the figure. The object of mass $m_1 = 5.0$ kg is released from rest at a height $h = 4.0$ m above the table. Find the maximum height above the table to which the 3.0-kg object rises (in m).



17. In the figure, the variation of components of acceleration of a particle of mass 1 kg is shown w.r.t. time. The initial velocity of the particle is $\vec{u} = (-3\hat{i} + 4\hat{j})$ m/s. Find the total work done by the resultant force on the particle in time interval from $t = 0$ to $t = 4$ seconds in J.

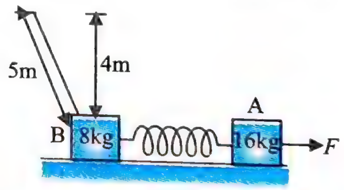


18. As shown in the figure, there is no friction between the horizontal surface and the lower block ($M = 3$ kg) but friction coefficient between both the blocks is 0.2. Both the blocks move together with initial speed V towards the spring, compresses it and due to the force exerted by the spring, moves in the reverse direction of the initial motion. Find the maximum value of V (in cm/s) so that during the motion, there is no slipping between the blocks. (use $g = 10$ m/s^2).

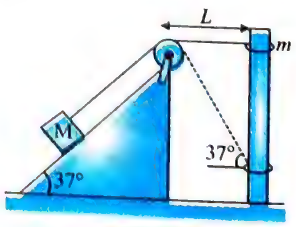


19. The potential energy (in SI units) of a particle of mass 2 kg in a conservative field is $U = 6x - 8y$. If the initial velocity of the particle is $\vec{u} = -1.5\hat{i} + 2\hat{j}$, then find the total distance (in m) travelled by the particle in first two seconds.

20. Two blocks having masses 8 kg and 16 kg are connected to the two ends of a light spring. The system is placed on a smooth horizontal floor. An inextensible string also connects B with ceiling as shown in the figure at the initial moment. Initially the spring has its natural length. A constant horizontal force F is applied to the heavier block as shown. What is the maximum possible value of F (in N) so the lighter block does not loose contact with ground.



21. A ring of mass $m = 1$ kg can slide over a smooth vertical rod. A light string attached to the ring passing over a smooth fixed pulley at a distance of $L = 0.7$m from the rod is shown in the figure. At the other end of the string, mass $M = 5$ kg is attached, lying over a smooth fixed inclined plane of inclination angle 37°. The ring is held in level with the pulley and released. Determine the velocity of ring (in m/s) when the string makes an angle ($\alpha = 37°$) with the horizontal. [sin 37° = 0.6]



Archives

JEE MAIN

Single Correct Answer Type

1. The potential energy function for the force between two atoms in a diatomic molecule is approximately given by $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are constants and x is the distance between the atoms. If the dissociation energy of the molecule is $D = [U_{(x=\infty)} - U_{\text{at equilibrium}}]$, D is

(1) $\frac{b^2}{2a}$ (2) $\frac{b^2}{12a}$ (3) $\frac{b^2}{4a}$ (4) $\frac{b^2}{6a}$

(AIEEE 2010)

2. This question has Statement I and Statement II. Of the four choices given after the statements, choose the one that best describes the two statements.

If two springs S_1 and S_2 of force constants k_1 and k_2, respectively, are stretched by the same force, it is found that more work is done on spring S_1 than on spring S_2.

Statement I: If stretched by the same amount, work done on S_1, will be more than that on S_2

Statement II: $k_1 < k_2$

(1) Statement I is false, Statement II is true.

(2) Statement I is true, Statement II is false.

(3) Statement I is true, Statement II is true, Statement II is the correct explanation for statement I.

(4) Statement I is true, Statement II is true, Statement II is not the correct explanation of Statement I.

(AIEEE 2012)

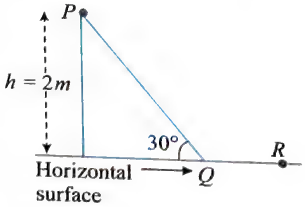
3. When a rubber-band is stretched by a distance x, it exerts a restoring force of magnitude $F = ax + bx^2$, where a and b are constants. The work done in stretching the unstretched rubber band by L is,

(1) $\frac{aL^2}{2} + \frac{bL^3}{3}$ (2) $\frac{1}{2}\left(\frac{aL^2}{2} + \frac{bL^3}{3}\right)$

(3) $aL^2 + bL^3$ (4) $\frac{1}{2}(aL^2 + bL^3)$

(JEE Main 2014)

4. A point particle of mass m, moves along the uniformly rough track PQR as shown in the figure. The coefficient of friction, between the particle and the rough track equals μ. The particle is released, from rest, from the point P and it comes to rest at a point R. The energies, lost by the ball, over the parts, PQ and QR, of the track, are equal to each other, and no energy is lost when particle changes direction from PQ to QR.



The values of the coefficient of friction μ and the distance $x(= QR)$, are, respectively close to:

(1) 0.2 and 6.5 m (2) 0.2 and 3.5 m

(3) 0.29 and 3.5 m (4) 0.29 and 6.5 m

(JEE Main 2016)

5. A person trying to lose weight by burning fat lifts a mass of 10 kg up to a height of 1 m 1000 times. Assume that the potential energy lost each time he lowers the mass is dissipated. How much fat will he use up considering the work done only when the weight is lifted up? Fat supplies 3.8×10^7 J of energy per kg which is converted to mechanical energy with a 20% efficiency rate. Take $g = 9.8$ m s^{-2}.

(1) 2.45×10^{-3} kg (2) 6.45×10^{-3} kg

(3) 9.89×10^{-3} kg (4) 12.89×10^{-3} kg

(JEE Main 2016)

6. A body of mass $m = 10^{-2}$ kg is moving in a medium and experiences a frictional force $F = -kv^2$. Its initial speed is $v_0 = 10$ ms^{-1}. If, after 10 s, its energy is $\frac{1}{8}mv_0^2$, the value of k will be

(1) 10^{-4} kg m^{-1} (2) 10^{-1} kg m^{-1}s^{-1}

(3) 10^{-3} kg m^{-1} (4) 10^{-3} kg s^{-1}

(JEE Main 2017)

7. A time dependent force $F = 6t$ acts on a particle of mass 1 kg. If the particle starts from rest, the work done by the force during the first 1 sec. will be

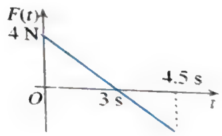
(1) 9 J (2) 18 J

(3) 4.5 J (4) 22 J **(JEE Main 2017)**

JEE ADVANCED

Single Correct Answer Type

1. A block of mass 2 kg is free to move along the x-axis. It is at rest and from $t = 0$ onwards, it is subjected to a time-dependent force $F(t)$ in the x-direction.



The force $F(t)$ varies with t as shown in the figure. The kinetic energy of the block after 4.5 s is

(1) 4.50 J (2) 7.50 J

(3) 5.06 J (4) 14.06 J **(IIT-JEE 2010)**

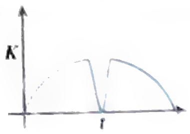
2. The work done on a particle of mass m by a force $K\left[\frac{x}{(x^2+y^2)^{3/2}}\hat{i} + \frac{y}{(x^2+y^2)^{3/2}}\hat{j}\right]$ (K being the constant of appropriate dimensions), when the particle is taken from the point $(a, 0)$ to the point $(0, a)$ along a circular path of radius a about the origin in the x-y plane is

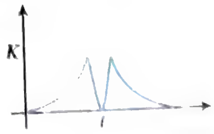
(1) $\frac{2K\pi}{a}$ (2) $\frac{K\pi}{a}$

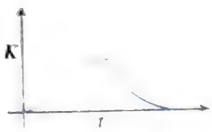
(3) $\frac{K\pi}{2a}$ (4) 0

(JEE Advanced 2

3. A tennis ball is dropped on a horizontal smooth surface. It bounces back to its original position after hitting the surface. The force on the ball during the collision is proportional to the length of compression of the ball. Which one of the following sketches describes the variation of its kinetic energy K with time t most appropriately? The figures are only illustrative and not to the scale.

(1) 

(2) 

(3) 

(4) 

(JEE Advanced 2014)

Multiple Correct Answers Type

1. A particle of mass m is initially at rest at the origin. It is subjected to a force and starts moving along the x-axis. Its kinetic energy K changes with time as $dK/dt = \gamma t$, where γ is a positive constant of appropriate dimensions. Which of the following statements is (are) true?
   (1) The force applied on the particle is constant
   (2) The speed of the particle is proportional to time
   (3) The distance of the particle from the origin increases linearly with time
   (4) The force is conservative

Matrix Match Type

1. A particle of unit mass is moving along the x-axis under the influence of a force and its total energy is conserved. Four possible forms of the potential energy of the particle are given in column I (a and U_0 are constants). Match the potential energies in column I to the corresponding statement(s) in column II.

	Column-I		Column-II
i.	$U_1(x) = \frac{U_0}{2}\left[1-\left(\frac{x}{a}\right)^2\right]^2$	**a.**	the force acting on the particle is zero at $x = a$.
ii.	$U_2(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2$	**b.**	the force acting on the particle is zero at $x = 0$.
iii.	$U_3(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 \exp\left[-\left(\frac{x}{a}\right)^2\right]$	**c.**	the force acting on the particle is zero at $x = -a$.
iv.	$U_4(x) = \frac{U_0}{2}\left[\frac{x}{a} - \frac{1}{3}\left(\frac{x}{a}\right)^3\right]$	**d.**	The particle experiences an attractive force towards $x = 0$ in the region $\lvert x\rvert < a$
		e.	The particle with total energy $\frac{U_0}{4}$ can oscillate about the point $x = -a$.

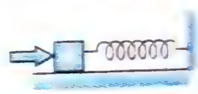
(JEE Advanced 2015)

Numerical Value Type

1. A light inextensible string that goes over a smooth fixed pulley as shown in the figure connects two blocks of masses 0.36 kg and 0.72 kg. Taking $g = 10$ m s^{-2}, find the work done (in joules) by the string on the block of mass 0.36 kg during the first second after the system is released froms rest.

(IIT-JEE 2009)

2. A block of mass 0.18 kg is attached to a spring of force constant 2 N m^{-1}. The coefficient of friction between the block and the floor is 0.1. Initially the block is at rest and the spring is unstretched. An impulse is given to the block as shown in the figure. The block slides a distance of 0.06 m and comes to rest for the first time. The initial velocity of the block in m s^{-1} is $V = N/10$. Then N is

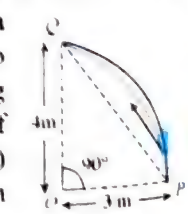


(IIT-JEE 2011)

3. A particle of mass 0.2 kg is moving in one dimension under a force that delivers a constant power 0.5 W to the particle. If the initial speed (in m/s) of the particle is zero, the speed (in m s^{-1}) after 5 s is

(JEE Advanced 2013)

4. Consider an elliptically shaped rail PQ in the vertical plane with $OP = 3$ m and $OQ = 4$ m. A block of mass 1 kg is pulled along the rail from P to Q with a force of 18 N, which is always parallel to line PQ (see the figure given). Assuming no frictional losses, the kinetic energy of the block when it reaches Q is ($n \times 10$) joules. The value of n is (take acceleration due to gravity = 10 ms^{-2})



(JEE Advanced 2014)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2) 2. (2) 3. (1) 4. (1) 5. (3)
6. (1) 7. (3) 8. (1) 9. (2) 10. (1)
11. (4) 12. (1) 13. (3) 14. (4) 15. (3)
16. (2) 17. (2) 18. (1) 19. (1) 20. (2)
21. (3) 22. (3) 23. (3) 24. (3) 25. (4)
26. (4) 27. (3) 28. (2) 29. (3) 30. (3)
31. (2) 32. (4) 33. (4) 34. (2) 35. (2)
36. (4) 37. (2) 38. (3) 39. (4) 40. (2)
41. (1) 42. (3) 43. (3) 44. (2) 45. (3)
46. (4) 47. (1) 48. (3) 49. (2) 50. (2)
51. (1) 52. (1) 53. (1) 54. (4) 55. (2)
56. (4) 57. (3) 58. (4) 59. (4) 60. (3)
61. (1) 62. (2) 63. (1) 64. (3) 65. (1)
66. (1) 67. (1) 68. (1) 69. (3) 70. (1)
71. (2) 72. (1) 73. (1) 74. (2) 75. (4)

Multiple Correct Answers Type

1. (1),(2) 2. (2),(4)
3. (2),(3) 4. (2),(3),(4)
5. (1),(2),(4),(5) 6. (1),(2),(3)
7. (1),(2),(3) 8. (1),(3)
9. (1),(3) 10. (1),(2),(3),(4)
11. (2),(3) 12. (1),(2),(4)
13. (2),(4) 14. (1),(2),(3),(4)
15. (1),(3),(4) 16. (1),(2),(3),(4)
17. (1),(2) 18. (1),(2)
19. (1),(2),(3) 20. (3),(4)
21. (1),(3) 22. (1),(3),(4)
23. (1),(3) 24. (2),(3),(4)
25. (1),(2),(3) 26. (1),(2),(3)
27. (1),(2),(4) 28. (1),(2),(3)
29. (2),(4) 30. (1),(2),(3)

Linked Comprehension Type

1. (4) 2. (1) 3. (4) 4. (3) 5. (3)
6. (1) 7. (2) 8. (2) 9. (2) 10. (1)
11. (2) 12. (3) 13. (4) 14. (3) 15. (1)
16. (3) 17. (1) 18. (4) 19. (4) 20. (3)
21. (2) 22. (2) 23. (4) 24. (1) 25. (3)
26. (3) 27. (2) 28. (4) 29. (4) 30. (2)
31. (3) 32. (4) 33. (1) 34. (4) 35. (3)
36. (1) 37. (4)

Matrix Match Type

1. i → b; ii → c; iii. → a, iv → c
2. i → a; ii → b, c, d; iii → b, c, d; iv → c
3. i → b; ii → a; iii → c; iv → a
4. i → a; ii → b; iii → b; iv → b
5. i → a, c; ii → b, c; iii → c; iv → b, d
6. i → b, d; ii → a, d; iii → c, d; iv → a, d
7. i → c; ii → b; iii → d; iv → a
8. i. → c; ii → d; iii → a; iv → b
9. (2) 10. (2) 11. (3) 12. (1) 13. (1)

Numerical Value Type

1. (7) 2. (2) 3. (7) 4. (4) 5. (5)
6. (4) 7. (2) 8. (9) 9. (10) 10. (0.5)
11. (4) 12. (3) 13. (2) 14. (5) 15. (7)
16. (5) 17. (10) 18. (20) 19. (15) 20. (30)
21. (0)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3) 2. (1) 3. (1) 4. (3) 5. (4)
6. (1) 7. (3)

JEE Advanced

Single Correct Answer Type

1. (3) 2. (4) 3. (2)

Multiple Correct Answers Type

1. (1, 2, 4)

Matrix Match Type

1. a → p, q, r, t; b → q, s; c → p, q, r, s; d → p, r, t

Numerical Value Type

1. (8) 2. (4) 3. (5) 4. (5)

9 Circular Motion

INTRODUCTION

We had studied simple linear motion till now. Now we will study and explore the same for circular motion. We will begin with the kinematics of circular motion and go on to study the dynamics of circular motion.

In the preceding chapters, we studied the application of Newton's laws of motion in situations involving linear motion. In this chapter, we will study circular motion and see how force is involved in creating a curvilinear path. We shall then learn to apply Newton's laws to objects traveling in circular paths.

KINEMATICS OF CIRCULAR MOTION

UNIFORM AND NON-UNIFORM CIRCULAR MOTION

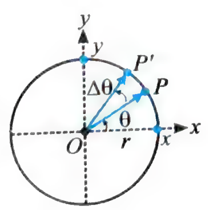
A particle is in **uniform circular motion** if it travels around a circle or a circular arc at constant (*uniform*) speed. When the speed of the particle varies with time, we call it "non-uniform circular motion".

In physics is defined acceleration as a change in the velocity, not a change in the speed (contrary to the everyday interpretation). In circular motion, the velocity vector is always changing in direction, so there is indeed an acceleration.

IMPORTANT TERMINOLOGY RELATED TO CIRCULAR MOTION

In linear kinematics we studied about position, displacement, velocity and acceleration now we will learn about these with reference to circular motion.

Angular position: To decide the angular position of a point in space we need to specify (i) origin and (ii) reference line. The angle made by the position vector w.r.t. origin, with the reference line is called **angular position.**



Suppose a particle P is moving in a circle of radius r and centre O. The angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle.

Angular displacement: Angle through which the position vector of the moving particle rotates in a given time interval is called its angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on above circle, its angular position θ changes. Suppose the point rotates through an angle $\Delta\theta$ in time, Δt, then $\Delta\theta$ is angular displacement.

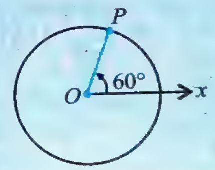
Important Points:

- Angular displacement is a dimensionless quantity.
- Its SI unit is radian
- Some other units are degree and revolution

  2π rad = $360°$ = 1 rev
- Angular position measured in counter clockwise direction from positive x-direction is taken as positive and vice versa.
- When the particle revolves in one plane in a circular path, the finite angular displacements can be treated as algebraic scalars.
- If we calculate θ in anticlockwise sense, we consider it positive and when measured in clockwise sense, it is negative.
- Hence the total angular displacement is the algebraic sum of the positive and negative angular displacements.

ILLUSTRATION 9.1

How do you describe the angular position of P as shown in the figure?

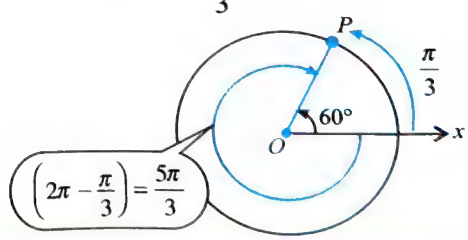


Sol. If we count the angle in anticlockwise sense, it is given as

$\phi = +\frac{\pi}{3}$ rad

Measuring the angle in clockwise sense,

$$\phi = -[2\pi - (\pi/3)] = -\frac{5\pi}{3} \text{ rad}$$

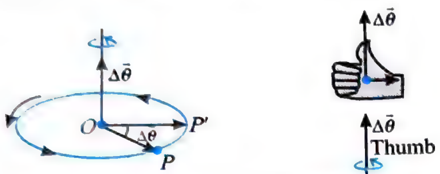


EXPRESSING ANGULAR DISPLACEMENT IN VECTOR FORM

Elementary angular displacement is a vector quantity, but finite angular displacement is a scalar. The addition of the elementary small angular displacements is commutative, $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$.

But in three-dimensional motion $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$. But when a particle moves in a plane, the finite angles may also be treated as vectors. **It means when a particle moves in a circular path, we can consider the finite angular displacement as vectors.** Direction of small angular displacement is decided by right hand thumb rule.

- When the fingers are directed along the motion of the point then thumb will represent the direction of angular displacement.
- $\Delta\vec{\theta}$ is directed parallel to (or along) the axis of revolution. Hence $\Delta\vec{\theta}$ is an axial vector.



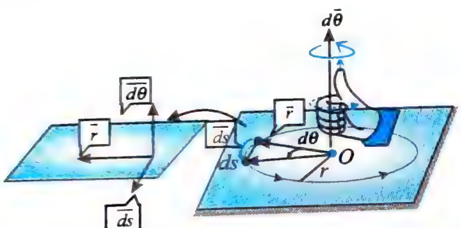
Curled fingers of right hand

(a) **(b)**

Relation Between Linear and Angular Quantities

We know arc length $ds = r\,d\theta$. We can express it in a vector form

$d\vec{s} = d\vec{\theta} \times \vec{r}$

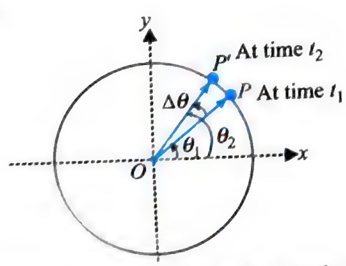


Average angular velocity: It is defined as total angular displacement divided by total time taken.

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Since finite angular displacement in circular motion can be considered as a vector hence average angular velocity taken as vector.

$$\vec{\omega}_{av} = \frac{\Delta\vec{\theta}}{\Delta E}$$

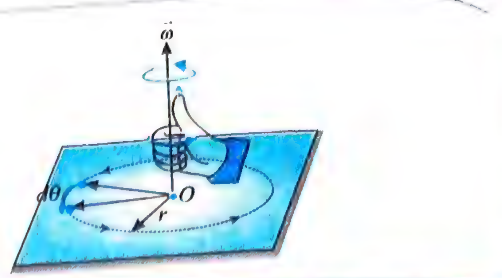


Instantaneous angular velocity: It is the limit of average angular velocity as Δt approaches zero. It is simply called angular velocity.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

The direction of angular velocity $\vec{\omega}$ is given by **right hand thumb rule.** If we wrap the fingers of the right hand in the sense of revolution (turning) of the point, the extended thumb will give us the direction of the angular velocity.

As $\vec{\omega}$ is directed parallel to (or along) the axis of revolution. Hence $\vec{\omega}$ is an axial vector.



Note: The magnitude of the angular velocity is given as angular speed.

Important Points:

- Angular velocity has dimension of $[T^{-1}]$ and SI unit rad/s.
- If a body makes n revolutions in t seconds then average angular velocity in radian per second will be $\omega_{av} = \frac{2\pi n}{t}$.

Frequency and Time Period

Frequency of circular motion is used when the particle moves with constant linear and angular speed. This is defined as number of complete revolutions of a particle in one second. Let us assume that the particle completes one revolution in time T seconds. Hence, in one second the particle has $1/T$ revolution, i.e., we call "frequency" given as

$$f = 1/T \qquad \text{...(i)}$$

Angular frequency: Since $\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega dt$

$$\int_0^{2\pi} d\theta = \omega \int_0^T dt$$

As ω = constant

$$[\theta]_0^{2\pi} = \omega[t]_0^T \Rightarrow [2\pi - 0] = \omega[T - 0]$$

This gives $\omega = \frac{2\pi}{T} = 2\pi f$

ILLUSTRATION 9.2

An insect moving in a circle travels N_1 revolutions in anticlockwise sense for a time T_1 and N_2 revolutions in clockwise sense for a time T_2

Find the angular speed averaged over the time $T_1 + T_2$

Sol. The angular displacement = $\theta = \theta_1 + \theta_2$.

$\theta_1 = 2\pi N_1$ and $\theta_2 = 2\pi N_2$.

$\theta = |2\pi N_1 - 2\pi N_2| = 2\pi |N_1 - N_2|$

Then, the average angular speed is $\omega = \frac{\theta}{t}$.

We have $\omega_{av} = \frac{2\pi}{T_1 + T_2} |N_1 - N_2|$

ILLUSTRATION 9.3

If angular displacement of a particle is given by $\theta = a - bt + ct^2$ then find its angular velocity.

Sol. Angular velocity is the rate of change of angular position,

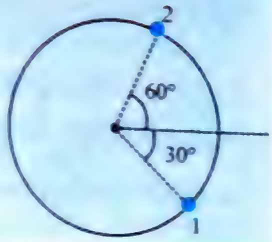
$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(a - bt + ct^2) = -b + 2ct.$$

Illustration 9.4

A particle moves from points 1 to 2 during a time 1/2 s. Find the average angular speed of the particle over 1/2 s

(i) in anticlockwise sense

(ii) in clockwise sense



Sol.

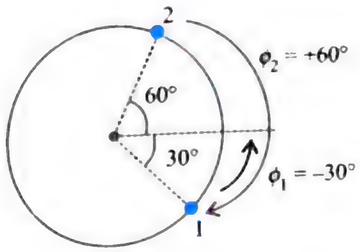
(i) Let us calculate $\Delta\phi$ to find ω_{av}.

Since $\Delta\phi = \phi_2 - \phi_1$.

$$\phi_2 = +60° \text{ and } \phi_1 = -30°$$

$$\Delta\phi = (+60°) - (-60°) = 90° = \frac{\pi}{2} \text{ rad}$$

$$\text{Hence } \omega_{av} = \frac{\Delta\phi}{\Delta t} = \frac{\pi/2}{1/2} = \pi \text{ rad/s}$$



(ii) If the particle moves from 1 to 2 in clockwise sense, the angular distance $\Delta\phi' = \frac{3\pi}{2}$ rad

$$\text{Hence } \omega_{av} = \frac{\Delta\phi'}{\Delta t} = \frac{3\pi/2}{1/2} = 3\pi \text{ rad/s}$$

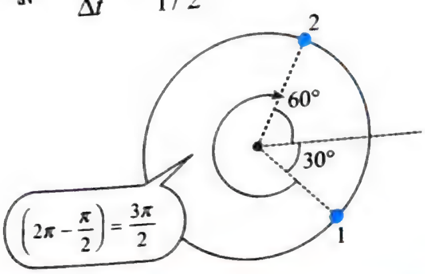
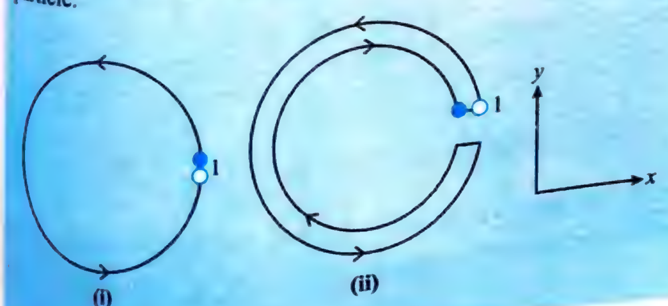


Illustration 9.5

A particle starts from the point 1 and reaches at the same point by revolving once in anticlockwise sense as shown in the Fig. (i) and once in anticlockwise sense and again in clockwise sense as shown in the Fig. (ii). If the time taken in both the cases is equal to 2 s, find the average angular velocity of the particle.



Sol. In Fig. (i) as the particle continuously rotates in anticlockwise sense $\theta = 2\pi$.

$$\text{Hence } \omega_{av} = \frac{\theta}{t} = \frac{2\pi}{2} = \pi \text{ rad/s}$$

$$\Rightarrow \quad \vec{\omega}_{av} = \pi\left(\hat{k}\right) \text{rad/s}$$

The angular displacement of the particle $\theta = \phi_2 - \phi_1$ where $\phi_1 = 0$ and $\phi_2 = 0$.

Then, we have $\theta = 0$, this gives $\omega_{av} = \frac{\theta}{t} = 0$.

Angular Acceleration: If ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as $\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$

Instantaneous Angular Acceleration

It is the limit of average angular velocity as Δt approaches zero.

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$$

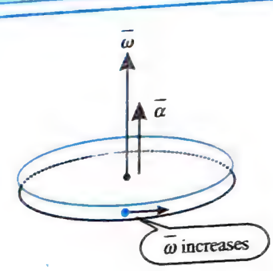
Hence, the angular acceleration is defined as the "rate of change in angular velocity".

$$\text{Since } \vec{\omega} = \frac{d\vec{\theta}}{dt}, \quad \therefore \ \vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$$

$$\text{Also } \vec{\alpha} = \omega\frac{d\vec{\omega}}{d\theta}$$

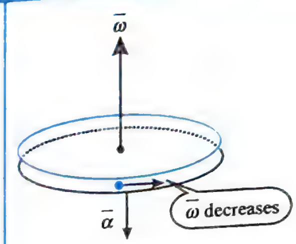
Important Points:

- Unit of angular acceleration is rad/s^2.
- Dimensional formula is $[T^{-2}]$.
- If $\alpha = 0$, circular motion is said to be uniform. $\theta = \omega t$



If $|\vec{\omega}|$ increases $\frac{d|\vec{\omega}|}{dt}$ is positive.

Then, $\vec{\alpha}$ is directed parallel to $\vec{\omega}$.



If $|\vec{\omega}|$ decreases $\frac{d|\vec{\omega}|}{dt}$ is negative.

Then, $\vec{\alpha}$ is directed anti parallel to $\vec{\omega}$

Motion with Constant Angular Acceleration

$\omega_0 \Rightarrow$ Initial angular velocity

$\omega \Rightarrow$ Final angular velocity

$\alpha \Rightarrow$ Constant angular acceleration

$\theta \Rightarrow$ Angular displacement

Translational motion	Circular motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$s = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$s_{n^{th}} = v_0 + \frac{a}{2}(2n-1)$	$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$

ILLUSTRATION 9.6

A fan is rotating with angular velocity 100 rev/sec. Then it is switched off. It takes 5 minutes to stop.

(a) Find the total number of revolution made before it stops. (Assume uniform angular retardation)

(b) Find the value of angular retardation.

(c) Find the average angular velocity during this interval.

Sol.

(a) $\theta = \left(\frac{\omega + \omega_0}{2}\right)t = \left(\frac{100+0}{2}\right) \times 5 \times 60 = 15000\,\text{rev}$

(b) $\omega = \omega_0 + \alpha t \Rightarrow 0 = 100 - \alpha(5 \times 60)$

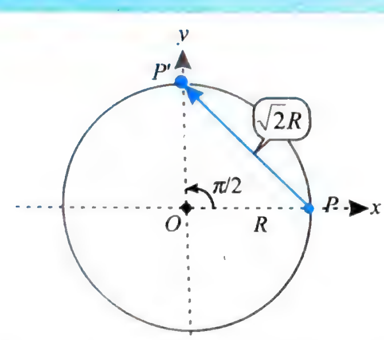
$\Rightarrow \quad \alpha = \frac{1}{3}\,\text{rev/sec}^2$

(c) $\omega_{av} = \frac{\text{Total angle of rotation}}{\text{Total time taken}} = \frac{15000}{50 \times 60} = 50\ \text{rev/sec}$

ILLUSTRATION 9.7

A particle is moving with constant speed in a circular path. Find the ratio of average velocity to its instantaneous velocity when the particle describes an angle $\theta = \frac{\pi}{2}$.

Sol.



Time taken to describe angle θ, $t = \frac{\theta}{\omega} = \frac{(\pi/2)}{(v/R)} = \frac{\pi R}{2v}$

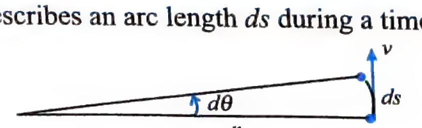
Average velocity $= \frac{\text{Total displacement}}{\text{Total time}} = \frac{\sqrt{2}R}{\pi R/2v} = \frac{2\sqrt{2}}{\pi}v$

Instantaneous velocity $= v$

The ratio of average velocity to its instantaneous velocity $= \frac{2\sqrt{2}}{\pi}$

Relation Between Linear and Angular Velocity

If a particle describes an arc length ds during a time dt,



Then, $ds = r\,d\theta$

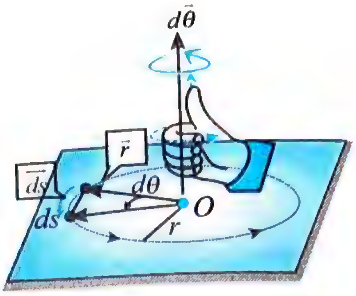
Then differentiating both sides with time, $\frac{ds}{dt} = r\frac{d\theta}{dt}$

$\Rightarrow \quad v = r\omega$

The above formula links the linear speed with angular speed.

In a vector form: As $d\vec{s} = d\vec{\theta} \times \vec{r}$

Then differentiating both sides with time, $\frac{d\vec{s}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$



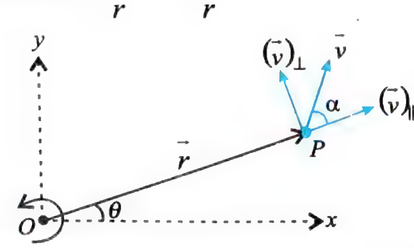
But $\frac{d\vec{s}}{dt} = \vec{v}$ and $\frac{d\vec{\theta}}{dt} = \vec{\omega}$, we have $\vec{v} = \vec{\omega} \times \vec{r}$

The above formula links the linear velocity with angular velocity in a vector form.

Relative Angular Velocity

Angular velocity is defined with respect to the point from which the position vector of the moving particle is drawn. Let a particle P is moving with velocity $\vec{v}$. Its position vector w.r.t. a point O is, $\vec{r}$ say. Take some fixed line as a reference line and let $\vec{r}$ makes an angle θ with this line as shown in figure. Then angular velocity of point P w.r.t. an observer on point O can be defined as $\omega = \frac{d\theta}{dt}$

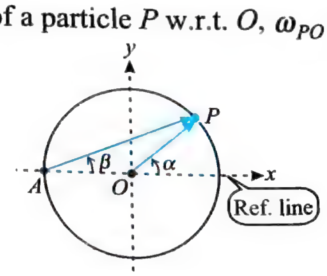
It is also given as $\omega = \frac{v_{\perp}}{r} = \frac{v\sin\alpha}{r}$.



Angular velocity of point 'P' with respect to 'O', $\vec{\omega} = \frac{|\vec{v}|_{\perp}}{|\vec{r}|}(\hat{k})$

Angular velocity of a given particle can be different about different points. Consider a particle P moving along a circular path shown in the figure given below. Here angular velocity of the particle P w.r.t. 'O' and 'A' will be different.

Angular velocity of a particle P w.r.t. O, $\omega_{PO} = \frac{d\alpha}{dt}$



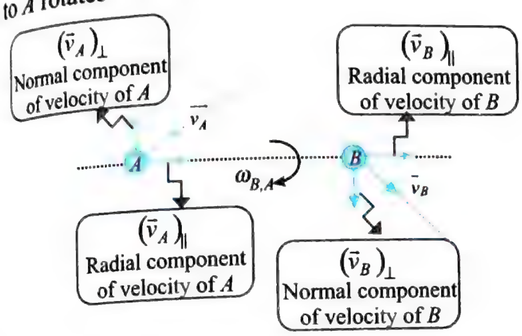
Angular velocity of a particle P w.r.t. A, $\omega_{PA} = \frac{d\beta}{dt}$

If two particles are moving with velocities $\vec{v}_1$ and $\vec{v}_2$, we can write the relative velocity of particle '1' w.r.t. particle '2' as $\vec{v}_{1,2} = (\vec{v}_1 - \vec{v}_2)$.

Now consider two particles has angular velocities $\vec{\omega}_1$ and $\vec{\omega}_2$. But we can not write angular velocity of particle '1' w.r.t. particle '2' as $\vec{\omega}_{1,2} = (\vec{\omega}_1 - \vec{\omega}_2)$

We should calculate the angular velocity of particle '1' w.r.t particle '2' as $\omega_{1,2} = \frac{|\vec{v}_{1,2}|_{\perp}}{r_{AB}}$.

Consider two particles A and B are moving. Relative angular velocity of a particle B with respect to another moving particle A is the angular velocity of the position vector of B with respect to A. That means it is the rate at which position vector of B with respect to A rotates at that instant.



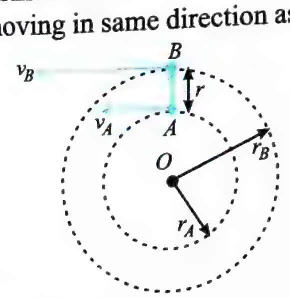
It is mathematically define as,

$$\omega_{B/A} = \frac{\text{Relative velocity of } B \text{ w.r.t. } A \text{ perpendicular to line } AB}{\text{Seperation between } A \text{ and } B}$$

$$\omega_{B,A} = \frac{(V_{BA})_{\perp}}{r_{BA}} = \frac{|(v_B)_{\perp} + (v_A)_{\perp}|}{r}$$

Important Points

- If two particles are moving on two different concentric circles with different velocities then angular velocity of B as observed by A will depend on their positions and velocities. Consider the case when A and B are closest to each other moving in same direction as shown in figure.

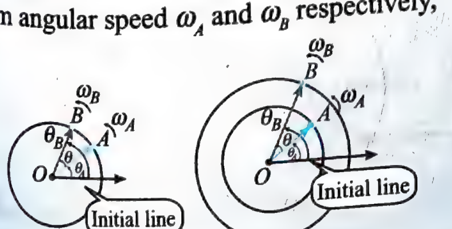


In this situation, $(v_{BA})_{\perp} = v_{rel} = |\vec{v}_B - \vec{v}_A| = v_B - v_A$

Separation between A and B, $r_{BA} = r_B - r_A$

$$\omega_{BA} = \frac{(v_{BA})_{\perp}}{r_{BA}} = \frac{v_B - v_A}{r_B - r_A}$$

- If two particles are moving on the same circle or different coplanar concentric circles in same direction with different uniform angular speed ω_A and ω_B respectively,



The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, $\theta = \theta_B - \theta_A$

The rate of change of angle between them

$$\frac{d\theta}{dt} = \frac{d\theta_B}{dt} - \frac{d\theta_A}{dt} = \omega_B - \omega_A$$

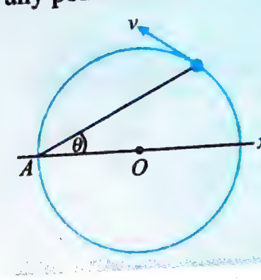
Here, $\omega_B - \omega_A$ is the subtraction of angular speeds. It is the rate of change of angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. This is not angular velocity of B w.r.t. A.

It means the time taken by one to complete one revolution around O w.r.t. the other

$$T = \frac{2\pi}{\omega_B - \omega_A} = \frac{2\pi}{\frac{2\pi}{T_B} - \frac{2\pi}{T_A}} = \frac{T_A T_B}{T_A - T_B}$$

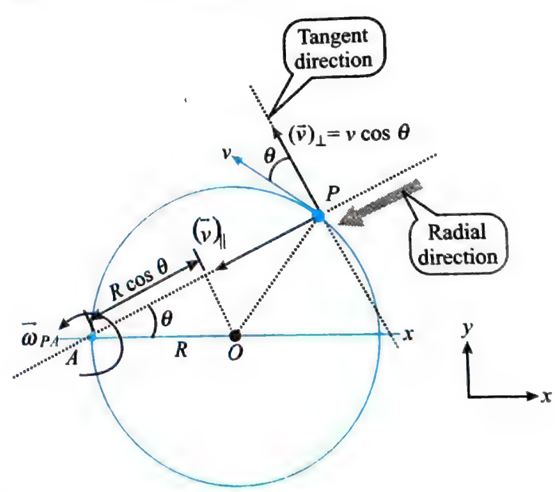
ILLUSTRATION 9.8

A particle is moving in a circle with constant speed v. If the radius of the circle is R, find the angular velocity of the particle relative to any point A on the perimeter of the circle.



Sol. Angular velocity of the particle P with respect to A.

$$\vec{\omega}_{PA} = \frac{(\vec{v}_{PA})_{\perp}}{AP}(\hat{k}) = \frac{v\cos\theta}{2R\cos\theta}(\hat{k}) = \frac{v}{2R}(\hat{k})$$



Angular velocity of P w.r.t. O

$$\vec{\omega}_{PO} = \frac{(\vec{v}_{P,O})_{\perp}}{OP}(\hat{k}) = \frac{v}{R}(\hat{k})$$

Important Observation:

Here we can observe $\vec{\omega}_{PO} = 2\vec{\omega}_{PA}$.

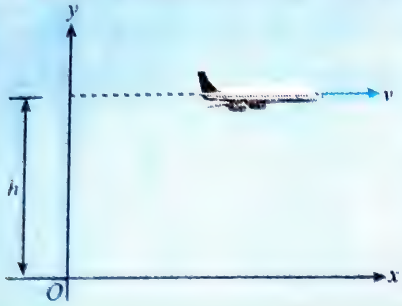
The angular velocity of a particle performing uniform circular motion about its centre is twice its angular velocity about any point situated on its circular path.

ILLUSTRATION 9.9

An aeroplane moves with constant velocity v parallel to x-axis at a height $y = h$. Find the

(a) angular velocity

(b) angular acceleration of the aeroplane relative to O as the function of time t. Assume that at $t = 0$, $x = 0$.



Sol.

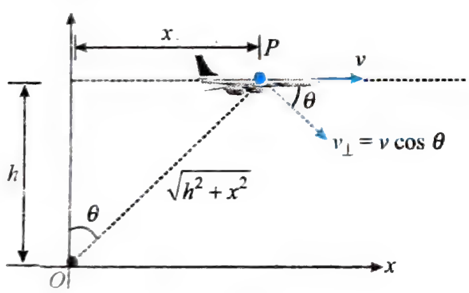
(a) Angular velocity, $\omega_{PO} = \dfrac{v_\perp}{OP}$

$$\Rightarrow \omega_{PO} = \frac{v\cos\theta}{\sqrt{h^2 + x^2}} \qquad \text{...(i)}$$

Here $x = vt$

and $\cos\theta = \dfrac{h}{\sqrt{h^2 + x^2}} = \dfrac{h}{\sqrt{h^2 + v^2t^2}}$

Hence $\omega = \dfrac{vh}{(h^2 + v^2t^2)}$



(b) Angular acceleration $\alpha = \dfrac{d\omega}{dt}$

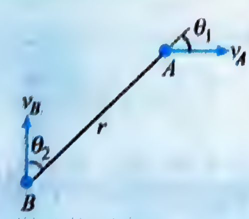
$$\omega = \frac{vh}{(h^2 + v^2t^2)}$$

$$\alpha = \frac{d}{dt}\left(\frac{vh}{(h^2 + v^2t^2)}\right) = vh\frac{d}{dt}(h^2 + v^2t^2)^{-1}$$

$$= vh\frac{(-1)}{(h^2 + v^2t^2)^2} \times v^2 \cdot 2t = \frac{-2v^3ht}{(h^2 + v^2t^2)^2}$$

ILLUSTRATION 9.10

Find the angular velocity of A with respect to B in the figure given below:



Sol. Angular velocity of A with respect to B; $\omega_{AB} = \dfrac{(v_{AB})_\perp}{r_{AB}}$

$(v_{A,B})_\perp = v_A \sin\theta_1 + v_B \sin\theta_2$ and $r_{AB} = r$

Hence, $\omega_{AB} = \dfrac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$

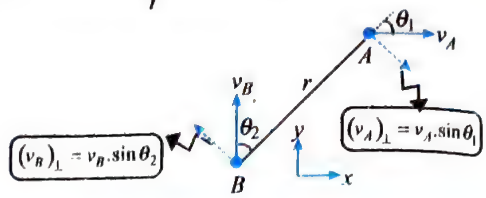
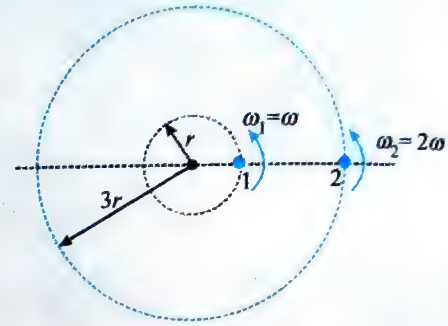


ILLUSTRATION 9.11

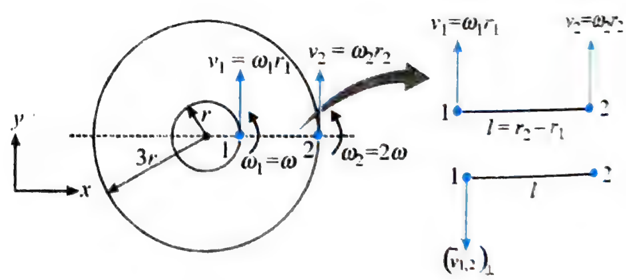
Two satellites 1 and 2 are orbiting with angular velocities ω and 2ω relative to the centre of earth respectively, in the same plane. If the radii of their orbits are r and $3r$ respectively, find the angular velocity of 1 relative to $2(\vec{\omega}_{1,2})$ and angular velocity of 2 relative to $1(\vec{\omega}_{2,1})$.



Sol. Angular velocity of '1' w.r.t. '2'

$$\vec{\omega}_{1,2} = \frac{(\vec{v}_{1,2})_\perp}{l} = \frac{(\omega_2 r_2 - \omega_1 r_1)}{(r_2 - r_1)}(\hat{k}) = \frac{6\omega r - \omega r}{(3r - r)}(\hat{k})$$

$$\Rightarrow \vec{\omega}_{1,2} = \frac{5\omega}{2}(\hat{k})$$



Angular velocity of '2' w.r.t '1'

$$\vec{\omega}_{2,1} = \frac{(\vec{v}_{2,1})_\perp}{l} = \frac{(\omega_2 r_2 - \omega_1 r_1)}{(r_2 - r_1)}(\hat{k})$$

$$= \frac{6\omega r - \omega r}{(3r - r)}(\hat{k}) \Rightarrow \vec{\omega}_{2,1} = \frac{5\omega}{2}(\hat{k})$$

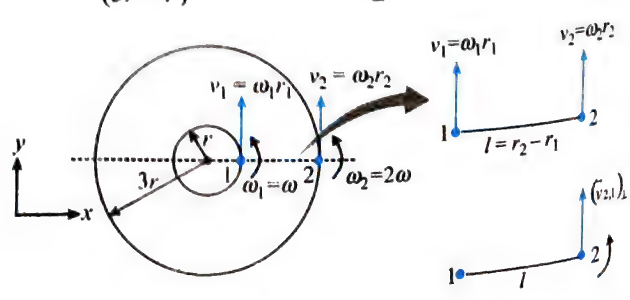


ILLUSTRATION 9.12

Find the time period of meeting of minute hand and second hand of a clock.

Sol. $\omega_{\min} = \dfrac{2\pi}{60}$ rad/ min and $\omega_{\sec} = \dfrac{2\pi}{1}$ rad/ min

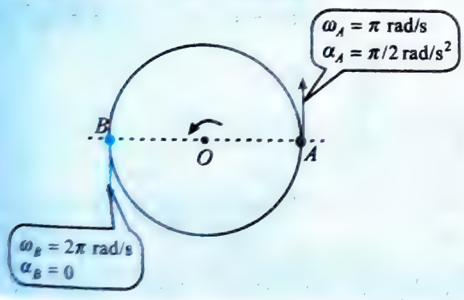
The rate of change of angle between minute hand and second hand is $\dfrac{d\theta}{dt} = \omega_{\sec} - \omega_{\min}$

$$(\omega_{\sec} - \omega_{\min}) \cdot T = 2\pi$$

$$\left(\frac{2\pi}{1} - \frac{2\pi}{60}\right) T = 2\pi \Rightarrow T = \frac{1}{\left(1 - \frac{1}{60}\right)} = \frac{60}{59}\text{min}$$

ILLUSTRATION 9.13

Two particles A and B move on a circle. Initially Particle A and B are diagonally opposite to each other. Particle A move with angular velocity π rad/s, angular acceleration $\pi/2$ rad/s^2 and particle B moves with constant angular velocity 2π rad/sec. Find the time after which both the particles A and B will collide.



Sol. Let the angle between OA and OB at any time is θ.

The rate of change of θ,

$$\dot{\theta} = \frac{d\theta}{dt} = \omega_B - \omega_A = 2\pi - \pi = \pi \text{ rad/sec}$$

Relative angular acceleration,

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \alpha_B - \alpha_A = -\frac{\pi}{2}\text{rad/ s}^2$$

If angular displacement is $\Delta\theta$, $\Delta\theta = \dot{\theta}t + \dfrac{1}{2}\ddot{\theta}t^2$

For A and B to collide angular displacement $\Delta\theta = \pi$

$$\Rightarrow \quad \pi = \pi t + \frac{1}{2}\left(\frac{-\pi}{2}\right)t^2$$

$$\Rightarrow \quad t^2 - 4t + 4 = 0$$

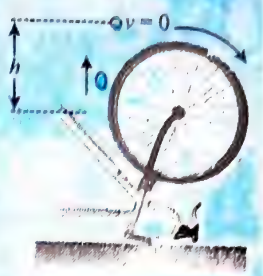
$$\Rightarrow \quad t = 2\text{sec}$$

$$\dot{\theta} = \pi \text{ rad/s}$$

$$\ddot{\theta} = -\frac{\pi}{2}\text{rad/s}$$

ILLUSTRATION 9.14

A cycle is placed upside down while its owner repairs the tire on the rear wheel. His friend spins the front wheel, of radius R, and observes that drops of water fly off tangentially in an upward direction when the drops are at the same level as the center of the wheel. He measures the height reached by drops moving vertically as shown in figure. A drop that breaks loose from the tire on one turn rises a distance h_1 above the tangent point. A drop that breaks loose on the next turn rises a distance $h_2 < h_1$ above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.



Sol. The mechanical energy is conserved in the drop–Earth system. At the instant, it comes off the wheel. The first drop has a velocity v_1 directed upward. The magnitude of this velocity is found from

$$\Delta K + \Delta U = 0 \Rightarrow \left(0 - \frac{1}{2}mv_1^2\right) + mgh_1 = 0$$

The angular velocity of the wheel at the instant the first drop leaves is

$$\omega_1 = \frac{v_1}{R} = \sqrt{\frac{2gh_1}{R^2}}$$

Similarly, for the second drop:

$$v_2 = \sqrt{2gh_2} \text{ and } \omega_2 = \frac{v_2}{R} = \sqrt{\frac{2gh_2}{R^2}}$$

The angular acceleration of the wheel is then

$$a = \frac{\omega_2^2 - \omega_1^2}{2\Delta\theta} = \frac{2gh/R^2 - 2gh_1/R^2}{2(2\pi)} = \frac{g(h_2 - h_1)}{2\pi R^2}$$

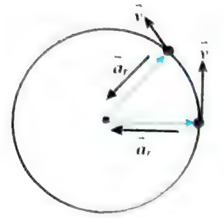
ACCELERATION IN CIRCULAR MOTION

Tangential and Radial Acceleration

In circular motion, the acceleration of the particle can be resolved into two components one along the radius and one along the tangent to its path. The component along the radius is called normal acceleration or radial or centripetal acceleration. The component along the tangent direction is called tangential acceleration.

Centripetal acceleration: It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.

Centripetal acceleration is always variable because it changes in direction. Centripetal acceleration is also called radial acceleration or normal acceleration.

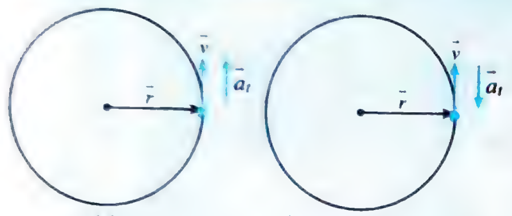


Tangential acceleration: When a particle moves in a curve, the magnitude of its velocity, that is, speed of the particle may change. Since the velocity changes its magnitude along the tangent to the path, the rate at which the magnitude of velocity changes along the tangent is called **tangential acceleration.**

$$a_t = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \alpha r = \text{Rate of change of speed.}$$

Important Points:

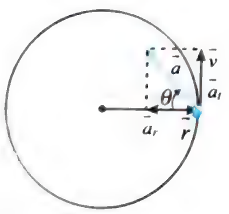
- If tangential acceleration is directed in direction of velocity then the speed of the particle increases. [Fig. (a)]
- If tangential acceleration is directed opposite to velocity then the speed of the particle decreases. [Fig. (b)]



(a) (b)

TOTAL ACCELERATION

Total acceleration is vector sum of centripetal acceleration and tangential acceleration.



$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_r + \vec{a}_t$$

$$a = \sqrt{a_t^2 + a_r^2}$$

$$\tan\theta = \frac{a_t}{a_r} \Rightarrow \theta = \tan^{-1}\left(\frac{a_t}{a_r}\right)$$

Important Points:

- In uniform circular motion, the speed of the particle is constant. Its tangential acceleration should be zero,

$$a_t = \frac{d|\vec{v}|}{dt} = 0$$

But its direction of motion is changing continuously. It means its velocity is changing. It should have centripetal acceleration or radial acceleration. Hence this net acceleration is equal to centripetal acceleration

$$|\vec{a}_{net}| = |\vec{a}_r|$$

- Differentiation of speed gives tangential acceleration.

$$a_t = \frac{d|\vec{v}|}{dt}$$

- Differentiation of velocity ($\vec{v}$) gives total acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

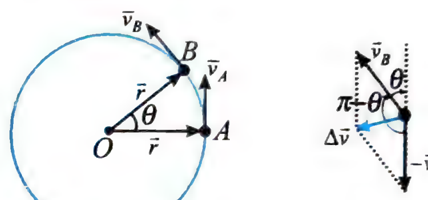
- $\left|\frac{d\vec{v}}{dt}\right|$ and $\frac{d|\vec{v}|}{dt}$ are not same physical quantity.

$\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e., magnitude of total acceleration

$\frac{d|\vec{v}|}{dt}$ is a rate of change of speed, i.e. tangential acceleration.

FINDING CENTRIPETAL ACCELERATION

Consider a particle which moves in a circle with constant speed v as shown in figure. When the particle describes (revolves) an angle θ, its velocity vector rotates through same angle θ. During this time let the velocity change from, $\vec{v}_A$ to $\vec{v}_B$.



$v_A = v_B = v$

$\therefore$ Change in velocity between the point A and B;

$$\Delta\vec{v} = \vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$$

Magnitude of change in velocity.

$$|\Delta\vec{v}| = \sqrt{v_B^2 + v_A^2 + 2v_A v_B \cos(\pi - \theta)}$$

$$|\Delta\vec{v}| = \sqrt{v^2 + v^2 - 2v^2\cos\theta} = \sqrt{2v^2(1-\cos\theta)}$$

$$= \sqrt{2v^2 \times 2\sin^2\frac{\theta}{2}} \Rightarrow |\Delta\vec{v}| = 2v\sin\frac{\theta}{2}$$

Magnitude of change in velocity, $|\Delta\vec{v}| = 2v\sin\frac{\theta}{2}$

Distance travelled by particle between A and $B = r\theta$

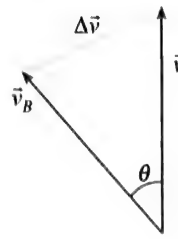
Hence time taken, $\Delta t = \frac{r\theta}{v}$

Magnitude of average net acceleration,

$$|\vec{a}_{net}|_{av} = \left|\frac{\Delta\vec{v}}{\Delta t}\right| = \frac{2v\sin\frac{\theta}{2}}{\left(\frac{r\theta}{v}\right)} = \frac{v^2}{r}\frac{2\sin\frac{\theta}{2}}{\theta}$$

If $\Delta t \to 0$, then θ is small, $\sin\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$

Let us find the direction of this acceleration. As we have taken very short time interval ($\Delta t \to 0$). During this interval θ tends to zero ($\Delta\theta \to 0$), $\vec{v}_B$ rotates towards $\vec{v}_A$. Consequently, $\Delta\vec{v}$ tends to be perpendicular to $\vec{v}_A$. Since $\vec{v}$ is tangential, $\Delta\vec{v}$ must be radially inward as shown in figure.



It means $\vec{a} = \dfrac{d\vec{v}}{dt}$ is directed towards the center of the circle.

Then, $\vec{a} = \dfrac{v^2}{R}\hat{r}$ where $\hat{r}$ = unit vector of position vector $\hat{r}$

The above acceleration is known as centripetal or normal or radial acceleration. It arises from the change in direction of velocity of the particle. As $\hat{r}$ changes its direction with respect to time, centripetal acceleration must change its direction.

Important Points:

- The centripetal acceleration or normal acceleration or radial acceleration is given by

$$|\vec{a}_r| = \frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 r$$

- Here we have derived the formula of centripetal acceleration under condition of constant speed, the same formula is applicable even when speed is variable.

ILLUSTRATION 9.15

A particle moves in a circle of radius 2.0 cm at a speed given by $v = 4t$, where v is in cm/s and t is in seconds.

(a) Find the tangential acceleration at $t = 1$s.

(b) Find total acceleration at $t = 1$s.

Sol.

(a) Tangential acceleration

$$a_t = \frac{dv}{dt} \text{ or } a_t = \frac{d}{dt}(4t) = 4\,\text{cm/s}^2$$

$$a_c = \frac{v^2}{R} = \frac{(4)^2}{2} = 8\ \text{cm/s}^2$$

(b) $\sqrt{a_t^2 + a_C^2} = \sqrt{(4)^2 + (8)^2} = 4\sqrt{5}\ \text{cm/s}^2$

Acceleration in Circular Motion: Vector Approach

Vector relation between velocity and angular velocity is given by,

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad \text{...(i)}$$

Differentiating the above equation with time, we have

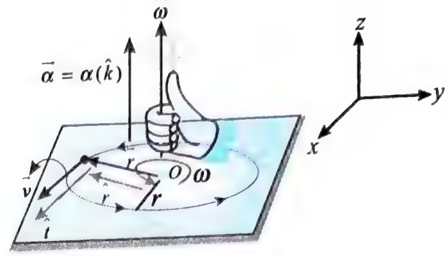
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

Using the formula of vector calculus,

$$\frac{d}{dt}(\vec{A} \times \vec{B}) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}$$

we have $\vec{a} = \dfrac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \dfrac{d\vec{r}}{dt}$

Substituting $\dfrac{d\vec{\omega}}{dt} = \vec{\alpha}$ and $\dfrac{d\vec{r}}{dt} = \vec{v}$



we have $\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$ ...(ii)

In above equation, the vector $\vec{a}$ has two components $\vec{\alpha} \times \vec{r}$ and $\vec{\omega} \times \vec{v}$. Let us explore the meaning of the first component

As $\vec{\alpha} = \alpha(\hat{k})$ and $\vec{r} = r(\hat{r})$, then

$$\vec{\alpha} \times \vec{r} = \alpha r(\hat{k} \times \hat{r}).$$

Since $\hat{k} \times \hat{r} = \hat{t}$, It means $\vec{\alpha} \times \vec{r} = \alpha r(\hat{t})$

Hence this is the tangential component of total acceleration $\vec{a}$, that is, tangential acceleration a_t.

Let us explore the meaning of the second component $(\vec{\omega} \times \vec{v})$

$$\vec{\omega} = \omega(\hat{k}) \text{ and } \vec{v} = v\hat{t}.$$

Then $\vec{\omega} \times \vec{v} = \omega v(\hat{k} \times \hat{t})$

As $\hat{k} \times \hat{t} = -\hat{r}$, it means, $\vec{\omega} \times \vec{v} = \omega v(-\hat{r})$

The component $\vec{\omega} \times \vec{v}$ is directed radially inward. It is the radial component of total acceleration, that is, centripetal acceleration.

Now it is clear total acceleration in non-uniform circular motion is given by the relation,

$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

where $\vec{a}_t = \vec{\alpha} \times \vec{r}$ is the tangential acceleration and $\vec{a}_r = \vec{\omega} \times \vec{v}$ is the centripetal acceleration.

The magnitude of tangential acceleration $|\vec{a}_t| = \alpha r$

and the magnitude of radial acceleration $|\vec{a}_r| = \omega . v$

We know $v = \omega r$ or $\omega = \dfrac{v}{r}$, it means, $|\vec{a}_r| = \dfrac{v}{r}.v = \dfrac{v^2}{r}$

ILLUSTRATION 9.16

A particle at the edge of a rotating disc speeds up from rest at a uniform angular acceleration α. If the radius of the disc is R, find the angular distance covered by the particle till it acquires a total acceleration a_0.

Sol. Using $\omega^2 = \omega_0^2 + 2\alpha\theta$

The angular displacement of the particle till it attains an velocity ω is

$$\theta = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

where $\omega_0 = 0$ (because the particle starts from rest)

Then, $\theta = \dfrac{\omega^2}{2\alpha}$ ...(i)

Let us calculate ω. The total acceleration of the particle is $a = \sqrt{a_t^2 + a_r^2}$

where $a_t = R\alpha$ and $a_r = R\omega^2$

If the maximum acceleration of the particle is a_0, we have $a_0 = \sqrt{R^2\alpha^2 + R^2\omega^4}$

This gives $\omega = \left(\dfrac{a_0^2 - R^2\alpha^2}{R^2}\right)^{\frac{1}{4}}$ ...(ii)

Substituting ω from Eq. (ii) in Eq. (i), we have

$$\theta = \frac{1}{2\alpha}\left[\left(\frac{a_0^2 - R^2\alpha^2}{R^2}\right)^{\frac{1}{4}}\right]^2 = \frac{\sqrt{a_0^2 - R^2\alpha^2}}{2R\alpha}$$

ILLUSTRATION 9.17

A particle moves in a circular path of radius R such that it speed v varies with distance s as $v = \alpha\sqrt{s}$ where α is a positive constant. Find the acceleration of the particle after traversing a distance s.

Sol. The total acceleration, $a = \sqrt{a_t^2 + a_r^2} = \sqrt{\left(\dfrac{dv}{dt}\right)^2 + \left(\dfrac{v^2}{R}\right)^2}$

where $v = \alpha\sqrt{s}$

Differentiating $v = \alpha\sqrt{s}$ with respect to time

$$\Rightarrow \quad \frac{dv}{dt} = \alpha\frac{1}{2}s^{-1/2}\frac{ds}{dt}$$

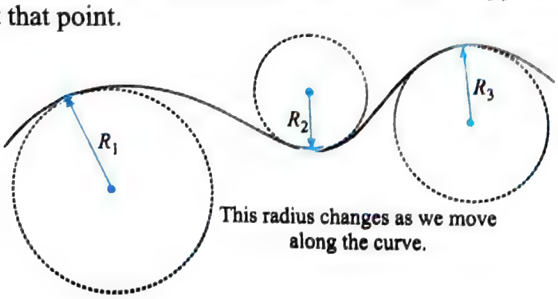
But $\dfrac{ds}{dt} = \alpha\sqrt{s} \Rightarrow a_t = \dfrac{dv}{dt} = \dfrac{\alpha^2}{2}$

The total acceleration $a = \sqrt{a_t^2 + a_r^2} = \sqrt{\left(\dfrac{dv}{dt}\right)^2 + \left(\dfrac{v^2}{R}\right)^2}$

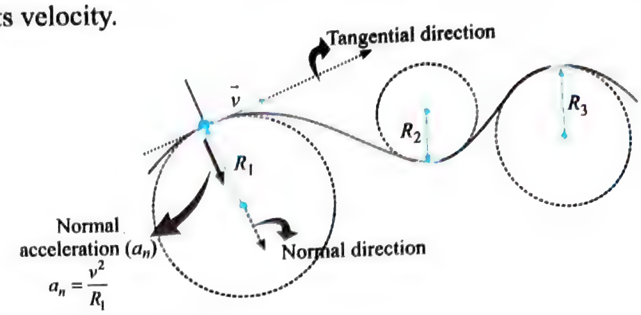
$$\Rightarrow \quad a = \sqrt{\left(\frac{\alpha^2}{2}\right)^2 + \left[\frac{\alpha\sqrt{s}}{R}\right]^2} = \alpha\sqrt{\frac{\alpha^2}{4} + \frac{s}{R^2}}$$

RADIUS OF CURVATURE

Any curved path can be assumed to be made of infinite circular arcs. We can draw a circle that closely fits nearby points on a local section of a curve. Radius of curvature at a point is the radius of the circular arc at a particular point which fits the curve at that point. Or we can say it is the radius of the approximating circle at that point.



When a particle moves in a curve, it has a tangential acceleration $a_t = \dfrac{d|\vec{v}|}{dt}$ due to change in its speed and it must have a radial (normal) acceleration $|\vec{a}_r| = v^2/R$ because of the change in direction of its velocity.

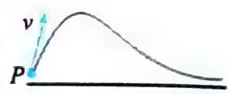


Hence we can write, $R = \dfrac{v^2}{a_n}$

where, R = radius of the effective circle at any instant when the particle has a velocity $\vec{v}$ and acceleration $\vec{a}$. This is known as **radius of curvature** of the path followed by the particle at the given position (or time).

ILLUSTRATION 9.18

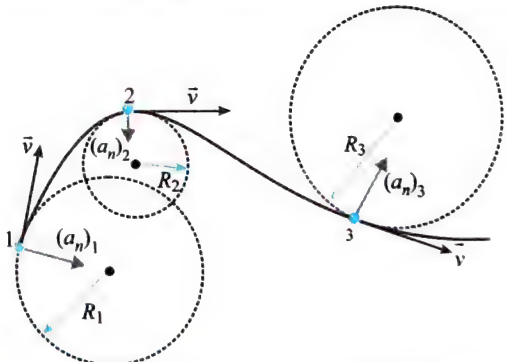
A particle P moves with a constant speed v in a curve as shown in the figure. Discuss the variation of magnitude of normal acceleration with time and distance.



Sol. Radius of curvature, $R = \dfrac{v^2}{a_n} \Rightarrow a_n = \dfrac{v^2}{R}$

or $\quad a_n \propto \dfrac{1}{R}$

Smaller the radius of curvature larger the centripetal acceleration.



Hence normal acceleration first increases and then decreases.

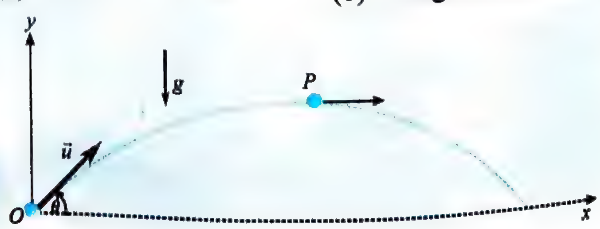
ILLUSTRATION 9.19

A particle is projected with velocity u at an angle θ with the horizontal.

(i) Find the tangential and normal acceleration of the particle at $t = 0$ and at highest point of its trajectory.

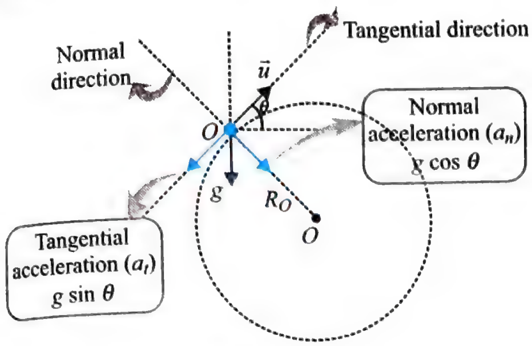
(ii) Find the radius of curvature:

(a) at O (b) at highest position P.



Sol.

(i) The direction of tangential acceleration is in the line of velocity and the direction of normal acceleration is perpendicular to velocity direction. The tangential and normal directions at O and P are shown in figure.

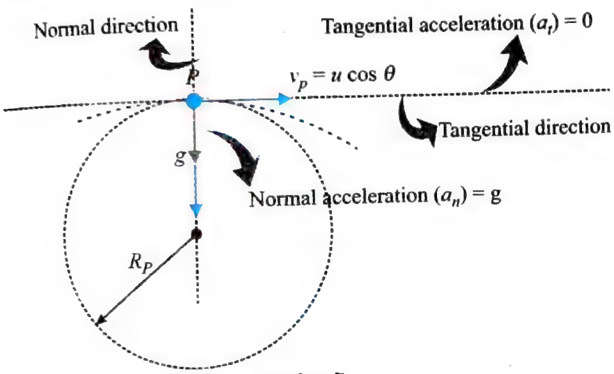


At Point O

At O: (at $t = 0$)

Tangential acceleration, $a_t = -g\sin\theta$

Normal acceleration, $a_n = g\cos\theta$



At Point P

At P (at highest point)

Tangential acceleration, $a_t = 0$

Normal acceleration, $a_n = g$

(ii) (a) Let radius of curvature at O be R_o. The normal acceleration at O be $g\cos\theta$.

$$a_n = \frac{v^2}{R} \Rightarrow R_o = \frac{v_o^2}{a_n} = \frac{v_o^2}{g\cos\theta}$$

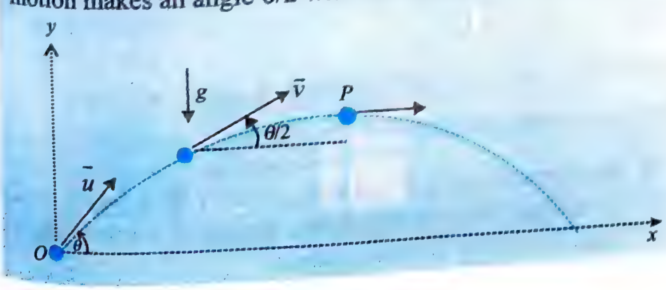
(b) Radius of curvature at P

Normal acceleration at P is 'g'

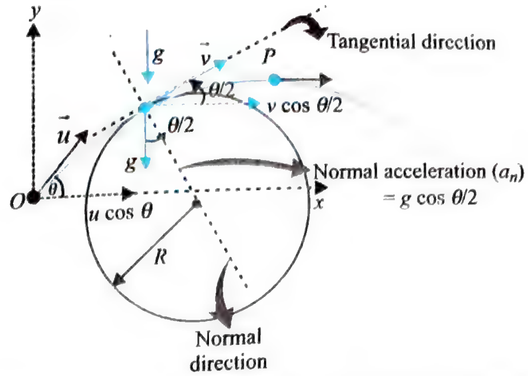
$$R_P = \frac{v^2}{a_n} = \frac{(v_o\cos\theta)^2}{g} = \frac{v_o^2\cos^2\theta}{g}$$

ILLUSTRATION 9.20

A particle is projected with velocity u at an angle θ with the horizontal. Find the radius of curvature where its line of motion makes an angle $\theta/2$ with horizontal.



Sol. The direction of normal acceleration is perpendicular to velocity direction as shown in figure.



The horizontal component of particle velocity will be constant

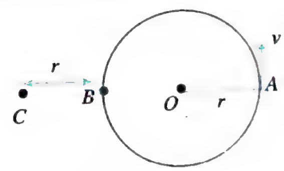
$$v\cos{}^{\theta}\!/_{2} = u\cos\theta \Rightarrow v = \frac{u\cos\theta}{\cos{}^{\theta}\!/_{2}}$$

Radius of curvature, $R = \dfrac{v^2}{a_n}$

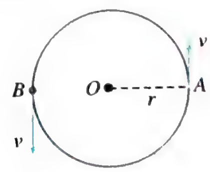
$$R = \frac{\left(\dfrac{u\cos\theta}{\cos{}^{\theta}\!/_{2}}\right)^2}{g\cos{}^{\theta}\!/_{2}} = \frac{u^2\cos^2\theta}{g\cos^3{}^{\theta}\!/_{2}}$$

CONCEPT APPLICATION EXERCISE 9.1

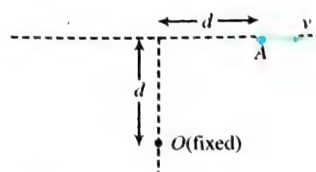
1. A particle is moving with constant speed in a circle as shown in figure. Find the angular velocity of the particle A with respect to fixed points B and C if angular velocity with respect to O is ω.



2. Particles A and B move with constant and equal speeds in a circle as shown in figure. Find the angular velocity of the particle A with respect to B, if the angular velocity of particle A w.r.t. O is ω.

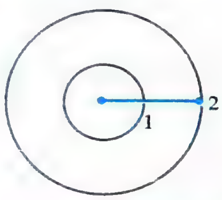


3. Find the angular velocity of A with respect to O at the instant shown in figure.



4. A particle travels in a circle of radius 20 cm at a speed that uniformly increases. If the speed changes from 5.0 m s^{-1} to 6.0 m s^{-1} in 2.0 s, find the angular acceleration.

5. Find the magnitude of the acceleration of a particle moving in a circle of radius 10 cm with uniform speed completing the circle in 4 s.
6. A particle in a circular path speeds up with a uniform rate between two diametrically opposite points of a circle of radius R. If its time of motion between these two points is equal to T, find the acceleration of the particle averaged over the time T.
7. The linear speed of a particle moving in a circle of radius R varies with time as $v = v_0 - kt$, where k is a positive constant. At what time the magnitudes of angular velocity and angular acceleration will be equal?
8. The angular velocity of a particle moving in a circle relative to the center of the circle is equal to ω. Find the angular velocity of the particle relative to a point on the circular path.
9. Two satellites 1 and 2 orbiting with the time periods T_1 and T_2, respectively, lie on the same line as shown in figure. After what minimum time, again the satellites will remain on the same line? Assume that the two satellites should lie in same side of the center of their concentric circular paths.



ANSWERS

1. $\dfrac{\omega}{2}, \dfrac{\omega}{3}$	2. ω	3. $\dfrac{v}{2d}$
4. 2.5 rad s^{-2}	5. 2.5 cm s^{-2}	6. $2\pi R/T^2$
7. $\dfrac{v_0 - k}{k}$	8. $-\omega/2$	9. $\left[\dfrac{T_1T_2}{T_1 - T_2}\right]$

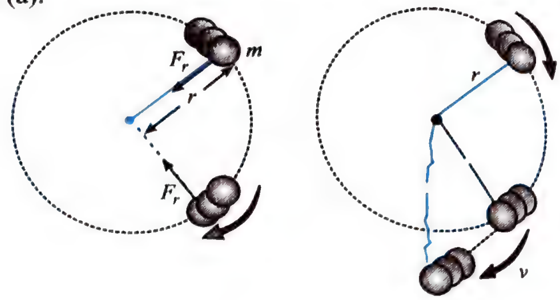
DYNAMICS OF CIRCULAR MOTION

CENTRIPETAL FORCE

The centripetal force is not another fundamental force of nature but is simply the net inward force needed to provide the centripetal acceleration necessary for circular motion.

Let us consider a puck of mass m that is tied to a string of length r and moves at constant speed in a horizontal, circular path as shown in figure. The weight of the puck is supported by a frictionless table, and the string is anchored to a peg at the center of the circular path of the puck.

Here a question arises, why does the puck move in a circle? According to Newton's first law, the puck would move in a straight line if there were no force on it; the string, however, prevents motion along a straight line by exerting on the puck a radial force $\vec{F}_r$ that makes it follow the circular path. This force is directed along the string toward the center of the circle as shown in Fig. (a).



(a) (b)

If we apply Newton's second law along the radial direction, the net force causing the centripetal acceleration can be related to the acceleration as follows:

$$F_c = ma_c = m\left(\frac{v^2}{r}\right) = \frac{mv^2}{r}$$

This force responsible for the centripetal acceleration, acts toward the center of the circular path and causes a change in the direction of the velocity vector. If this force should vanish, the object would no longer move in its circular path; instead, it would move along a straight-line path tangent to the circular path as shown in Fig. (b).

According to Newton's second law, when a particle accelerates, there must be a net force acting on the particle in the direction of its acceleration. Following this law, as the particle in uniform circular motion accelerates radially inwards, the particle must experience a net force directed radially inwards (centripetal). Hence, we call this **centripetal force**.

In some cases, many forces may act on a revolving particle such that two or more forces or components of some forces may push or pull the particle in circular paths towards their centres. In these cases, we can say that "the net force that is responsible in pushing or pulling the particle towards the centre of circular path" is known as **centripetal force**.

The centripetal force is not a special force. It can be any field force such as gravitational, electrostatics, magnetic, and others. This can be also be any constant force (reaction force), i.e., friction and normal reaction, etc. The string force and spring force can take the credit of pulling or pushing a particle in a circular track as described in this topic. Hence, any force or component of any force along the radial direction or sum (resultant) of two or many forces acting on a particle directed towards the centre of a circle may be called centripetal force.

As $F = ma$, centripetal force = mass × centripetal acceleration

i.e. $$F = \frac{mv^2}{r} = m\omega^2 r \quad \text{...(ii)}$$

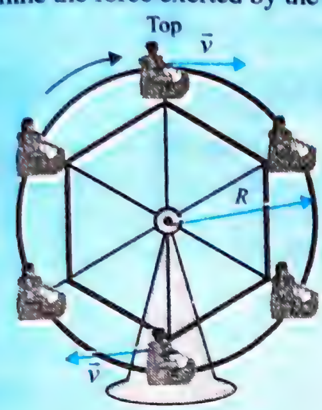
APPLICATIONS OF NEWTON'S LAWS OF MOTION IN CIRCULAR MOTION

- Remember mv^2/r is not a force itself. It is just the value of the net force acting along the inside normal which is responsible for circular motion.

- This force may be friction, normal, tension, spring force, gravitational force or a combination of them.
- So to solve any problem in uniform circular motion we identify all the forces acting along the normal (towards center), calculate their resultant and equate it to mv^2/r.
- If circular motion is non-uniform then in addition to above step we also identify all the forces acting along the tangent to the circular path, calculate their resultant and equate it to $m\,dv/dt$ or $m\,d|\vec{v}|/dt$.

ILLUSTRATION 9.21

A child of mass m rides on a Ferris wheel as shown in figure. The child moves in a vertical circle of radius R at a constant speed of v. Determine the force exerted by the seat on the child:



(i) at the bottom of the ride,
(ii) at the top of the ride.

Sol. As the speed of the child is constant, we can categorize this situation as one involving a particle (the child) in uniform circular motion, acted upon by two forces the gravitational force and normal reaction exerted by the seat.

(i) From free body diagram of forces acting on the child at the bottom of the ride, it is clear that the net upward force on the child that provides required centripetal force for circular motion.

Now let us apply Newton's second law to the child in the radial direction when he is at the bottom of the ride:

$$F_{net} = N_{bottom} - mg = m\left(\frac{v^2}{R}\right)$$

$$\Rightarrow N_{bottom} = mg + m\frac{v^2}{R} = mg\left(1 + \frac{v^2}{Rg}\right)$$

Hence, the magnitude of the force N_{bottom} exerted by the seat on the child is greater than the weight of the child. So, the child experiences an apparent weight that is greater than his true weight

(ii) In the FBD at top of the ride, it is clear the net downward force that provides the required centripetal acceleration.

Now let us apply Newton's second law to the child in the radial direction when he is at the top of the ride:

$$F_{net} = mg + N_{top} = m\left(\frac{v^2}{R}\right)$$

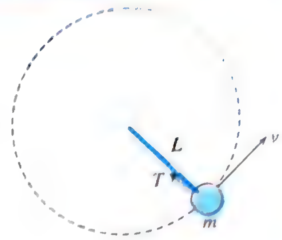
$$\Rightarrow N_{top} = mg - m\frac{v^2}{R} = mg\left(1 - \frac{v^2}{Rg}\right)$$

Hence, the magnitude of the force N_{top} exerted by the seat on the child is lesser than the weight of the child. So, the child experiences an apparent weight that is lesser than his true weight.

ILLUSTRATION 9.22

A block of mass 1 kg is tied to a string of length 1 m, the other end of which is fixed. The block is moved on a smooth horizontal table with constant speed 10 m s^{-1}. Find the tension in the string.

Sol. The ball in the figure does not accelerate vertically. Therefore, we model it as a particle in equilibrium in the vertical direction. It experiences a centripetal acceleration in the horizontal direction. In this case, centripetal force is provided by tension.



$$\Sigma F_y = N - mg = 0$$

$$N = mg \quad \text{...(i)}$$

$$\Sigma F_x = T = ma_c = \frac{mv^2}{r} \quad \text{...(ii)}$$

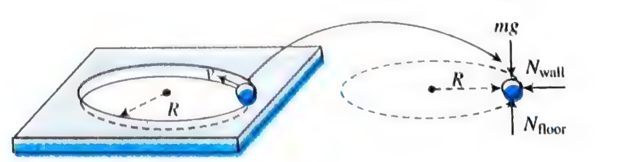
$$T = \frac{mv^2}{r} = \frac{1\times10^2}{1} = 100\,\text{N}$$

ILLUSTRATION 9.23

A ball of mass m moves with speed v against a smooth, fixed vertical circular groove of radius R kept on smooth horizontal surface. Find the:

(a) normal reaction of the floor on the ball.
(b) normal reaction of the vertical wall on the ball.

Sol. The ball in figure does not accelerate vertically. Here centripetal force is provided by normal reaction of vertical wall. It experiences a centripetal acceleration in the horizontal direction (radial). In this case, centripetal force is provided by tension.



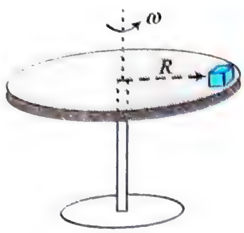
$$\Sigma F_y = N_{Floor} - mg = 0$$

$$N_{Floor} = mg \quad \text{...(i)}$$

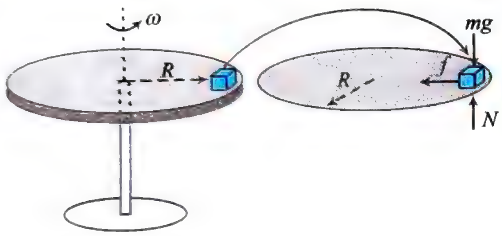
$$\Sigma F_x = N_{wall} = ma_c = \frac{mv^2}{R} \quad \text{...(ii)}$$

ILLUSTRATION 9.24

A block of mass m is kept on the edge of a horizontal turn table of radius R, which is rotating with constant angular velocity ω (along with the block) about its axis. If coefficient of friction is μ, find the friction force between block and table.



Sol. The block in figure does not accelerate vertically. Therefore, we model it as a particle in equilibrium in the vertical direction. It experiences a centripetal acceleration in the horizontal direction. In this case, centripetal force is provided by friction.



$$\Sigma F_y = N - mg = 0$$

$$N = mg \quad \text{...(i)}$$

$$\Sigma F_x = f = ma_c = m\omega^2 R \quad \text{...(ii)}$$

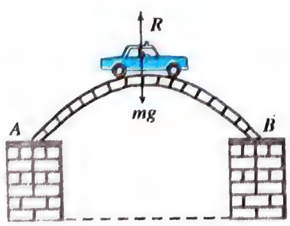
Friction force = centripetal force = $m\omega^2 R$

ILLUSTRATION 9.25

A car of mass m moving over a convex bridge of radius r. Find the normal reaction acting on car when it is at the highest point of the bridge.

Sol. The motion of the motor car over a convex bridge AB is the motion along the segment of a circle AB (figure).

The centripetal force is provided by the difference of weight mg of the car and the normal reaction R of the bridge.



$$\therefore \quad mg - R = \frac{mv^2}{r}$$

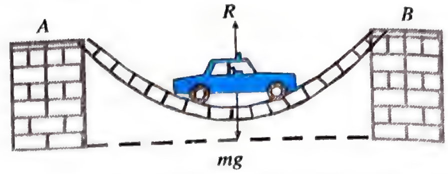
or $$R = mg - \frac{mv^2}{r}$$

Clearly $R < mg$, i.e., the weight of the moving car is less than the weight of the stationary car.

ILLUSTRATION 9.26

A car of mass m moving over a concave bridge of radius r. Find the normal reaction acting on car when it is at the lowest point of the bridge.

Sol. The motion of the motor car over a concave bridge AB is the motion along the segment of a circle AB (figure).



The centripetal force is provided by the difference of normal reaction R of the bridge and weight mg of the car.

$$\therefore \quad R - mg = \frac{mv^2}{r}$$

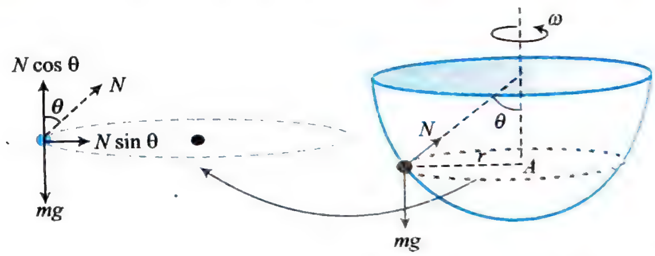
or $$R = mg + \frac{mv^2}{r}$$

Clearly $R > mg$, i.e., the weight of the moving car is greater than the weight of the stationary car.

ILLUSTRATION 9.27

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is θ, find the angular speed at which the bowl is rotating.

Sol. Let ω be the angular speed of the rotation of the bowl. Two forces are (a) normal reaction, N and (b) weight, mg.



The ball is rotating in a circle of radius r (= $R \sin \theta$) with center at A at an angular speed ω. Thus,

$$N \sin \theta = mr\omega^2$$

$$= mR\omega^2 \sin \theta$$

$$N = mR\omega^2 \quad \text{...(i)}$$

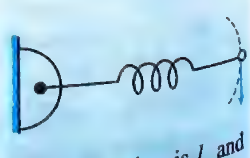
and $$N \cos \theta = mg \quad \text{...(ii)}$$

Dividing Eqs. (i) by (ii), we get

$$\frac{1}{\cos\theta} = \frac{\omega^2 R}{g} \quad \text{or} \quad \omega = \sqrt{\frac{g}{R\cos\theta}}$$

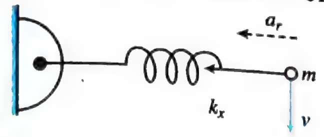
ILLUSTRATION 9.28

A particle of mass m is moving with a constant speed v in a circular path in a smooth horizontal plane (plane of the paper) by a spring force as shown in figure. If the natural length of the spring is l_0 and stiffness of the spring is k, find the elongation of the spring.



Sol. If the particle executes uniform circular motion, its speed must be uniform because there is no tangential force to

speed it up. Here, the spring force kx acting on the particle is the centripetal force caused by the elongation x of the spring.



Equation of motion:

$$\Sigma F_r = F_{SP} = ma_r \quad \text{...(i)}$$

But centripetal acceleration

$$a_r = \frac{v^2}{R} \text{ (where } R = \text{radius of circular path)} \quad \text{...(ii)}$$

Radius of rotation: $R = l_0 + x$ ...(iii)

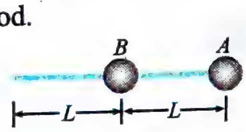
$$|F_{SP}| = kx$$

Put in (i), $kx = \dfrac{mv^2}{(l_0 + x)} \Rightarrow kx^2 + kl_0 x - mv^2 = 0$

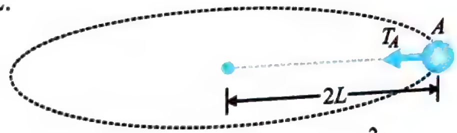
This gives $x = \dfrac{\sqrt{k^2 l_0^2 + 4mv^2 k} - kl_0}{2k}$

ILLUSTRATION 9.29

Ball A is attached to one end of a rigid massless rod, while an identical ball B is attached to the center of the rod, as figure illustrates. Each ball has a mass $m = 0.50$ kg, and the length of each half of the rod is $L = 0.50$ m. This arrangement is held by the empty end and is whirled around in a horizontal circle at a constant rate, so each ball is in uniform circular motion. Ball A travels at a constant speed of $v_A = 4.0$ m/s. Find the tension in each half of the rod.



Sol. **For Ball A:** Only a single tension force of magnitude T_A acts on ball A. It points to the left in the drawing and is due to the tension in the rod between the two balls. This force alone provides the centripetal force keeping ball A on its circular path of radius $2L$.

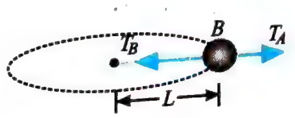


Ball A $\quad \underbrace{T_A}_{\text{Centripetal force}, F_C} = \dfrac{mv_A^2}{2L}$

The tension in the right half of the rod:

$$T_A = \frac{mv_A^2}{2L} = \frac{(0.50)(4.0)^2}{2(0.5)} = 8.0 \text{ N} \quad \text{...(i)}$$

For Ball B: Two tension forces act on ball B. One has a magnitude T_B and points to the left in the figure. It is due to the tension in the left half of the rod. The other has a magnitude T_A and points to the right. It is due to the tension in the right half of the rod. The centripetal force acting on ball B points toward the center of the circle and is the vector sum of these two forces, or $T_B - T_A$.



Ball B $\quad \underbrace{T_B - T_A}_{\text{Centripetal force}, F_C} = \dfrac{mv_B^2}{L}$ ...(ii)

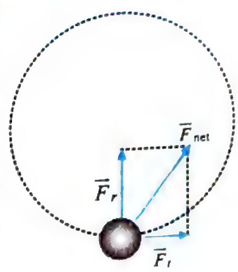
The angular speed of both the balls should be equal as the time period of rotation of both the balls is equal.

$$\omega = \frac{v_A}{2L} = \frac{v_B}{L} \Rightarrow v_B = \frac{v_A}{2} = \frac{4}{2} = 2.0 \text{ m/s} \quad \text{...(iii)}$$

From Eqs. (i),(ii) and (iii) we get the tension in the left half of the rod:

$$T_B = T_A + \frac{mv_B^2}{L} = 8.0 + \frac{(0.50)(2.0)^2}{0.5} = 12 \text{ N}$$

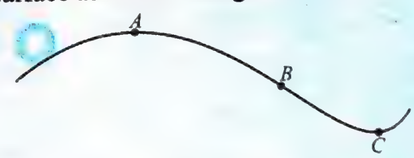
NON-UNIFORM CIRCULAR MOTION



If a particle moves with varying speed in a circular path, there is, in addition to the radial component of acceleration, a tangential component having magnitude $|dv/dt|$. It means, the force acting on the particle must also have a tangential and a radial component. The total acceleration of the particle should be $\vec{a} = \vec{a}_r + \vec{a}_t$. It means the total force exerted on the particle will be $\vec{F}_{net} = \vec{F}_r + \vec{F}_t$ as shown in figure. The vector $\vec{F}_r$ is directed toward the center of the circle and is responsible for the centripetal acceleration. The vector $\vec{F}_t$ tangent to the circle is responsible for the tangential acceleration, which represents a change in the particle's speed with time.

ILLUSTRATION 9.30

A bead of mass m slides along a curved wire lying on a horizontal surface as shown in figure.



(i) If the bead slides at constant speed along the wire, draw the vectors representing the force exerted by the wire on the bead at points A, B, and C.

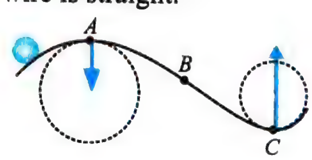
(ii) Now consider the bead in figure speeds up with constant tangential acceleration as it moves toward the right. Draw the vectors representing the force on the bead at points A, B, and C.

Sol.

(i) Because the speed is constant, the only direction the force can have is that of the centripetal acceleration.

As $F_{centripetal} = N_{wire} = \dfrac{mv^2}{R}$

The force is larger at C than at A because the radius of curvature at C is smaller. There will be no force at B because the wire is straight.



(ii) In this case in addition to the forces in the centripetal direction in part (a), there are now tangential forces to provide the tangential acceleration. The tangential force is the same at all three points because the tangential acceleration is constant.

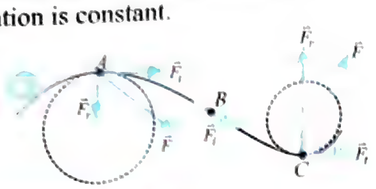


ILLUSTRATION 9.31

A simple pendulum is constructed by attaching a bob of mass m to a string of length L fixed at its upper end. The bob oscillates in a vertical circle. It is found that the speed of the bob is v when the string makes an angle θ with the vertical. Determine:

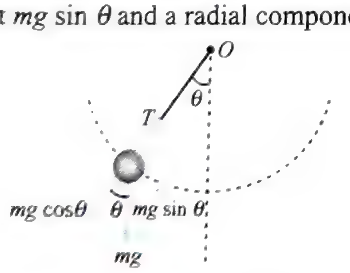
(i) the tangential acceleration of the sphere,

(ii) the tension in the cord,

(iii) the magnitude of net force on the bob at the instant.

Sol. In this case, the speed of the sphere is not uniform, at most points along the path, a tangential component of acceleration arises from the gravitational force exerted on the sphere. Here the motion of the sphere as a particle under a net force and moving in a circular path, but it is not a particle in *uniform* circular motion. We need to use the techniques discussed in this section on nonuniform circular motion.

(i) From the free body diagram of the sphere, it can be observed that two forces acting on the sphere are the gravitational force exerted by the Earth and the tension force exerted by the cord.
Let us resolve gravitational force (mg) into a tangential component $mg \sin\theta$ and a radial component $mg \cos\theta$.



Applying Newton's second law to the sphere in the tangential direction:

$$\Sigma F_t = mg\sin\theta = ma_t \Rightarrow a_t = g\sin\theta$$

(ii) Apply Newton's second law to the forces acting on the sphere in the radial direction,

$$F_r = T - mg\cos\theta = \frac{mv^2}{l} \Rightarrow T = mg\left(\frac{v^2}{lg} + \cos\theta\right)$$

(iii) Net acceleration of the bob

$$a_{\text{net}} = \sqrt{a_t^2 + a_r^2} = \sqrt{(g\sin\theta)^2 + \left(\frac{v^2}{l}\right)^2}$$

Hence net force acting on the bob,

$$|\vec{F}_{\text{net}}| = ma_{\text{net}} = m\sqrt{g^2\sin^2\theta + \frac{v^4}{l^2}}$$

ILLUSTRATION 9.32

The kinetic energy of a particle of mass m moving along a circle of radius r depends on the distance covered as $K = as^2$ where a is a constant. Find the centripetal and tangential forces and also the total force acting on the particle as a function of s.

Sol. The kinetic energy of a particle, $K = \frac{1}{2}mv^2 = as^2$

$$\text{or } v^2 = \frac{2as^2}{m} \quad \text{...(i)}$$

$$\text{Centripetal force, } F_C = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{2as^2}{m}\right) = \frac{2as^2}{r} \quad \text{...(ii)}$$

$$\text{Tangential acceleration, } a_t = \frac{dv}{dt} = \frac{d}{dt}\left(\sqrt{\frac{2a}{m}}\,s\right) = \left(\sqrt{\frac{2a}{m}}\right)\frac{ds}{dt}$$

$$\Rightarrow a_t = \sqrt{\frac{2a}{m}}\,v = \sqrt{\frac{2a}{m}}\left(\sqrt{\frac{2a}{m}}\,s\right) = \frac{2as}{m}$$

$$\text{Tangential force, } F_t = ma_t = m\left(\frac{2as}{m}\right) = 2as \quad \text{...(iii)}$$

$$\text{Total force, } F = \sqrt{F_t^2 + F_c^2} = 2as\sqrt{1 + \left(\frac{s}{r}\right)^2}$$

ILLUSTRATION 9.33

A block of mass m is kept on rough horizontal turn table at a distance r from center of table. Coefficient of friction between turn table and block is μ. Now turn table starts rotating with uniform angular acceleration α. Find the time after which slipping occurs between block and turn table.

Sol. Tangential acceleration of the block, $a_t = \frac{dv}{dt} = \alpha r$

Speed of the particle after time t: $\int_0^v dv = (\alpha r)\int_0^t dt \Rightarrow v = \alpha rt$

Centripetal acceleration, $a_c = \frac{v^2}{r} = \alpha^2 rt^2$

Net acceleration $a_{\text{net}} = \sqrt{a_t^2 + a_c^2} = \sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

Net force acting on the block, $F_n = ma_{\text{net}} = m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4}$

If the block starts slipping, $f = \mu_s mg$

$$m\sqrt{\alpha^2 r^2 + \alpha^4 r^2 t^4} = \mu_s mg \Rightarrow (\alpha^4 r^2 t^4) = \mu^2 g^2 - \alpha^2 r^2$$

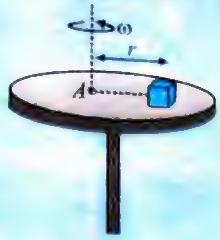
$$t = \left(\frac{\mu^2 g^2 - \alpha^2 r^2}{\alpha^4 r^2}\right)^{1/4} \Rightarrow t = \left[\left(\frac{\mu g}{\alpha^2 r}\right)^2 - \left(\frac{1}{\alpha}\right)^2\right]^{1/4}$$

ILLUSTRATION 9.34

A small block is supported by a turn-table. The friction coefficient between block and surface is μ.

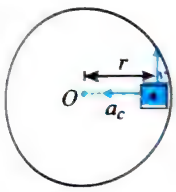
(a) If turn-table rotates at constant angular speed ω, what can be the maximum angular speed for which the block does not slip?

(b) If the angular speed is increased uniformly from rest with an angular acceleration α, at what speed will the block slip?

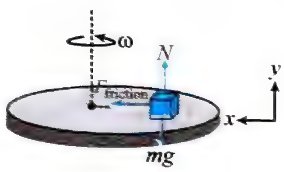


Sol.

(a) The weight of the block should be balanced by the normal reaction by the turn table on the block. The block moves in the circular path, the friction force acting on the block provides required centripetal force for its circular motion.



Top view



Side view

Since angular speed is constant, the block has centripetal acceleration only. From Newton's second law in radial direction,

$$F_{net} = f = ma_r = \frac{mv^2}{r} = m\omega^2 r \qquad \text{...(i)}$$

If the block does not slip the friction should be static in nature $f \le \mu_s mg$

$$m\omega^2 r \le \mu_s mg \Rightarrow \omega \le \sqrt{\frac{\mu_s g}{r}}$$

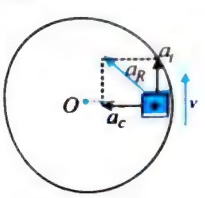
Hence, maximum angular speed, $\omega_{max} = \sqrt{\dfrac{\mu_s g}{r}}$

(b) When the turn-table rotates with angular acceleration, the block has centripetal as well as tangential acceleration.

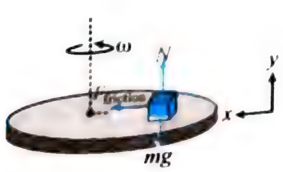
Therefore $\vec{a}_{net} = \vec{a}_c + \vec{a}_t$

$$|\vec{a}_R| = \sqrt{a_c^2 + a_t^2} = \sqrt{(\omega^2 r)^2 + (r\alpha)^2}$$

Resultant acceleration of block is parallel to surface of turn-table. The only force that is parallel to surface is force of friction.



Top view



Side view

Net force acting on the block,

$$F_n = ma_{net} = m\sqrt{(\omega^2 r)^2 + (r\alpha)^2}$$

If the block does not slip the friction should be static in nature $f \le \mu_s mg$

$$m\sqrt{(\omega^2 r)^2 + (r\alpha)^2} \le \mu_s mg \Rightarrow (\omega^2 r)^2 \le (\mu_s g)^2 - (r\alpha)^2$$

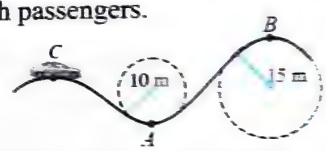
$$\Rightarrow \quad \omega \le \left[\left(\frac{\mu_s g}{r}\right)^2 - \alpha^2\right]^{1/4}$$

Hence maximum angular speed, $\omega_{max} = \left[\left(\dfrac{\mu_s g}{r}\right)^2 - \alpha^2\right]^{1/4}$

CONCEPT APPLICATION EXERCISE 9.2

1. In a rotor, a hollow vertical cylindrical structure rotates about its axis and a person rests against the inner wall. At a particular speed of the rotor, the floor below the person is removed and the person hangs resting against the wall without any floor. If the radius of the rotor is 2 m and the coefficient of static friction between the wall and the person is 0.2, find the minimum speed at which the floor may be removed. Take $g = 10\ m/s^2$.

2. A roller-coaster car has a mass of 500 kg when fully loaded with passengers.



   (a) If the vehicle has a speed of 20.0 m s^{-1} at point A, what is the force exerted by the track on the car at this point?
   (b) What is the maximum speed the vehicle can have at point B and still remain on the track?

3. A 60 kg woman is on a large vertical swing of radius 20 m. The swing rotates with constant speed.
   (a) At what speed would she feel weightless at the top?
   (b) At this speed, what is her apparent weight at the bottom?

4. An air puck of mass m_1 is tied to a string and allowed to revolve in a circle of radius R on a frictionless horizontal table. The other end of the string passes through a small hole in the center of the table, and a load of mass m_2 is tied to the string. The suspended load remains in equilibrium while the puck on the tabletop revolves.

   (a) Find the tension in the string.
   (b) Find the radial force acting on the puck.
   (c) Find the speed of the puck.
   (d) Qualitatively describe what will happen in the motion of the puck if the value of m_2 is somewhat increased by placing an additional load on it.
   (e) Qualitatively describe what will happen in the motion of the puck if the value of m_2 is instead decreased by removing a part from the hanging load.

5. A ceiling fan has a diameter (of the circle through the outer edges of the three blades) of 120 cm and rpm 1500 at full speed. Consider a particle of mass 1 g sticking at the outer end of a blade.

How much force does it experience when the fan runs

(a) How much force does it experience when the fan runs at full speed?

(b) Who exerts this force on the particle?

(c) How much force does the particle exert on the blade along its surface?

6. A block of mass m is kept on a horizontal ruler. The friction coefficient between the ruler and the block is μ. The ruler is fixed at one end and the block is at a distance L from the fixed end. The ruler is rotated about the fixed end in the horizontal plane through the fixed end.

(a) What can the maximum angular speed be for which the block does not slip?

(b) If the angular speed of the ruler is uniformly increased from zero at an angular acceleration α, at what angular speed will the block slip?

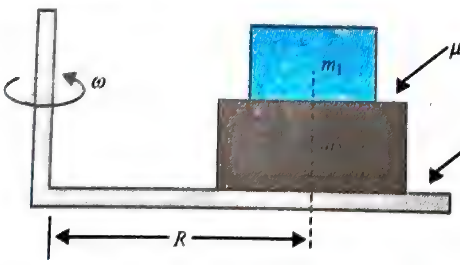
7. A car is moving with uniform speed over a circular bridge of radius R which subtends an angle of 90° at its center. Find the minimum possible speed so that the car can cross the bridge without losing the contact any where.

8. A circular table with smooth horizontal surface is rotating at an angular speed ω about its axis. A groove is made on the surface along a radius, and a small particle is gently placed inside the groove at a distance l from the center. Find the speed of the particle with respect to the table as its distance from the center becomes L.

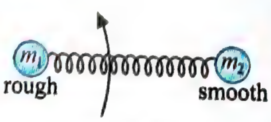
9. There are two blocks of masses m_1 and m_2. m_1 is placed on m_2 on a table which is rotating with an angular velocity ω about the vertical axis. The coefficients of friction between the blocks is μ_1 and between m_2 and table is μ_2 ($\mu_1 < \mu_2$). If the blocks are placed at distance R from the axis of rotation, for relative sliding between the surfaces in contact, find the:

(a) frictional force at the contacting surface

(b) maximum angular speed ω.



10. A block of mass m_1 connected with another block of mass m_2 by a light spring of natural length l_0 and stiffness k is kept stationary on a rough horizontal surface. The coefficient of friction between m_1 and surface is μ and the block m_2 is smooth. The block m_2 is moved with certain speed so as to execute uniform circular motion around the block m_1 in horizontal plane. Find the (a) maximum angular speed of the block m_2 relative to m_1 (b) acceleration of m_2 in (a).



ANSWERS

1. 10 m/s

2. (a) 2.50×10^4 N (b) $5\sqrt{6}$ m s^{-1}

3. (a) $10\sqrt{2}$ m s^{-1} (b) 1200 N

4. (a) $m_2 g$ (b) $m_2 g$ (c) $\sqrt{\dfrac{m_2 gR}{m_1}}$ (d) Gains speed (e) Spiral outwards

5. (a) $\left(\dfrac{3\pi^2}{2}\right)$N (b) Friction (c) $\left(\dfrac{3\pi}{2}\right)^2$ N

6. (a) $\sqrt{\dfrac{\mu g}{2}}$ (b) $\left[\left(\dfrac{\mu g}{L}\right)^2 - \alpha^2\right]^{\frac{1}{4}}$

7. $\left(\dfrac{Rg}{\sqrt{2}}\right)^{\frac{1}{2}}$

8. $\omega(L^2 - l^2)^{\frac{1}{2}}$

9. (a) $(m_1 + m_2)\omega^2 R$ (b) $\sqrt{\dfrac{\mu_2 g}{R}}$

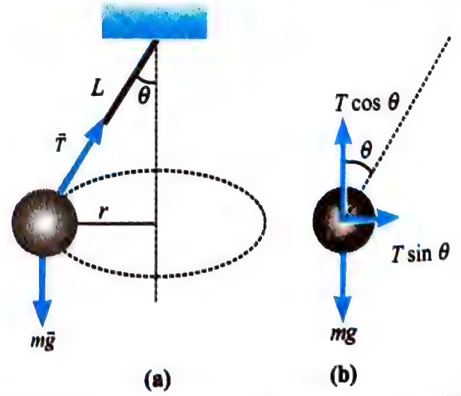
10. (a) $\sqrt{\dfrac{\mu k m_1 g}{m_2(l_0 k + \mu m_1 g)}}$ (b) $\dfrac{\mu m_1 g}{m_2}$

CONICAL PENDULUM

A small ball of mass m is suspended from a string of length L. The ball revolves with constant speed v in a horizontal circle of radius r as shown in figure. Because the string sweeps out the surface of a cone, the system is known as a **conical pendulum.**

The string sweeps out a cone and that the ball moves in a horizontal circle. Let us find an expression for v.

The ball in figure does not accelerate vertically. Therefore, we model it as a particle in equilibrium in the vertical direction. It experiences a centripetal acceleration in the horizontal direction, so it is modeled as a particle in uniform circular motion in this direction.



Let θ represent the angle between the string and the vertical. In the diagram of forces acting on the ball in figure, the force $\vec{T}$ exerted by the string on the ball is resolved into a vertical component $T\cos\theta$ and a horizontal component $T\sin\theta$ acting toward the center of the circular path. Apply the particle in equilibrium model in the vertical direction:

$$\sum F_y = T\cos\theta - mg = 0$$

or $T\cos\theta = mg$ ...(i)

$$\sum F_x = T\sin\theta = ma_c = \frac{mv^2}{r}$$ ...(ii)

Divide Eq. (ii) by Eq. (i) and use $\tan\theta = \frac{v^2}{rg}$

Solve for v: $v = \sqrt{rg\tan\theta}$

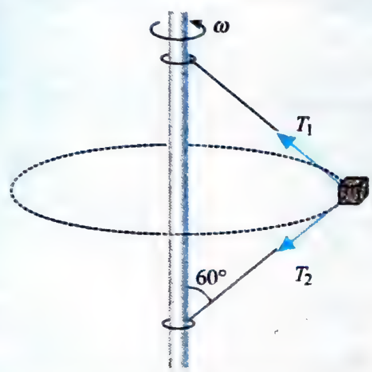
Incorporate $r = L\sin\theta$ from the geometry in figure:

$$v = \sqrt{Lg\sin\theta\tan\theta}$$

Notice that the speed is independent of the mass of the ball.

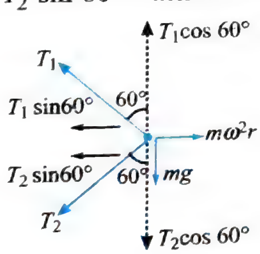
ILLUSTRATION 9.35

A small block is connected to one end of two identical massless strings of length $16\frac{2}{3}$ cm each with their other ends fixed to a vertical rod. If the ratio of tensions T_1/T_2 be 4 : 1, then what will be the angular velocity ω of the block?



Sol. For horizontal equilibrium of the block,

$$T_1\sin 60° + T_2\sin 60° = m\omega^2 r = m\omega^2 l\sin 60°$$



$$T_1 + T_2 = m\omega^2 l$$ ...(i)

For vertical equilibrium of the block,

$$T_1\cos 60° = T_2\cos 60° + mg$$ ...(ii)

Dividing Eqs. (i) by (ii),

$$\frac{T_1+T_2}{T_1-T_2} = \frac{\omega^2 l}{2g} \Rightarrow \frac{\frac{T_1}{T_2}+1}{\frac{T_1}{T_2}-1} = \frac{\omega^2 l}{2g}$$

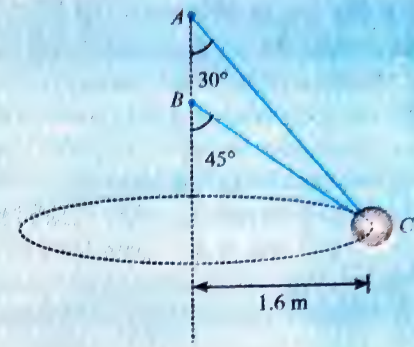
$$\Rightarrow \frac{4+1}{4-1} = \frac{\omega^2 l}{2g}$$

$$\omega^2 = \frac{10g}{3l} = \frac{10\times 9.8\times 3}{3\times 50\times 10^{-2}} = 196$$

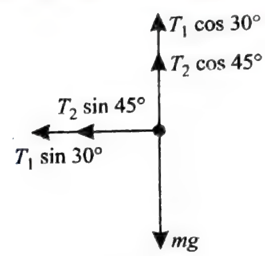
$$\Rightarrow \omega = 14 \text{ rad s}^{-1}$$

ILLUSTRATION 9.36

Two wires AC and BC are tied at C of small sphere of mass 5 kg, which revolves at a constant speed v in the horizontal speed v in the horizontal circle of radius 1.6 m. Find the minimum value of v.



Sol. From force diagram shown in figure,



$$T_1\cos 30° + T_2\cos 45° = mg$$ ...(i)

$$T_1\sin 30° + T_2\sin 45° = \frac{mv^2}{r}$$ ...(ii)

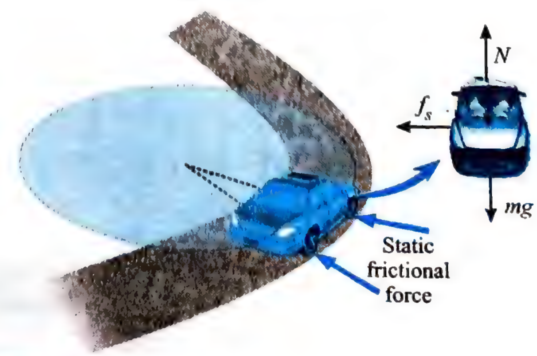
After solving Eqs. (i) and (ii), $T_1 = \dfrac{mg - \frac{mv^2}{r}}{\frac{(\sqrt{3}-1)}{2}}$

But $T_1 > 0$, $\dfrac{mg - \frac{mv^2}{r}}{\frac{\sqrt{3}-1}{2}} > 0 \Rightarrow mg > \dfrac{mv^2}{r}, v < \sqrt{rg}$

$$v_{max} = \sqrt{rg} = \sqrt{1.6\times 10} = 40 \text{ m/s}$$

TURNING OF A VEHICLE ON HORIZONTAL CIRCULAR ROAD

A car moving on a flat horizontal road negotiates a curve as shown in figure. If the radius of the curve is R and the coefficient of static friction between the tires and dry pavement is μ_s, let us calculate maximum speed the car can have and still make the turn successfully.



Imagine that the curved roadway is part of a large circle so that the car is moving in a circular path.

As the size of car is much smaller than radius of curve we model the car as a particle in uniform circular motion in the horizontal direction. The car is not accelerating vertically, so it is modeled as a particle in equilibrium in the vertical direction.

The force that enables the car to remain in its circular path is the force of static friction. (It is static because no slipping occurs at the point of contact between road and tires. If this force of static friction were zero—for example, if the car were on an icy road—the car would continue in a straight line and slide off the curved road.) The maximum speed v_{max} the car can have around the curve is the speed at which it is on the verge of skidding outward. At this point, the friction force has its maximum value $f_{s,max} = \mu_s N$. Apply equation in the radial direction for the maximum speed condition:

$$f_{s,max} = \mu_s N = \frac{mv_{max}^2}{r} \quad \text{...(i)}$$

The car is in equilibrium the vertical direction:

$$\sum F_y = 0 \Rightarrow N - mg = 0 \Rightarrow N = mg \quad \text{...(ii)}$$

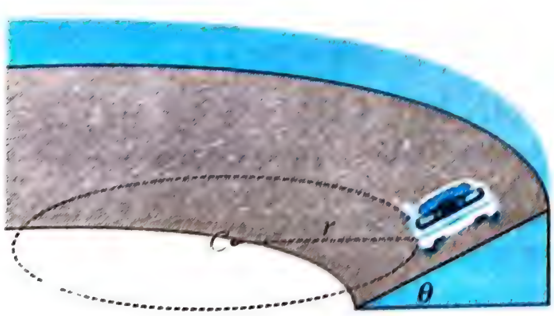
From Eqs. (i) and (ii), we get

$$v_{max} = \sqrt{\frac{\mu_s NR}{m}} = \sqrt{\frac{\mu_s (mg)R}{m}} = \sqrt{\mu_s gR}$$

Notice that the maximum speed does not depend on the mass of the car, which is why curved highways do not need multiple speed limits to cover the various masses of vehicles using the road.

BANKING OF ROADS

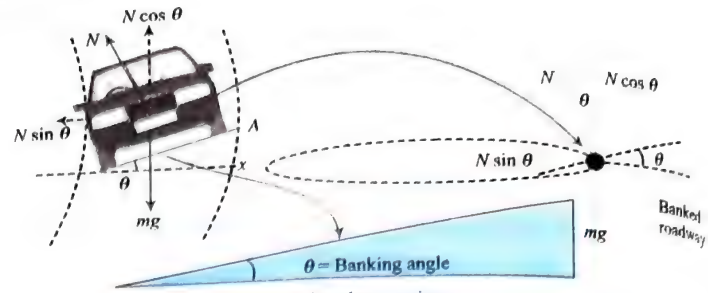
The curved roadways are designed in such a way that a vehicle will not have to rely on friction to round the curve without skidding. In other words, a car moving at the designated speed can negotiate the curve even when the road is covered with ice. For achieving this such roads are usually *banked*, which means that the roadway is tilted toward the inside of the curve as seen in the figure.



Banking of curved road provides following purpose:

- To contribute in providing necessary centripetal force.
- To reduce frictional wear and tear of tyres.
- To avoid skidding
- To avoid overturning of vehicles.

Suppose the designated speed for the curved-road ramp is to be v and the radius of the curve is R. Let us find At what angle should the curve be banked.



Cross section of roads at turning

The car is modeled as a particle in equilibrium in the vertical direction and a particle in uniform circular motion in the horizontal direction.

On a level (unbanked) road, the force that causes the centripetal acceleration is the force of static friction between car and road. If the road is banked at an angle θ as in figure, however, the normal force $\vec{N}$ has a horizontal component toward the center of the curve. Because the ramp is to be designed so that the force of static friction is zero, only the component $N_x = N\sin\theta$ causes the centripetal acceleration. Write Newton's second law for the car in the radial direction, which is the x direction:

$$\sum F_r = N\sin\theta = \frac{mv^2}{R} \quad \text{...(i)}$$

Apply the particle in equilibrium model to the car in the vertical direction:

$$\sum F_y = N\sin\theta - mg = 0$$

$$N\cos\theta = mg \quad \text{...(ii)}$$

Divide Equation (i) by Equation (ii): $\tan\theta = \dfrac{v^2}{Rg}$ ...(iii)

ILLUSTRATION 9.37

A circular track has a radius of 10 m. If a vehicle goes round it at an average speed of 18 km/hr what should be the proper angle of banking.

Sol. We know angle of banking is given by

$$\tan\theta = \frac{v^2}{rg} = \frac{(5)^2}{10\times 10} = \frac{1}{4}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{4}\right)$$

ILLUSTRATION 9.38

A turn of radius 20 m is banked for the vehicles going at a speed of 36 km/h. If the coefficient of static friction between the road and the tyre is 0.4, what are the possible speeds of a vehicle so that it neither slips down nor skids up ?

Sol. Angle of banking for designed speed ...(i)

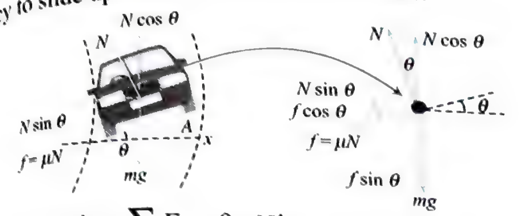
$$\tan\theta = \frac{v_0^2}{Rg}$$

$$v_0 = 36 \text{ km}/h = 10 \text{ m/s}$$

$$\Rightarrow \quad \tan\theta = \frac{v_0^2}{Rg} = \frac{10^2}{20\times 10} = \frac{1}{2}$$

The vehicle may have tendency to slide up or down depending on speed of the vehicle. If speed of the vehicle is more it has tendency to slide up and visa-versa.

For speed greater than designed speed: The vehicle has tendency to slide up friction will act downward.



In vertical direction, $\sum F_y = 0,\ N'\cos\theta - \mu N'\sin\theta = mg$

In horizontal direction: $\sum F_r = \dfrac{mv_{max}^2}{R}$

$$N\sin\theta + \mu N\cos\theta = \frac{mv_{max}^2}{R} \qquad \text{...(ii)}$$

$$N\cos\theta - \mu N\sin\theta = mg \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii),

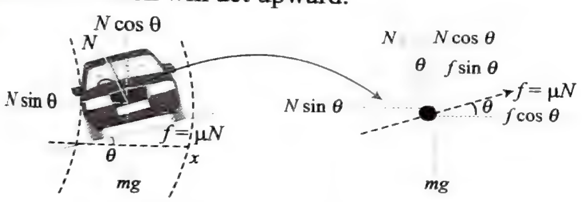
$$\frac{N(\sin\theta + \mu\cos\theta)}{N(\cos\theta - \mu\sin\theta)} = \frac{\frac{mv_{max}^2}{R}}{mg} \Rightarrow \frac{(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta} = \frac{v_{max}^2}{Rg}$$

$$\Rightarrow \left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right) = \frac{v_{max}^2}{Rg} \qquad \text{...(iv)}$$

$$\Rightarrow \left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right) = \left(\frac{0.5 + 0.4}{1 - 0.4 \times 0.5}\right) = \frac{v_{max}^2}{20 \times 10}$$

$$\Rightarrow v_{max} = 15 \text{ m/s}$$

For speed lesser than designed speed: The vehicle has tendency to slide down friction will act upward.



In vertical direction: $\sum F_y = 0,\ N'\cos\theta + \mu N'\sin\theta = mg \quad \text{...(v)}$

In horizontal direction $\sum F_r = \dfrac{mv_{max}^2}{R}$

$$N'\sin\theta - \mu N'\cos\theta = \frac{mv'^2}{R} \qquad \text{...(vi)}$$

From Eqs. (v) and (vi), we get $\dfrac{(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta} = \dfrac{v'^2}{Rg}$

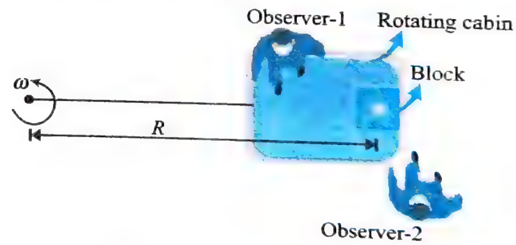
$$\left(\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right) = \frac{v'^2}{Rg} \Rightarrow v_{min} = 10\sqrt{\frac{1}{6}} \text{ m/s}$$

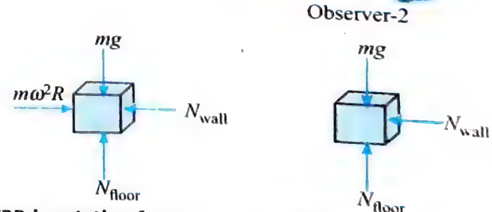
CENTRIFUGAL FORCE

Suppose you are sitting in your car at rest and then step on the gas pedal, you feel as if you are being pressed back into your seat. A fictitious force appears to act on you. In fact, this sensation of the fictitious force that presses you in the backward direction comes from the inertia of your body, which is accelerated forward by your car. This example describes a fictitious force due to a change in the car's speed.

Another fictitious force is due to the change in the direction of the velocity vector. To understand the motion of a system that is non-inertial because of a change in direction, consider you are sitting by the side of a door of a car while it negotiates a curve, you feel as if you are pushed away from the centre of the curve and hits the door. Here a question arises, what causes you to move toward the door? A popular but incorrect explanation is that a force acting away from the centre pushes you outward from the center of the circular path. Although often called the **"centrifugal force,"** it is a fictitious force. Since you are sitting in an accelerating frame you think that you are not accelerating towards the centre of the circle, rather you are centrifuged. But the actual situation can be viewed from ground frame. The actual thing is that "you are moving in a circular path for same time" by the inward reaction force (pressing) offered by the inner wall of the car. You may feel as if you are pressed against the walls of the vehicle. According to Newton's 3rd law, the wall is pushing your muscles in contact, towards the centre of the circular path. In fact, the reaction force given by the wall of the car is the centripetal force and in the rotating frames $\vec{F}_{pseudo} = -m\vec{a}_{cp}$ is the centrifugal force which never exists in inertial frame.

Consider a block of mass m is kept against a wall in a cabin, rotating with angular velocity ω. It is being observed by two observers, observer '1' is observing the block from the cabin while observer '2' is observing it from ground. If observer '1' analyze the dynamics of the block kept at a distance R from the axis of rotation, he has to assume that a force $m\omega^2 R$ react radially outward on the particle. Only then he can apply Newton's laws of motion to the block in the rotating frame. This radially outward pseudo force is called the centrifugal force.





FBD in rotating frame **FBD in ground frame**

For observe '1': $N_{wall} - m\omega^2 R = 0$

For observe '2' $N_{wall} = m\omega^2 R$

Important Points:

- As centrifugal force exists in rotating (accelerating) frames, it is a pseudo force. Hence, centrifugal force has no action-reaction pair. The centrifugal force is adopted to solve the problems in the rotating frame.
- Centripetal forces are real forces (arising from interaction) whereas centrifugal forces are imaginary or pseudo force (arising from the rotation of the reference frames). The centrifugal force acting on a particle depends only on

"mass m", angular velocity $\vec{\omega}$ of the rotating frame (but not the angular velocity of the particle) and the perpendicular distance of the particle from the axis of rotation of the reference frame.

- The centrifugal force is directed "radially" outwards from the axis of rotation of the reference frame (or observer) along the line drawn from the particle perpendicular to the axis of rotation.

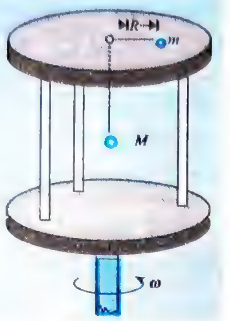
ILLUSTRATION 9.39

A particle of mass m connected with a hanging bob of mass M by an inextensible string is stationary relative to the rotating platform. The coefficient of static friction between the particle and platform is μ. Find the:

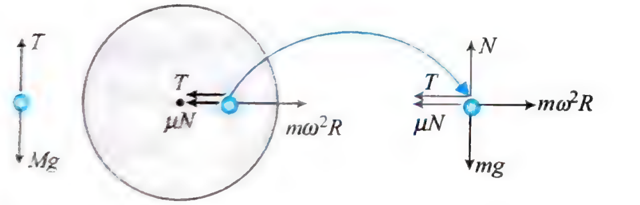
(a) maximum angular speed ω_{max},

(b) minimum angular speed ω_{min}

(for $M > \mu m$)



Sol. The particle kept on rotating platform is not sliding. The friction on the particle should be static in nature. If we analyze the block from rotating platform, we should apply a pseudo force (centrifugal force) directed "radially" outwards from the axis of rotation. For maximum angular velocity, the particle has the tendency to slide away from axis of rotation, it results the direction of friction force towards axis of rotation as shown in FBD. The particle is stationary in rotating platform.



FBD of M **Top view** **FBD of m from rotating frame**

For the particle: $T + \mu N = m\omega^2 R$ ...(i)

For the hanging bob: $T = Mg$ ...(ii)

From Eqs. (i) and (ii),

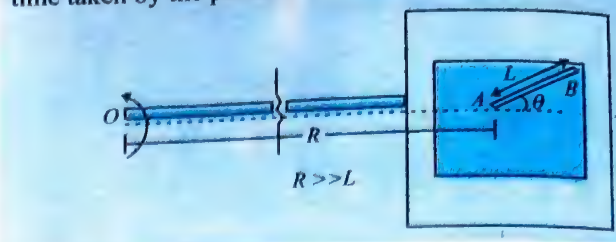
$$Mg + \mu mg = m\omega^2 R \Rightarrow \omega_{max} = \sqrt{\frac{(M+\mu m)g}{mR}}$$

For minimum angular velocity of the direction of friction will be towards radial direction (away from center)

Hence $\omega_{min} = \sqrt{\dfrac{(M-\mu m)g}{mR}}$

ILLUSTRATION 9.40

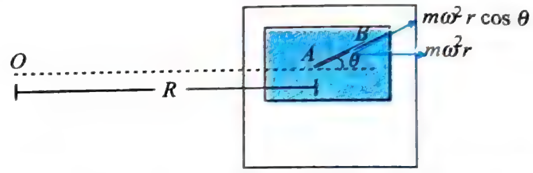
A table with smooth horizontal surface is fixed in a cabin that rotates with a uniform angular velocity ω in a circular path of radius R. A smooth groove AB of length $L(<<R)$ is made on the surface of the table. The groove makes an angle θ with the radius OA of the circle in which the cabin rotates. A small particle is kept at the point A in the groove and is released to move along AB. Find the time taken by the particle to reach the point B.



Sol. As the table rotates with the uniform angular velocity ω, hence particle also rotates with the same angular velocity. We will analyze this problem in the reference frame of table.

The centrifugal force acting on the particle is in a direction parallel to OA, hence, its component along the length of the groove $= m\omega^2 r \cos\theta$

If a be the acceleration of the particle along the groove, then $a = \omega^2 r \cos\theta$



As the length of groove AB $L(<<R)$, we can take the radius of rotation of the particle while moving on groove constant and equal to $r = R$.

Hence the acceleration a of the particle along the groove will be constant then $a = \omega^2 R \cos\theta$

Hence the time required by the particle to reach the end B of the groove, using

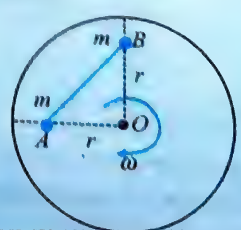
$$L = ut + \frac{1}{2}at^2$$

Here $u = 0$ and $a = \omega^2 R \cos\theta$

$$\Rightarrow \quad t = \sqrt{\frac{2L}{\omega^2 R\cos\theta}}$$

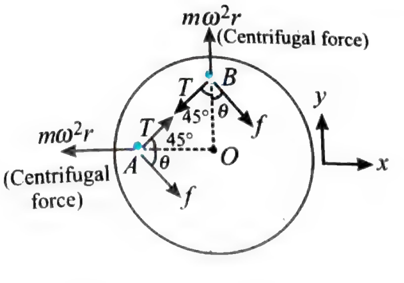
ILLUSTRATION 9.41

Figure shows top view of a circular rotating table, rotating with speed ω. Two particles connected by a string are kept on two mutually perpendicular radii. Coefficient of friction is μ. What can be the maximum angular speed of the table so that particles do not slip on it?



Sol. We will solve this problem in the reference frame of table. Here the centrifugal force on the particles will act in radial away as shown in free body diagrams of the blocks.

Friction force is static, therefore it is variable. Let friction act at an angle θ with radial direction as shown in figure.



In the impending state of motion, $f_{max} = \mu N$

For particle A,

In y-direction: $f\cos\theta + T\cos 45° = mr\omega^2$ ...(i)

In x-direction: $f\sin\theta = T\sin 45°$ ...(ii)

From Eqs. (i) and (ii) we eliminate T, to obtain

$$mr\omega^2 = f(\sin\theta + \cos\theta) \quad \text{...(iii)}$$

We can write Eq. (iii) as

$$mr\omega^2 = f\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)$$

$$= f\sqrt{2}(\sin\theta\cos 45° + \cos\theta\sin 45°)$$

$$= \mu mg\sqrt{2}\sin(45° + \theta)$$

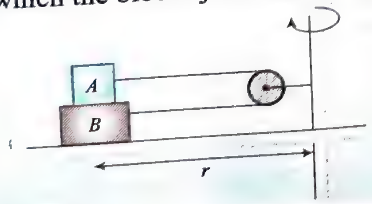
$$\omega^2 = \frac{\sqrt{2}\mu g}{r}\sin(45° + \theta)$$

Since maximum value of $\sin(45° + \theta) = 1$

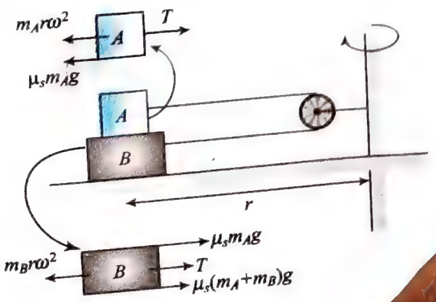
$$\therefore \quad \omega_{max.} = \sqrt{\frac{\sqrt{2}\mu g}{r}}$$

ILLUSTRATION 9.42

In figure shown two blocks of mass m_A and m_B. If the blocks are rotating at radius r as shown in figure are kept on a rough table. Consider friction (μ_s) between all the contact surfaces pulley is frictionless. Determine the angular speed of the turn-table for which the blocks just begin to slide.



Sol. We will solve this problem in the reference frame of table. Here the centrifugal force on the blocks will act in radial away as shown in free body diagrams of the blocks.



Equation for block B:

$$T + \mu_s(m_A + m_B)g + \mu_s m_A g = m_B r\omega^2 \quad \text{...(i)}$$

Equation for block A:

$$T = \mu_s m_A g + m_A r\omega^2 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we eliminate T to obtain

$$2\mu_s m_A g + \mu_s(m_A + m_B)g = (m_B - m_A)r\omega^2$$

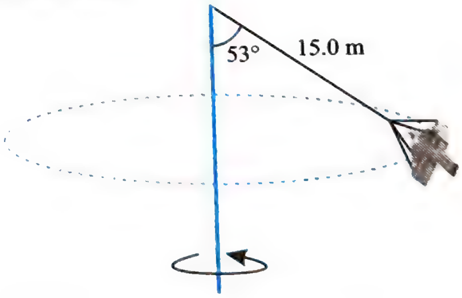
$$\omega = \left[\frac{\mu_s g(3m_A + m_B)}{r(m_B - m_A)}\right]^{1/2}$$

CONCEPT APPLICATION EXERCISE 9.3

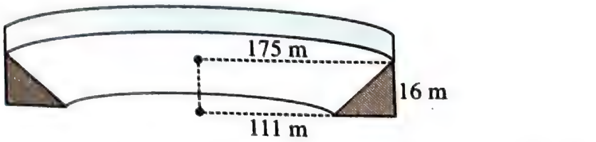
1. A large mass M and a small mass m hang at the two ends of a string that passes over a smooth tube. The mass m moves around a circular path which lies in a horizontal plane. The length of the string from the mass m to the top of the tube is l and θ is the angle the length makes with the vertical. What should be the frequency of rotation of the mass m so that the mass M remains stationary?

2. An 900-kg race car can drive around an unbanked turn at a maximum speed of 50 m/s without slipping. The turn has a radius of curvature of 200 m. Air flowing over the car exerts a downward-pointing force (called the *downforce*) of 11000 N on the car. **(a)** What is the coefficient of static friction between the track and the car's tires? **(b)** What would be the maximum speed if no downforce acted on the car?

3. A "swing" ride at a carnival consists of chairs that are swung in a circle by 15.0 m cables attached to a vertical rotating pole, as the drawing shows. Suppose the total mass of a chair and its occupant is 200 kg. **(a)** Determine the tension in the cable attached to the chair. **(b)** Find the speed of the chair.

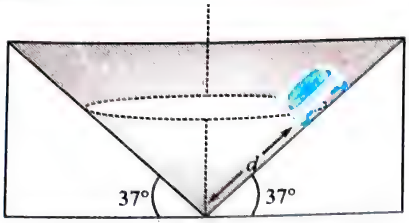


4. On a banked race track, the smallest circular path on which cars can move has a radius of 111 m, while the largest path has a radius of 175 m, as the drawing illustrates. The height of the outer wall is 16 m. Find **(a)** the smallest and **(b)** the largest speed at which cars can move on this track without relying on friction.

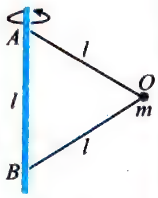


5. A racetrack has the shape of an inverted cone, as the drawing shows. On this surface the cars race in circles that are parallel to the ground. For a speed of 34.0 m/s, at what value of the distance d should a driver locate his car

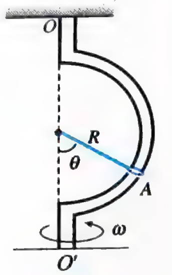
if he wishes to stay on a circular path without depending on friction?



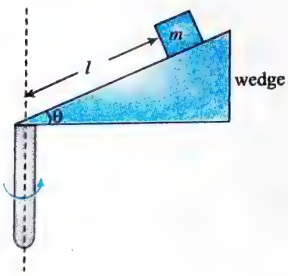
6. A particle of mass m is connected with two inextensible strings AO and BO of equal lengths l. These two strings are finally attached to a vertical rod at points A and B as shown in figure. Distance between A and B is also l. The setup is rotated with constant angular speed ω with rod as the axis. **(a)** Find the values of ω for which the particle remains at point B. **(b)** Find the range of values of ω for which tension (T_1) in the string AO is greater than mg but the other string remains slack. **(c)** Find the value of ω for which tension (T_1) in string AO is twice the tension (T_2) in the string BO.



7. A sleeve A can slide freely along a smooth rod bent in the shape of a half circle of radius R. The system is set in rotation with a constant angular velocity ω about a vertical axis OO'. Find the angle θ corresponding to the steady position of the sleeve.



8. A small wedge whose base is horizontal is fixed to a vertical rod as shown in figure. The sloping side of the wedge is frictionless and the wedge is spun with a constant angular speed ω about vertical axis as shown in the figure. Find the:



(a) value of angular speed ω for which the block of mass m just does not slide down the wedge.

(b) normal reaction on the block by wedge when block does not slip relative to wedge.

ANSWERS

1. $\frac{1}{2\pi}\sqrt{\frac{Mg}{ml}}$

2. (a) $\frac{9}{16}$ **(b)** $15\sqrt{5}$ m/s

3. (a) $\frac{10000}{3}$ N **(b)** $4\sqrt{10}$ m/s

4. (a) $\frac{\sqrt{1110}}{2}$ m/s **(b)** $\frac{5\sqrt{70}}{2}$ m/s

5. 150 m

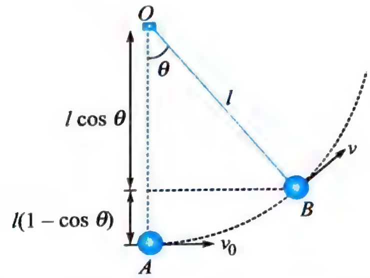
6. (a) $\omega = \sqrt{\frac{g}{l}}$ **(b)** $\sqrt{\frac{g}{l}} < \omega \le \sqrt{\frac{2g}{l}}$ **(c)** $\sqrt{\frac{6g}{l}}$

7. $\cos^{-1}\left(\frac{g}{\omega^2 R}\right)$

8. (a) $\sqrt{\frac{g\sin\theta}{l\cos^2\theta}}$ **(b)** $mg\sec\theta$

MOTION IN A VERTICAL CIRCLE

Consider a small bob of mass m attached to a string of length l tied at a pivot shown in figure.



If the bob is given an initial speed v_0 as shown in figure, it starts following the circular path shown by dashed line. As it moves up, due to gravity, its speed decreases. When it is at an angular displacement θ from the initial position, we can find its speed by using conservation of mechanical energy principle

At points A and B, we have, $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\right) + mgl(1-\cos\theta) = 0 \quad \text{...(i)}$$

$$v^2 = v_0^2 - 2gl(1-\cos\theta)$$

or $$v = \sqrt{v_0^2 - 2gl(1-\cos\theta)} \quad \text{...(ii)}$$

Equation (ii) gives the velocity of bob during circular motion at an angular displacement θ from the initial position.

For tangential acceleration at any position on vertical circle

$$a_t = \frac{mg\sin\theta}{m} = g\sin\theta \quad \text{...(iii)}$$

Normal acceleration, $a_n = \frac{v^2}{l}$

$$\Rightarrow \quad a_n = \frac{v^2 - 2gl(1-\cos\theta)}{l} \quad \text{...(iv)}$$

If initial velocity v_0, imparted to the bob is very small, then after traversing a small angular amplitude it will return back as velocity is not sufficient to make the complete revolution and it will start oscillations. Let the angular amplitude be $\theta = \alpha$, at which its velocity becomes zero. Angle α can be obtained from Eq. (ii) by substituting $v = 0$ in it as

$$0 = \sqrt{v_0{}^2 - 2gl(1-\cos\alpha)}$$

or $$v_0^2 - 2gl + 2gl\cos\alpha = 0$$

or $$\cos\alpha = \frac{2gl - v_0^2}{2gl} \quad \text{...(v)}$$

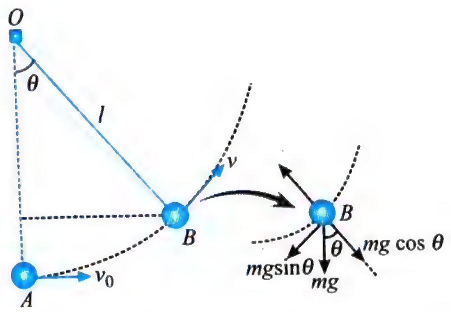
During circular motion tension in the string is also varying. See figure, when the bob is at an angular position θ, there are two forces acting on it. Tension T toward centre of circle and mg in downward direction. The net force toward the center of circle is $T - mg\cos\theta$, which is responsible for providing the required centripetal force. Thus when the bob is at an angular displacement θ, we have

$$T - mg\cos\theta = \frac{mv^2}{l}$$

or $$T = mg\cos\theta + \frac{mv^2}{l}$$

Substituting the value of v from Eq. (ii), we get

$$T = \frac{mv_0^2}{l} - 2mg + 3mg\cos\theta \quad \text{...(vi)}$$



From Eq. (vi), we can calculate the tension in thread when it makes an angle θ with the downward vertical. During circular motion sometimes it is possible that tension in thread becomes zero. It can not occur in lower half of the circle but it can be possible when particle is making revolution in upper half of the circle. If it happens at an angle $\theta = \phi$, then from Eq. (iv)

$$0 = \frac{mv_0^2}{l} - 2mg + 3mg\cos\phi$$

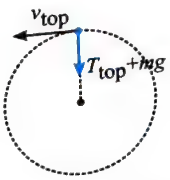
or $$v_0^2 - 2gl + 3gl\cos\phi = 0$$

or $$\cos\phi = \frac{2gl - v_0^2}{3gl} \quad \text{...(vii)}$$

Condition of Completing Vertical Circle

The bob will complete the circle only and only if tension is never zero (except momentarily, if at all) if tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity.

From Eq. (vi), it can be observed that tension decreases with increase in θ because $\cos\theta$ decreases as θ increases and v decreases with height. Hence tension is minimum at the top most position. i.e. $T_{min} = T_{top}$. However, if tension is momentarily zero at highest point the body would still be able to complete the circle.



Hence, condition for completing the circle (or looping the loop)

$$T_{top} + mg = \frac{mv_{top}^2}{l} \quad \text{...(viii)}$$

For looping the loop, $T_{top} \geq 0$, from (viii)

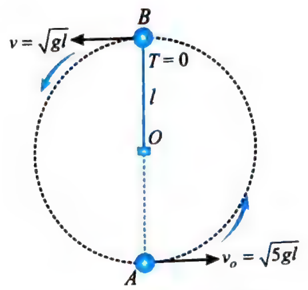
$$\frac{mv_{top}^2}{l} \geq mg \quad \Rightarrow v_{top} \geq \sqrt{gl} \quad \text{...(vii)}$$

Condition for looping the loop is $v_{top} \geq \sqrt{gl}$

Minimum velocity required at top most position to complete circle is $\sqrt{gR}$ where R = Radius of the circle.

At top most position $\theta = 180°$ and $v = \sqrt{gl}$, then from Eq. (ii)

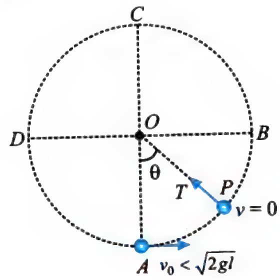
$$\sqrt{gl} = \sqrt{v_0^2 - 2gl(1-\cos\theta)} \Rightarrow v_0 \geq \sqrt{5gl}$$



i.e., for looping the loop, minimum velocity at lowest point must be greater than or equal to $\sqrt{5gR}$ where R = Radius of the circle.

Condition for Oscillation: The Speed Becomes Zero Before Tension

If velocity of the particle at lowest point $v_0 < \sqrt{2gl}$, the ball never rises above the level of the center O i.e. the body is confined to move within B and D.



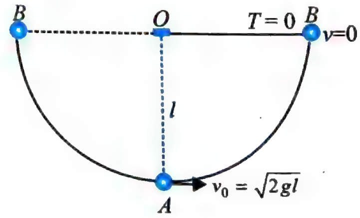
In this case tension cannot be zero, since a component of gravity acts radially outwards. Hence the string will not go slack, and the ball will reverse back as soon as its speed becomes zero. Its motion will be oscillatory motion.

Special Case: If velocity given at lowest position is $v_0 = \sqrt{2gl}$

From Eq. (v), $\cos\alpha = 0$ or $\alpha = \frac{\pi}{2}$

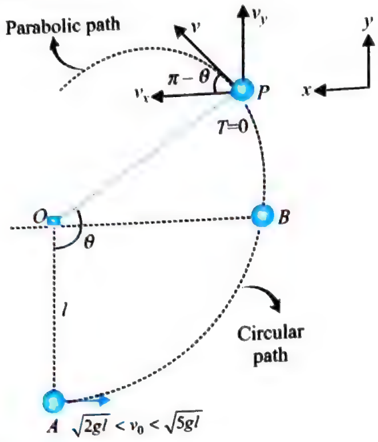
From Eq. (vii), $\cos\phi = 0$ or $\phi = \frac{\pi}{2}$

It means that the velocity of bob and tension in thread becomes zero simultaneously at an angular displacement $\pi/2$, or when thread becomes horizontal. Thus, bob will oscillate in lower half of the circle, as shown in figure.



Condition For Leaving the Circle: Tension Becomes Zero Before Speed

If $\sqrt{2gl} < v_0 < \sqrt{5gl}$, the ball rises above the level of center O i.e. it goes beyond point B $(90° < \theta < 180°)$.



In this case a component of gravity will always act towards center, hence centripetal acceleration or speed will remain nonzero. Hence tension becomes zero first.

As soon as, tension becomes zero at any point, string will go slack and subsequently, the only force acting on the body is gravity. Hence its subsequent motion will be similar to that of a projectile. In this case motion is a combination of circular and projectile motion.

Special Case: the velocity given at lowest position $v_0 = \sqrt{4gl}$

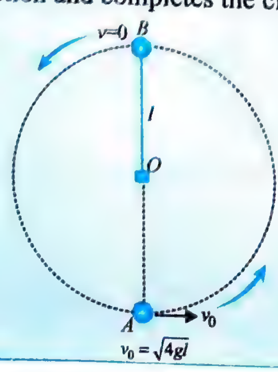
From Eq. (v), $\cos\alpha = -1$ or $\alpha = \pi$

From Eq. (vii), $\cos\phi = -\dfrac{2}{3}$ or $\phi = \cos^{-1}\left(-\dfrac{2}{3}\right) \Rightarrow \phi < 180°$

It means the initial velocity of bob is sufficient to carry the bob to highest point but tension in thread becomes zero before reaching the top position and afterward it will no longer be in circular motion. As soon as thread will slack at angle $\theta = \cos^{-1}\left(-\dfrac{2}{3}\right)$ particle becomes free to move and it will follow the projectile path as shown in figure.

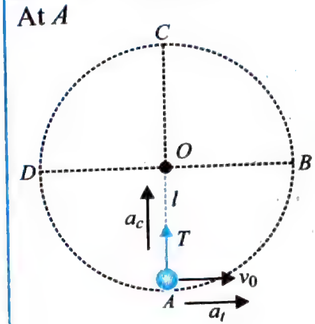
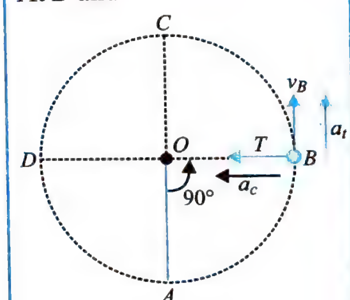
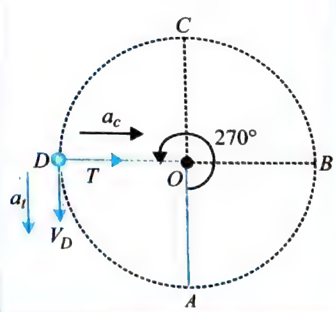
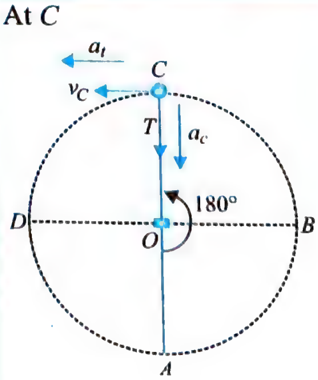
Important Point:

- If instead of thread we use a light rod and if the velocity given at lowest position $v_0 = \sqrt{4gl}$, it cannot slack during revolution and the bob will be able to move to the topmost point of the circle When it reaches the topmost point its velocity becomes zero but due to its inertia, it will fall in forward direction and completes the circle.



- Thus in cases when particle is restricted to move along the circular path, $\sqrt{4gl}$ is the sufficient velocity for the particle at the bottom most point to complete the circle. If particle is not restricted to its path, it will leave the path at an angle $\theta = \cos^{-1}\left(-\dfrac{2}{3}\right)$

If velocity at lowest point is just enough for looping the loop, value of various quantities. (True for a point mass attached to a string or a mass moving on a smooth vertical circular track.)

At A 	Tension at lowest position from Eq. (v) $T = \dfrac{m(\sqrt{5gl})^2}{l} + mg\cos 0° = 6mg$ Tangential acceleration $a_t = g\sin 0° = 0$ Normal acceleration $a_n = \dfrac{v_0^2}{l} = \dfrac{(\sqrt{5gl})^2}{l} = 5g$
At B and D  	Velocity at B Eq. from Eq. (i) $v_B = \sqrt{5gl - 2gl(1-\cos 90°)} = \sqrt{3gl}$ Tension at B from Eq. (iv) $T_B = \dfrac{mv_B^2}{l} + mg\cos 90° = \dfrac{m(3gl)}{l} + 0 = 3mg$ Tangential acceleration $\lvert a_t \rvert = g\sin\theta = g\sin 90° = g$ Normal acceleration $a_n = \dfrac{v_B^2}{l} = 3g$
At C 	Velocity at C from Eq. (i) $v_C = \sqrt{5gl - 2gl(1-\cos 180°)} = \sqrt{gl}$ Tension at C from Eq. (iv) $T_C = \dfrac{m(\sqrt{gl})^2}{l} + mg\cos 180° = 0$ Tangential acceleration $\lvert a_t \rvert = g\sin\theta = g\sin 180° = 0$ Normal acceleration $a_n = \dfrac{v_C^2}{l} = \dfrac{(\sqrt{gl})^2}{l} = g$

CONDITION FOR LOOPING THE LOOP IN SOME OTHER CASES

- A man is moving on a smooth vertical circular track.

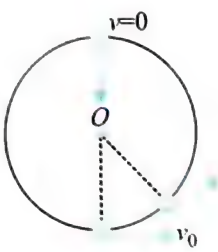
Condition for just looping the loop:

Minimum velocity at highest point $v_{top} = \sqrt{gr}$

Minimum velocity at lowest point $v_{bottom} = \sqrt{5gr}$

- A bead attached to a ring and rotated.

Condition for just looping the loop, velocity $v = 0$ at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).

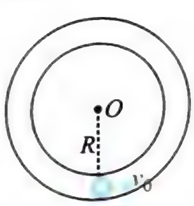


By energy conservation, velocity at lowest point = $\sqrt{4gl}$

$v_{min} = \sqrt{4gl}$ (for completing the circle)

- A block rotated between smooth surfaces of a pipe.

Condition for just looping the loop, velocity $v = 0$ at highest point (even if normal is zero, the bead will not lose contact with the track, normal can act radially outward).



By energy conservation, velocity at lowest point = $\sqrt{4gl}$

$v_{min} = \sqrt{4gl}$ (for completing the circle)

ILLUSTRATION 9.43

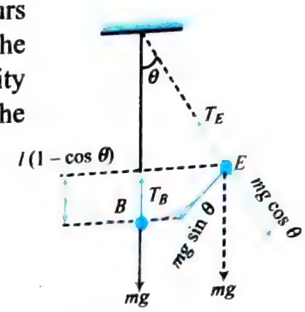
A particle tied to the end of a string oscillates along a circular arc in a vertical plane. The other end of the string is fixed at the center of the circle. If the string has a breaking strength of twice the weight of the particle,

(a) find the maximum distance that the particle can cover in one cycle of oscillation. The length of the string is 50 cm.

(b) find the tension in the extreme position.

(c) find the acceleration of the particle at bottom and extreme position.

Sol. As the maximum tension occurs at the lowest position, tension at the bottom can be $2mg$. Let u be the velocity at bottom. Considering forces on the particle at the bottom:

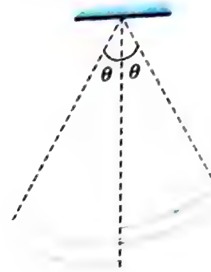


$$T - mg = \frac{mu^2}{l}$$

$$2mg - mg = \frac{mu^2}{l} \Rightarrow u = \sqrt{lg}.$$

(a) Let θ = angular amplitude from bottom to the extreme, using conservation of mechanical energy $\Delta K + \Delta U = 0$

$\Rightarrow$ Loss in KE = Gain in GPE



$$\frac{1}{2}mu^2 = mg(l - l\cos\theta)$$

$$\Rightarrow \quad \frac{1}{2}mlg = mg(l - l\cos\theta)$$

$$(1 - \cos\theta) = \frac{1}{2} \Rightarrow \theta = 60° = \frac{\pi}{3}$$

Length of arc covered in one cycle

$$= 4(l\theta) = 4\left(\frac{1}{2} \times \frac{\pi}{3}\right) = \frac{2\pi}{3}\ m$$

(b) At extremes, the speed is zero.

$$T_E - mg\cos\theta = \frac{mv^2}{l} = 0$$

Balancing radial forces we get: $T_E = mg\cos\theta = mg/2$

(c) At extremes :

Radial acceleration $a_r = \frac{v^2}{l} = 0 \Rightarrow a_r = 0\ \text{m s}^{-2}$

And tangential acceleration,

$$a_t = \frac{F_t}{m} = \frac{mg\ \sin 60°}{m} = g\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \text{Net acceleration, } \sqrt{a_t^2 + a_r^2} = g\frac{\sqrt{3}}{2}$$

At bottom: There are no tangential forces. Hence, $a_t = 0$.

$$a_r = \frac{u^2}{r} = \frac{(\sqrt{gl})^2}{l} = g$$

$\Rightarrow$ Net acceleration = g

ILLUSTRATION 9.44

The bob of a pendulum at rest is given a sharp hit to impart a horizontal velocity $\sqrt{10gl}$, where l is the length of the pendulum. Find the tension in the string when

(a) the string is horizontal

(b) the bob is at its highest point

(c) the string makes an angle of 60° with the upward vertical.

at lowest point we have given $v_1 = \sqrt{10gl}$ at A and

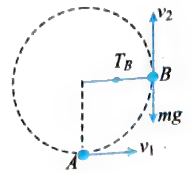
Sol. Let at lowest point we have given $v_1 = \sqrt{10gl}$ at A and when the string is horizontal the velocity at B is v_2.

Using conservation of mechanical energy, $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2\right) + (mgl) = 0$$

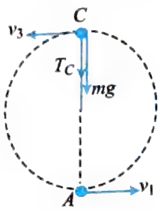
$$\frac{1}{2}m(10\,gl) = \frac{1}{2}mv_2^2 + mgl \Rightarrow v_2^2 = 8gl$$

(a) Now using force equation, the tension in the string at horizontal position,



$$T = \frac{mv^2}{R} = m\frac{8gl}{l} = 8mg$$

(b) Similarly, using conservation of mechanical energy between A and C,



$$\Delta K + \Delta U = 0$$

Let the velocity at C be v_3,

$$\left[\frac{1}{2}mv_3^2 - \frac{1}{2}mv_1^2\right] + (mg\,2l) = 0$$

$$\Rightarrow \quad \frac{1}{2}m\,10gl = \frac{1}{2}mv_3^2 + 2mgl$$

$$\Rightarrow \quad v_3^2 = 6\,gl$$

So, the tension in the string at C is given by

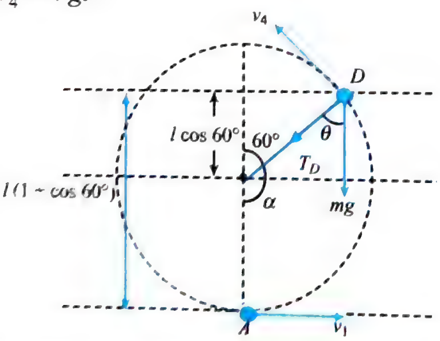
$$T_C = \frac{mv_3^2}{l} - mg = 5mg$$

(c) Let the velocity at point D be v_4.

Again using conservation of mechanical energy between A and D, $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}mv_4^2 - \frac{1}{2}mv_1^2\right) + [mgl(1 + \cos 60°)] = 0$$

$$\Rightarrow \quad v_4^2 = 7\,gl$$



So, the tension in the string is

$$T_D = \frac{mv^2}{l} - mg\cos 60°$$

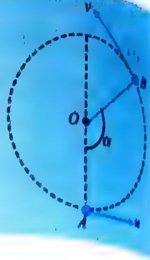
$$= m\frac{(7gl)}{l} - 0.5mg$$

$$= 7mg - 0.5mg = 6.5mg$$

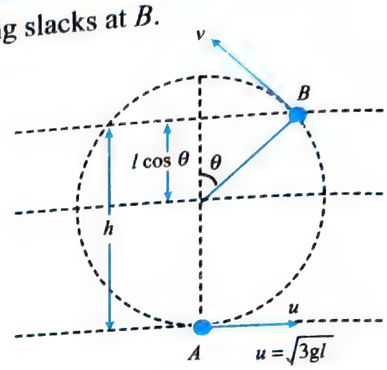
The bob of a stationary pendulum is given a sharp hit to impart it a horizontal speed of $\sqrt{3gl}$.

ILLUSTRATION 9.45

The bob of a pendulum at rest is given a sharp hit to impart a horizontal velocity $u = \sqrt{3gl}$ where l is the length of the pendulum. Find the angle rotated by the string before it becomes slack.



Sol. Let string slacks at B.



Applying COME at A and B, $\Delta U + \Delta K = 0$

$$mgh + \left[\frac{1}{2}mv^2 - \frac{1}{2}mu^2\right] = 0$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = -mgh$$

$$v^2 = u^2 - 2g\,(l + l\cos\theta)$$

When string slacks tension in the string becomes zero. The component of the weight in radial direction provide centripetal force at this position. From FBD of bob, we can write

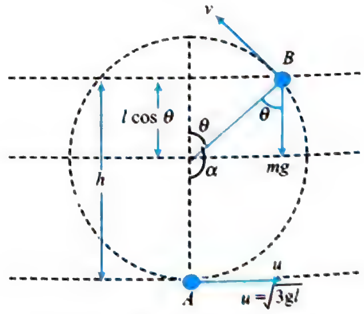
$$\frac{mv^2}{l} = mg\cos\theta$$

$$v^2 = lg\cos\theta \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$3gl - 2g(l + l\cos\theta) = gl\cos\theta$$

$$3\cos\theta = 1 \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$



Hence, the angle rotated by the string before it becomes slack is

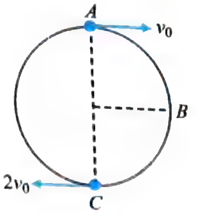
$$\Rightarrow \quad \alpha = \pi - \theta = \pi - \cos^{-1}\left(\frac{1}{3}\right)$$

ILLUSTRATION 9.46

A small ball attached with one end of an inextensible thread is moving in a vertical circle. Ratio of its maximum to minimum velocity is 2 : 1.

Calculate tension in thread and acceleration of the ball at the moment when velocity of the ball is directed vertically downward. ($g = 10\ \text{m s}^{-2}$).

Sol. Let a ball of mass m be moving in a vertical circle of radius r, as shown in figure. Velocity of the ball is minimum when ball passes through highest point A and maximum when it passes through the lowest point C.



Let velocity of ball at A be v_0 then that at C will be equal to $2v_0$.

According to law of conservation of energy.

KE at C = KE at A + Loss of energy from A to C

$$\frac{1}{2}m(2v_0)^2 = \frac{1}{2}mv_0^2 + mg(2r)$$

$$v_0^2 = \frac{4}{3}gr \quad \text{...(i)}$$

Velocity of the ball is directed vertically downward when the ball passes through B. Let velocity at this point be v. Then according to law of conservation of energy.

KE at B = KE at A + Loss of energy from A to B

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + mg(r) \quad \text{...(ii)}$$

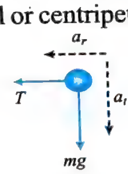
From Eqs. (i) and (ii),

$$v^2 = \frac{4}{3}gr + 2gr = \frac{10}{3}gr$$

Now considering FBD at B,

$$T = \frac{mv^2}{r} = \frac{10}{3}mg$$

$mg = ma_t$ or tangential acceleration, $a_t = g$ normal or centripetal acceleration.



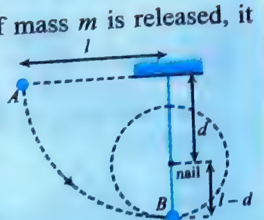
$$a_n = \frac{v^2}{r} = \frac{10}{3}g$$

Net acceleration at ball of B will be equal to the vector sum of these two.

$$a = \sqrt{a_t^2 + a_n^2} = \frac{g}{3}\sqrt{109}$$

ILLUSTRATION 9.47

In the given system, when the ball of mass m is released, it will swing down the dotted arc.



(a) How fast will it reach the lowest point in its swing? A nail is located at a distance d below the point of suspension.

(b) Show that d must at $0.6l$, if the ball is to swing completely around a circle centered along the nail.

(c) If $d = 0.6l$, find the change in tension in the string just after it touches the nail.

Sol.

(a) Radius of the circle centered at nail = $l - d$. To complete the circle centered at nail, the speed at the bottom must be at least = $\sqrt{5g(l-d)}$

From A to B conserving mechanical energy $\Delta K + \Delta U = 0$:

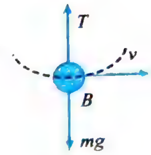
Loss in GPE = Gain in KE

$$mgl = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gl}$$

(b) To complete the circle: $\sqrt{2gl} = \sqrt{5g(l-d)}$

$$\Rightarrow 5l - 5d = 2l \Rightarrow d = \frac{3}{5}l = 0.6l$$

(c) Just before touching the nail, the ball is moving in a circle of radius l.



$$\Rightarrow T_{\text{before}} - mg = \frac{mv^2}{l}$$

$$T_{\text{before}} = mg + \frac{mv^2}{l} = mg + 2mg = 3mg$$

Just after touching the nail, the ball is moving in a circle of radius $(l - d)$.

$$T_{\text{after}} - mg = \frac{mv^2}{(l-d)}$$

$$T_{\text{after}} = mg + \frac{mv^2}{l-d} = mg + \frac{m(2gl)}{0.4l}$$

$\Rightarrow$ Tension = $6mg$

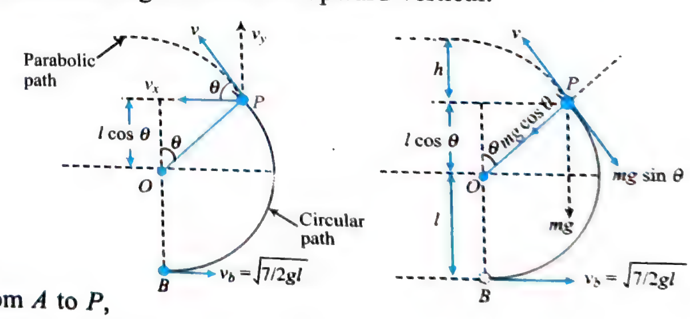
Hence, the tension in the string changes from $3mg$ to $6mg$ as it touches the nail.

ILLUSTRATION 9.48

A block is tied to one end of a light string of length l whose other end is fixed to a rigid support. The block is given a speed of $\sqrt{7gl/2}$ from the lowermost position. Find the height and speed at which the block leaves the circle. Also find the maximum height to which it rises finally.

Sol. As velocity of the block v_b, $\sqrt{2gl} < v_b < \sqrt{5gl}$.

The block will leave the circle at some point P, where the radius OP makes an angle θ with the upward vertical.



From A to P,

$\Delta K + \Delta U = 0$ or $\Delta K = -\Delta U$

Loss in KE = Gain in GPE

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv^2 = mg\,(l + l\cos\theta) \quad \text{...(i)}$$

From the force diagram: $T + mg\cos\theta = \dfrac{mv^2}{l}$

As the block leaves the circle at P, $T = 0$.

$$mg\cos\theta = \frac{mv^2}{l} \quad \text{...(ii)}$$

Putting the value of v^2 from Eq. (ii) in Eq. (i),

$$\frac{1}{2}mv_h^2-\frac{1}{2}m(gl\cos\theta)=mg\,l(1+\cos\theta)$$

$$\Rightarrow\quad \frac{1}{2}m\left(\frac{7}{2}gl\right)=\frac{mgl}{2}(2+3\cos\theta)$$

$$\Rightarrow\quad \cos\theta=\frac{1}{2}\quad\Rightarrow\quad\theta=60^\circ$$

Hence, the block leaves the circle and string slack at a height $h=l+l\cos 60^\circ=1.5l$ from the bottom. The velocity at that moment v is $v=\sqrt{lg\cos\theta}=\sqrt{\frac{lg}{2}}$

After the string becomes slack, the block moves as a projectile in parabolic path. Now, further height attained

$$=\frac{v^2\sin^2\theta}{2g}=\frac{lg}{4g}\left(\frac{3}{4}\right)=\left(\frac{3}{16}\right)l$$

Thus, total height attained from the bottom is

$$\frac{3}{2}l+\frac{3}{16}l=\frac{27}{16}l$$

Motion of a Particle on a Spherical Surface

Consider a particle of mass m is placed at the top of a sphere of radius R. Now it is projected with initial velocity v_0, it starts moving on the circular path of the sphere for some distance and at some point, it breaks off the sphere and follow the projectile trajectory after this.

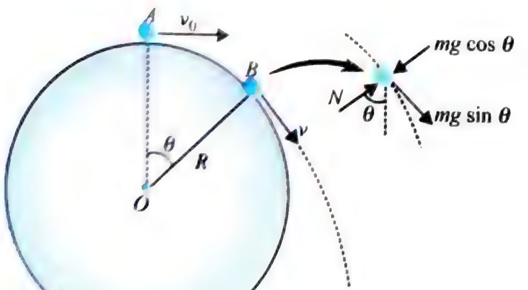
Let v be the speed of the particle, when it moves an angle θ from the vertical diameter. We can find its speed by using conservation of mechanical energy principle

At points A and B, we have, $\Delta K+\Delta U=0$

$$\left(\frac{1}{2}mv^2-\frac{1}{2}mv_0^2\right)-mgR(1-\cos\theta)=0 \qquad \text{...(i)}$$

$$v^2=v_0^2+2gR(1-\cos\theta)$$

or $$v=\sqrt{v_0^2+2gR(1-\cos\theta)} \qquad \text{...(ii)}$$



During its circular motion the net force on body toward centre is ($mg\cos\theta-N$) which provides the necessary centripetal force for circular motion. N is the normal reaction between sphere and the particle Thus we have

$$mg\cos\theta-N=\frac{mv^2}{R} \qquad \text{...(iii)}$$

The normal reaction becomes zero when the contact between the particle and spherical surface breaks off. From Eq. (iii), we have

$$mg\cos\theta=\frac{mv^2}{R} \qquad \text{...(iv)}$$

Substituting the value of v from Eq. (ii), we have

$$Rg\cos\theta=v_0^2+2gR(1-\cos\theta)$$

or $$\cos\theta=\frac{2gR+v_0^2}{3gR} \qquad \text{...(v)}$$

If we know the initial velocity given to the particle, then from above equation we can find the angle at which the particle leaves the spherical surface and starts projectile motion in gravity.

ILLUSTRATION 9.49

A particle of mass m is kept on the top of a smooth sphere of radius R. It is given a sharp impulse which imparts it a horizontal speed v.

(a) Find the normal force between the sphere and the particle just after the impulse.

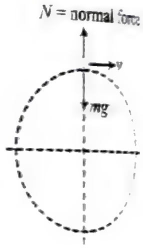
(b) What should be the minimum value of v for which the particle does not slip on the sphere ?

(c) Assuming the velocity v to be half the minimum calculated in part (b), find the angle made by the radius through the particle with the vertical when it leaves the sphere.

Sol.

(a) Radius $=R$, horizontal speed $=v$

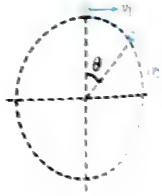
From the free body diagram



$$mg-N=\frac{mv^2}{R}$$

$$\Rightarrow\quad N=mg-\frac{mv^2}{R}$$

(b) When the particle is given maximum velocity so that the the particle does not slip on the sphere.



$$N\ge 0$$

$$\frac{mv^2}{R}=mg$$

$$v=\sqrt{gR}$$

(c) If the body is given velocity v_1 at the top such that

$$v_1=\frac{\sqrt{gR}}{2}\quad\Rightarrow\quad v_1^2=\frac{gR}{4}$$

Let the velocity be v_2 when it leaves contact with the surface

So, $$\frac{mv_2^2}{R}=mg\cos\theta$$

$$\Rightarrow\quad v_2^2=Rg\cos\theta \qquad \text{...(i)}$$

Again $$\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2=mgR(1-\cos\theta) \qquad \text{...(ii)}$$

$$\Rightarrow\quad v_2^2=v_1^2+2gR(1-\cos\theta)$$

From Eqs. (i) and (ii),

$$Rg\cos\theta=\frac{Rg}{4}+2gR(1-\cos\theta)$$

$$\Rightarrow \cos\theta = \frac{1}{4} + 2 - 2\cos\theta$$

$$\Rightarrow 3\cos\theta = \left(\frac{9}{4}\right)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{4}\right)$$

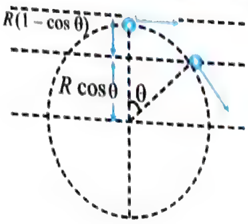
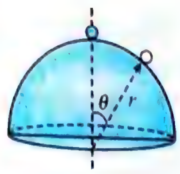


ILLUSTRATION 9.50

A point mass m starts from rest and slides down the surface of a frictionless hemisphere of radius r as shown in figure. Measure angle from the vertical and potential energy from the top.



(a) Find the change in potential energy of the mass with angle.

(b) Find the kinetic energy as a function of angle.

(c) Find the radial and tangential accelerations as a function of angle.

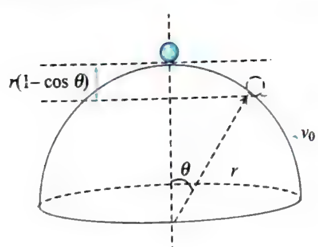
(d) Find the angle at which the mass flies off the hemisphere.

(e) If there is friction between the mass and hemisphere, does the mass fly off at a greater or lesser angle than in part (d)?

Sol.

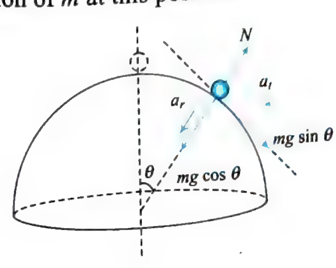
(a) As the particle is moving down w.r.t. earth, hence, the potential energy of the particle will decrease.
Change in potential energy,
$$\Delta U = U_\theta - U_0 = -mgr(1-\cos\theta) \quad \text{...(i)}$$

(b) The force acting on the point mass are a conservative force mg and a non-conservative force N. Here $W_N = 0$ because all along the motion from the $\theta = 0$ to $\theta = \theta$, the velocity of the mass is perpendicular to N. Consequently, the mechanical energy of m remains constant.



$$\Delta K + \Delta U = 0$$
$$(K_\theta - K_0) + \Delta U = 0$$
$$K_\theta = K_0 - \Delta U$$
$$= 0 - [-mgr(1-\cos\theta)]$$
$$= mgr(1-\cos\theta) \quad \text{...(ii)}$$

(c) Figure shows the free body diagram of m at θ. The radial acceleration of m at this position



$$a_r = \frac{v_\theta^2}{r} = \frac{2gr(1-\cos\theta)}{r}$$

$$\Rightarrow \quad a_r = 2g(1-\cos\theta)$$

Force equation for m in the tangential direction is

$$mg\sin\theta = ma_t \Rightarrow a_t = g\sin\theta$$

(d) Angle at which the mass flies off the sphere:
Force equation in the radial direction,

$$mg\cos\theta - N = \frac{mv^2}{r} \quad \text{...(iii)}$$

As the mass slide, down the sphere, its speed increases. So the right hand side of Eq. (iii), mv^2/r, increases with increasing θ. Also $mg\cos\theta$ decreases as θ increases, (for $0 < \theta < \pi/2$, $\cos\theta$ decreases as θ increases).
N must decrease with increasing θ. The angle at which $N \to 0$, the mass loses contact with the sphere, it flies off it. Let this happen when $\theta \to \alpha$. At $\theta \to \alpha$. $N \to 0$, $v \to v_\alpha$.
Under this limiting condition, Eq. (iii) reduces to

$$mg\cos\alpha - 0 = \frac{mv_\alpha^2}{r} \quad \text{...(iv)}$$

Conservation of mechanical energy between $\theta = 0$ and $\theta = \alpha$ gives, $\Delta K + \Delta U = 0$

$$\Rightarrow \quad \left[\frac{1}{2}mv_\alpha^2 - 0\right] + [-mgr(1-\cos\alpha)] = 0 \quad \text{...(v)}$$

From Eqs. (iv) and (v),

$$\Rightarrow \quad \frac{\left(\frac{mv_\alpha^2}{r}\right)}{\left(\frac{1}{2}mv_\alpha^2\right)} = \frac{mg\cos\alpha}{mgr(1-\cos\alpha)}$$

$$\Rightarrow \quad \frac{\left(\frac{1}{r}\right)}{\left(\frac{1}{2}\right)} = \frac{\cos\alpha}{r(1-\cos\alpha)}$$

$$\Rightarrow \quad 2 = \frac{\cos\alpha}{1-\cos\alpha}$$

$$\Rightarrow \quad 2 - 2\cos\alpha = \cos\alpha$$

$$\cos\alpha = \frac{2}{3} \quad \Rightarrow \quad \alpha = \cos^{-1}\left(\frac{2}{3}\right)$$

(e) If friction is present, the speed at $\theta = \alpha$ will be less than v_α defined in part (d), and $mg\cos\alpha = mv^2/r$ will not be satisfied. In fact, then $mg\cos\alpha > mv^2/r$ as $v < v_\alpha$ and, consequently, Newton's second law of motion will demand that.

$$N = mg\cos\alpha - \frac{mv^2}{r} \quad \Rightarrow \quad N > 0$$

For N to vanish, both θ and v must increase a little more. Therefore with friction present, the mass will fly off the sphere at a greater angle than in part (d).

ILLUSTRATION 9.51

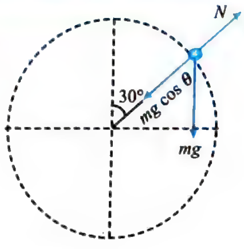
A particle of mass m is kept on a fixed, smooth sphere of radius R at a position where the radius through the particle makes an angle of 30° with the vertical. The particle is released from this position.

(a) What is the force exerted by the sphere on the particle just after the release ?

(b) Find the distance travelled by the particle before it leaves contact with the sphere.

Sol.

(a) When the particle is released from rest, the centripetal force is zero.



$$N = mg\cos\theta = mg\cos 30° = \frac{\sqrt{3}}{2}mg$$

(b) When the particle leaves contact with surface $N = 0$,

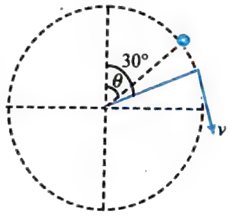
$$\frac{mv^2}{R} = mg\cos\theta$$

$$v^2 = Rg\cos\theta \quad \text{...(i)}$$

$$\left(\frac{1}{2}\right)mv^2 = mgR(\cos 30° - \cos\theta)$$

$$v^2 = 2Rg\left(\frac{\sqrt{3}}{2} - \cos\theta\right) \quad \text{...(ii)}$$

From Eqs. (i) and (ii),



$$Rg\cos\theta = 2Rg\left[\frac{\sqrt{3}}{2} - \cos\theta\right]$$

$$3\cos\theta = \sqrt{3}$$

$$\cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

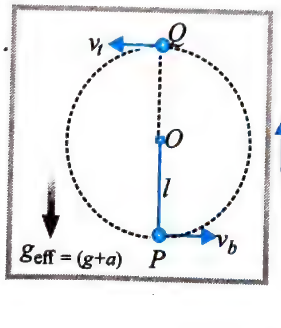
So the distance travelled the particle before leaving contact.

$$L = R\left(\theta - \frac{\pi}{6}\right) \text{ [because } 30° = \pi/6]$$

SOME SPECIAL CASES OF CIRCULAR MOTION

CIRCULAR MOTION IN NON-INERTIAL FRAME OF REFERENCE

Case I: Circular motion in a lift moving up with same acceleration.



Let a bob of mass m is suspended by a light string which is connected to a fixed point O, and system is moving up with acceleration a.

In frame of reference of the lift the effective value of 'g' will be $(g + a)$ in vertical downward direction.

The tension will be minimum at position $Q(T_Q = 0)$ and maximum at position 'P'.

Now applying Newton's second law at position Q;

$$mg_{eff} = \frac{mv^2}{l} \Rightarrow v = \sqrt{g_{eff}l} \quad \text{...(i)}$$

The classical method of finding the velocity at the bottom-most point would be applying the work energy theorem as conservation of mechanical energy is only applicable in inertial frame of reference.

But applying the conservation of mechanical energy here using the concept of g_{eff} makes the calculations a lot easier. The conservation of mechanical energy principle can be applied in the non-inertial frame of reference by taking it in the line of right of g_{eff}. Now applying conservation of mechanical energy at points P and Q.

$$\Delta K = -\Delta U$$

$$\left(\frac{1}{2}mv_0^2 - \frac{1}{2}mv_t^2\right) = -(-mg_{eff}\cdot 2l)$$

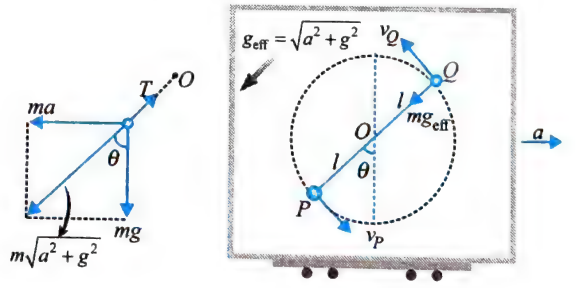
$$v_b^2 = v_t^2 + 4g_{eff}l \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $v_b = \sqrt{5g_{eff}l}$ ...(iii)

We can observe the values of the velocities of P and Q are in the similar pattern as we have calculated in initial frame of surface.

Case II: Circular motion in a trolley moving horizontally with acceleration.

For finding the magnitude and direction of effective gravity we need to analyze the bob in the trolley frame.



The magnitude of effective gravity

$$g_{eff} = \frac{T}{m} = \sqrt{a^2 + g^2}$$

and $\tan\theta = \dfrac{a}{g}$ or $\theta = \tan^{-1}\left(\dfrac{a}{g}\right)$

Tension can be zero at point Q. Applying Newton's second law at Q.

$$mg_{eff} = \frac{mv_Q^2}{l} \Rightarrow v_Q = \sqrt{g_{eff}l} \quad \text{...(i)}$$

Now applying conservation of mechanical energy between points P and Q

$$\Delta K = -\Delta U$$

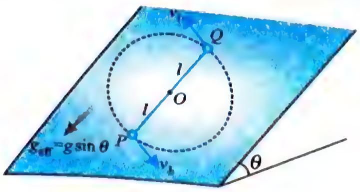
$$\frac{1}{2}mv_P^2 - \frac{1}{2}mv_Q^2 = -[-mg_{eff}\cdot 2l]$$

$$v_P^2 = v_Q^2 + 4g_{eff}l = 5g_{eff}l$$

or $\quad v_P = \sqrt{5g_{\text{eff}}l}$ ...(ii)

Hence we can observe the values of velocities, we calculated in Eqs. (i) and (ii) are similar as in case-I.

Circular Motion on an Inclined Plane



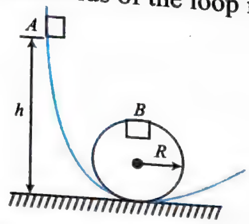
Here the value of g_{eff} will be $g \sin\theta$, down the inclined plane. The tension will be minimum (say zero) at Q and maximum at point P. Here we can write the velocities of the particle at P and Q as

$$v_Q = \sqrt{g_{\text{eff}}R} = \sqrt{g\sin\theta \cdot R}$$

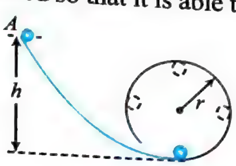
and $\quad v_P = \sqrt{5g_{\text{eff}}R} = \sqrt{5g\sin\theta \cdot R}$

Concept Application Exercise 9.4

1. The block on the loop shown in figure slides without friction. At what height from A it starts so that it passes against the track at B with a net upward force equal to its own weight? The radius of the loop is R.



2. A block is released from rest at the top of an inclined plane which later curves into a circular track of radius r as shown in figure. Find the minimum height h from where it should be released so that it is able to complete the circle.

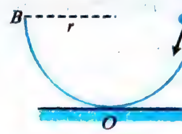


3. Two point masses m are connected the light rod of length l and it is free to rotate in vertical plane as shown in figure. Calculate the minimum horizontal velocity is given to mass so that it completes the circular motion in vertical lane.

4. A simple pendulum of length l and mass m free to oscillate in vertical plane. A nail is located at a distance $d = l - a$ vertically below the point of suspension of a simple pendulum. The pendulum bob is released from the position where the string makes an angle of 90° from vertical. Discuss the motion of the bob if **(a)** $l = 2a$ and **(b)** $l = 2.5a$.

5. A bob of mass m suspended by a light inextensible string of length l from a fixed point. The bob is given a speed of $\sqrt{6gl}$. Find the tension in the string when string deflects through an angle 120° from the vertical.

6. AOB is a smooth semicircular track of radius r. A block of mass m is given a velocity $\sqrt{2rg}$ parallel to track at point A. Calculate normal reaction between block and track when block reaches at point O.



7. A cabin is moving upwards with a constant acceleration g. A boy standing in the cabin wants to whirl a particle of mass m in a vertical circle of radius l. (Mass is attached to an ideal string.) Calculate minimum velocity which should be provided at lowermost point (w.r.t. cabin) so that particle can just complete the circle.

Cabin moving upward with acceleration g

8. A ball is attached to a horizontal cord of length L whose other end is fixed. **(a)** If the ball is released, what will be its speed at the lowest point of its path? **(b)** A peg is located a distance h directly below the point of attachment of the cord. If $h = 0.75L$, what will be the speed of the ball when it reaches the top of its circular path about the peg?

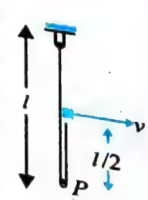
9. A heavy particle hanging from a fixed point by a light inextensible string of length l is projected horizontally with speed $\sqrt{gl}$. Find the speed of the particle and the inclination of the string to the vertical at the instant when the tension in the string equals the weight of the particle.

10. A particle is suspended from a fixed point by a string of length 5 m. It is projected from the equilibrium position with such a velocity that the string slackens after the particle has reached a height 8 m above the lowest point. Find the velocity of the particle just before the string slackens. Find also to what height the particle can rise further.

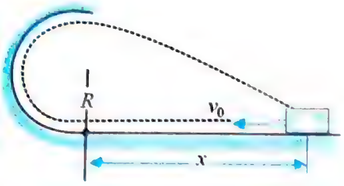
11. A small box of mass m is kept on a fixed, smooth sphere of radius R at a position where the radius through the box makes an angle of 30° with the vertical. The box is released from the position. **(a)** What is the force exerted by the sphere on the box just after the release? **(b)** Find the distance travelled by the box before it leaves contact with the sphere.

12. Figure shows a smooth track, a part of which is a circle of radius r. A block of mass m is pushed against a spring of spring constant k fixed at the left end and is then released. Find the initial compression of the spring so that the block presses the track with a force mg when it reaches the point P, where the radius of the track is horizontal.

13. A bob of mass m is projected with a horizontal velocity $v = \sqrt{gl/2}$ as shown in the figure. In consequence, it moves in a circular path in a vertical plane by the inextensible string which passes over the smooth fixed peg. Find the maximum angle that the bob swings in the left hand side.



14. A block is projected with a speed v_0 such that it strikes the point of projection after describing the path as shown by the dotted line. If friction exists for the path of length d and the vertical circular path is smooth, assuming μ = coefficient of frication



(a) Find v_0.

(b) What is the minimum value of v_0?

15. A small ball is suspended from point O by a thread of length l. A nail is driven into the wall at a distance of $l/2$ below O, at A. The ball is drawn aside so that the thread takes up a horizontal position at the level of point O and then released.

(a) At what angle from the vertical of the ball's trajectory, will the tension in the thread disappear?

(b) What will be the highest point from the lowermost point of circular track, to which it will rise?

ANSWERS

1. $3R$ **2.** $\frac{5}{2}r$ **3.** $\sqrt{\frac{48gl}{5}}$

5. $\frac{5}{2}mg$

6. $5mg$ **7.** $\sqrt{10gr}$ **8. (a)** $\sqrt{2gL}$ **(b)** $\sqrt{gL}$

9. $\sqrt{\frac{gl}{3}}, \cos^{-1}\left(\frac{2}{3}\right)$

10. 5.42 m s^{-1}; 0.96 m

11. (a) $\frac{\sqrt{3}}{2}mg$ **(b)** $\left[\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)-\frac{\pi}{6}\right]R$

12. $\sqrt{\frac{3mgr}{k}}$

13. $\cos^{-1}\left(\frac{1}{6}\right)$

14. (a) $\sqrt{\frac{d^2g}{4R}+2g[\mu d+2R]}$

(b) $\sqrt{5gR}$

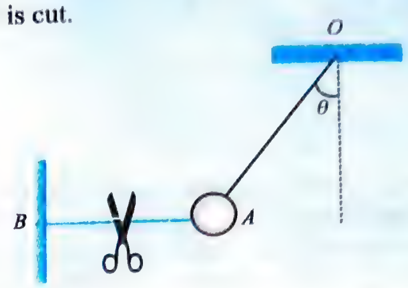
15. (a) $\cos^{-1}\left(\frac{2}{3}\right)$

(b) $\frac{50l}{54}$

Solved Examples

EXAMPLE 9.1

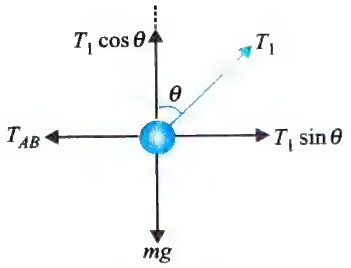
A bob of mass m is suspended by two strings as shown in figure. Find the ratio of the tension in string OA before and just after AB is cut.



Sol. Before cutting of string AB, the bob is in static equilibrium i.e., $a = 0$.

Resolving in vertical and horizontal direction,

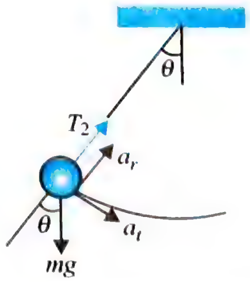
$$T_1 \cos\theta = mg \Rightarrow T_1 = \frac{mg}{\cos\theta} \quad \text{...(i)}$$



When string AB is cut, the bob moves in a circular path around O, with radius equal to the length of the string OA.

Now the acceleration of the bob cannot be taken zero, rather we have to consider acceleration in radial and tangential direction.

Resolving the force in radial and tangential direction.



In radial direction, $T_2 - mg\cos\theta = ma_r$

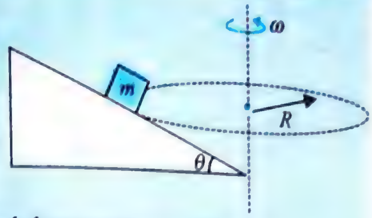
But just after cutting the string AB, speed of the bob is zero

As radial acceleration, $a_r = \frac{v^2}{l} = 0$

$$\therefore\ T_2 - mg\cos\theta = 0 \Rightarrow T_2 = mg\cos\theta \quad \text{...(ii)}$$

From Eqs. (i) and (ii), $\frac{T_1}{T_2} = \frac{mg/\cos\theta}{mg\cos\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$

EXAMPLE 9.2

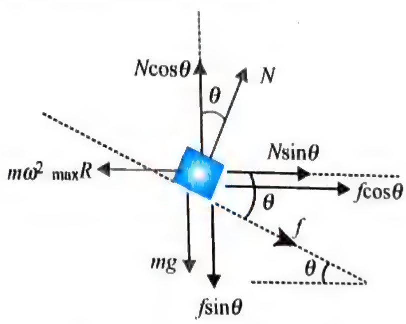


A small block is placed on a rough triangular shaped wedge which is rotating as shown such that the block undergoes circular path of radius R. The coefficient of friction between the block and the wedge is μ. Find the range of angular speed ω so that the block does not slip with respect to the wedge.

Sol. Let us analyze the situation from the frame of reference rotating with the object. For this we need to consider a centrifugal force $m\omega^2R$ acting on the block radial outward.

(i) **Maximum angular speed (ω_{max})**

Greater the value of ω greater the value of centrifugal force. In this case the block has tendency to slide up the inclined plane hence the friction force acts down the incline. As the block is not sliding on the plane.



In vertical direction: $N\cos\theta - f\sin\theta = mg$ ...(i)

Under limiting condition i.e. when the block is about to slip,

$f = \mu N$ ...(ii)

From (i) and (ii)

$$N(\cos\theta - \mu\sin\theta) = mg \Rightarrow N = \frac{mg}{\cos\theta - \mu\sin\theta} \quad ...(iii)$$

In horizontal direction: $N\sin\theta + (\mu N)\cos\theta = m\omega_{max}^2 R$

$$\Rightarrow \quad N(\sin\theta + \mu\cos\theta) = m\omega_{max}^2 R \quad ...(iv)$$

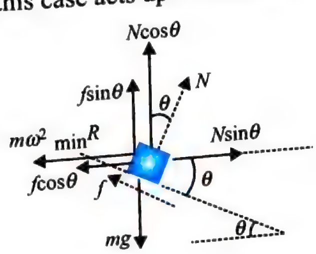
From Eqs. (iii) and (iv) we get,

$$mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) = m\omega_{max}^2 R$$

$$\therefore \quad \omega_{max} = \sqrt{\frac{g}{R}\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)} \quad ...(v)$$

(ii) **For minimum angular speed (ω_{min})**

Lesser is ω, lesser is centrifugal force. In this case the block has tendency to slide down the inclined plane, thus friction in this case acts up the incline.



In vertical direction: $N\cos\theta + f\sin\theta = mg$ ...(i)

Under limiting condition i.e. when the block is about to slip,

$f = \mu N$ ...(ii)

From Eqs. (i) and (ii)

$$N(\cos\theta + \mu\sin\theta) = mg \Rightarrow N = \frac{mg}{\cos\theta + \mu\sin\theta} \quad ...(iii)$$

In horizontal direction: $N\sin\theta - (\mu N)\cos\theta = m\omega_{min}^2 R$

$$\Rightarrow \quad N(\sin\theta - \mu\cos\theta) = m\omega_{min}^2 R \quad ...(iv)$$

From Eqs. (iii) and (iv), we get

$$mg\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right) = m\omega_{min}^2 R \quad ...(v)$$

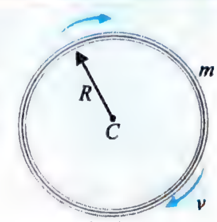
$$\Rightarrow \quad \omega_{min} = \sqrt{\frac{g}{R}\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right)}$$

Hence required range is

$$\sqrt{\frac{g(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)}} \le \omega \le \sqrt{\frac{g(\sin\theta + \mu\cos\theta)}{R(\cos\theta - \mu\sin\theta)}}$$

EXAMPLE 9.3

A circular loop of thick rope of mass m and radius R placed on a smooth table is revolving with a speed v about a vertical axis coinciding with the symmetry axis of the rope. Find the tension in the rope.

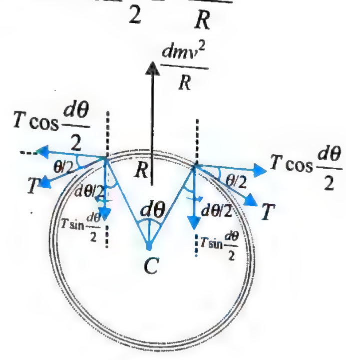


Sol. Let us consider an observer is also revolving with the loop with same angular speed as loop. If he observes a small element of the rope, he will notice a centrifugal force in radial outward direction. If we increase the angular velocity, the centrifugal force increases and result in the tension in the rope.

Now consider a small elemental length dl on the rope as shown in figure, which subtend an angle $d\theta$ at the center. This element (say mass = dm) experiences the centrifugal force along radially outward direction, given as

$$dF_c = \frac{dmv^2}{R}$$

As shown in figure, tension acts at the edges of this dl tangentially away from the element. If we resolve the two tensions along and perpendicular to the element, the components $T\cos\frac{d\theta}{2}$, will cancel each other and the perpendicular components which are in radially inward direction, $2T\sin\frac{d\theta}{2}$ and balances the centrifugal force, thus we have: $2T\sin\frac{\theta}{2} = \frac{dmv^2}{R}$



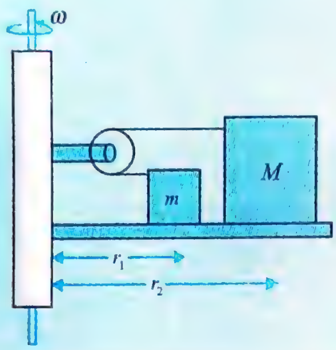
As $d\theta$ is very small, we can use $\sin d\theta = d\theta$, thus

$$T\,d\theta = \left(\frac{m}{2\pi R}\times R d\theta\right)\times\frac{v^2}{R} \qquad \left[dm = \frac{m}{2\pi R}\times R d\theta\right]$$

Hence, $T = \dfrac{mv^2}{2\pi R}$

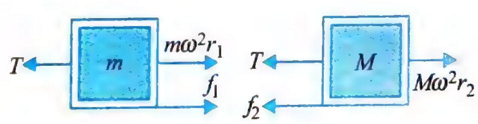
EXAMPLE 9.4

Two blocks of masses m and M are connected by a chord passing around a frictionless pulley which is attached to a rotating frame, which rotates about a vertical axis with an angular velocity ω. If the coefficient of friction between the two masses and the surface be μ_1 and μ_2 respectively, determine the value of ω, at which the block starts sliding radially ($M > m$).



Sol. Evidently, the larger block of mass experiences more centrifugal force radially outwards, compared to the block of smaller block m, [$M > m$ and $r_2 > r_1$].

Figure shows their F.B.D.



Owing to the larger force experienced by block of mass M, it tends to fly off radially.

In the situation of limiting equilibrium, we have

$$T = m\omega^2 r_1 + f_1 \Rightarrow T + f_2 = M\omega^2 r_2$$

[where f_1 and f_2 are frictional forces for the two blocks and the surface]

$$f_1 = \mu_1 mg \Rightarrow f_2 = \mu_2 Mg$$

The above two equations get reduced to

$$T = m\omega^2 r_1 + \mu_1 mg \qquad \text{...(i)}$$

$$T + \mu_2 Mg = M\omega^2 r_2 \qquad \text{...(ii)}$$

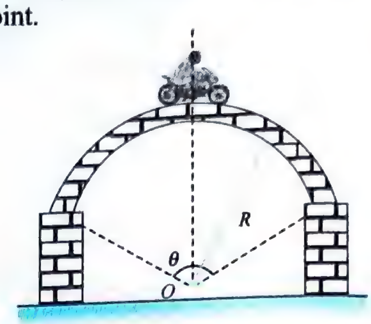
Subtracting Eqs. (i) and (ii)

$$\mu_2 Mg = M\omega^2 r_2 - m\omega^2 r_1 - \mu_1 mg$$

$$\omega^2 = \frac{g[\mu_1 m + \mu_2 M]}{Mr_2 - mr_1} \Rightarrow \omega = \left[\frac{g(\mu_1 m + \mu_2 M)}{Mr_2 - mr_1}\right]^{1/2}$$

EXAMPLE 9.5

A motorcycle has to move with a constant speed on an overbridge which is in the form of a circular arc of radius R and has a total length L. Suppose the motorcycle starts from the highest point.



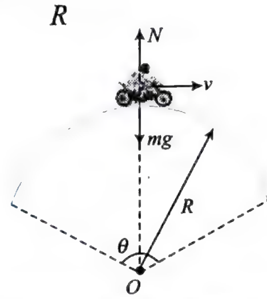
(a) What can its maximum velocity be for which the contact with the road is not broken at the highest point?

(b) If the motorcycle goes at speed $1/\sqrt{2}$ times the maximum found in part (a), where will it lose the contact with the road?

(c) What maximum uniform speed can it maintain on the bridge if it does not lose contact anywhere on the bridge?

Sol.

(a) $\theta = \dfrac{\text{Arc length}}{\text{Radius}} = \dfrac{L}{R}$



From free-body diagram at top position, equation of motion is $mg - N = \dfrac{mv^2}{R}$

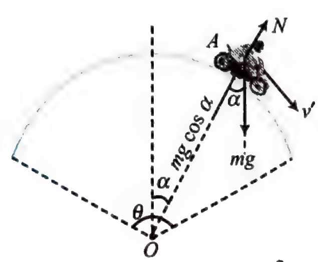
$$\Rightarrow \quad N = mg - \frac{mv^2}{R}$$

If the contact at highest point does not loose,

$$N > 0,\ mg > \frac{mv^2}{R}$$

$$v^2 < gR \Rightarrow v < \sqrt{gR}$$

(b) If the velocity of motorcyclist $v' = \dfrac{1}{\sqrt{2}} v = \dfrac{\sqrt{gR}}{\sqrt{2}}$



At position A, $mg\cos\alpha - N' = \dfrac{mv'^2}{R}$

$$\Rightarrow \quad N' = mg\cos\alpha - \frac{m}{R}\left(\sqrt{\frac{gR}{2}}\right)^2$$

$$N' = mg\cos\alpha - \frac{mgR}{2R} = mg\cos\alpha - \frac{mg}{2}$$

If he does not loose contact with road, $N' > 0$

$$mg\cos\alpha - \frac{mg}{2} > 0 \Rightarrow \cos\alpha > \frac{1}{2} \Rightarrow \alpha < \frac{\pi}{3}$$

Hence, arc length, $l' = \frac{R\pi}{3}$ from top.

(c) The equation of motion of motorcyclist in the function of α measured from vertical,

$$mg\cos\alpha - N'' = \frac{mv''^2}{R}$$

The normal reaction will be minimum at the end of arc. Hence, critical position is at the end of arc, where

$$\alpha = \frac{\theta}{2} = \frac{L}{2R}$$

$$mg\cos\left(\frac{L}{2R}\right) - N'' = \frac{mv''^2}{R}$$

$$\Rightarrow \quad N'' = mg\cos\left(\frac{L}{2R}\right) - \frac{mv''^2}{R}$$

If the motorcycle does not loose contact even at the end of arc, $N'' > 0$

$$mg\cos\left(\frac{L}{2R}\right) - \frac{mv''^2}{R} > 0$$

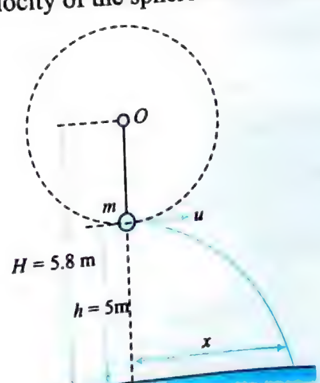
$$\Rightarrow \quad g\cos\left(\frac{L}{2R}\right) > \frac{v''^2}{R}$$

$$\Rightarrow \quad v'' < \sqrt{gR\cos\left(\frac{L}{2R}\right)}$$

EXAMPLE 9.6

A small sphere tied to the string of length 0.8 m is describing a vertical circle so that the maximum and minimum tensions in the string are in the ratio 3 : 1. The fixed end of the string is at a height of 5.8 m above ground.

(a) Find the velocity of the sphere at the lowest position.



(b) If the string suddenly breaks at the lowest position, when and where will the sphere hit the ground?

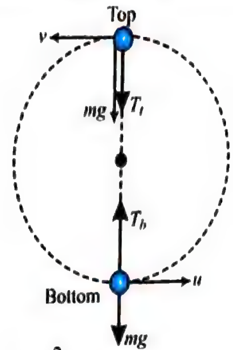
Sol.

(a) Let u and v the speeds of sphere at the bottom and the top positions and m be the mass.

radius of circle = length of string = r = 0.8 m.

T_b = tension at the bottom or the maximum tension

T_t = tension at the top or the minimum tension



$$T_b - mg = \frac{mu^2}{r}$$

$$T_t + mg = \frac{mv^2}{r} \quad \Rightarrow \quad T_b = 3\,T_t$$

$$\left(\frac{mu^2}{r} + mg\right) = 3\left(\frac{mv^2}{r} - mg\right)$$

$$\Rightarrow \quad (3v^2 - u^2) = 4rg \qquad \text{...(i)}$$

Using conservation of energy,

loss in KE from bottom to the top = gain in GPE

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mg(2r)$$

$$\Rightarrow \quad v^2 = u^2 - 4rg \qquad \text{...(ii)}$$

Using Eqs. (i) and (ii), we get

$$3(u^2 - 4gr) - u^2 = 4gr$$

$$\Rightarrow \quad 2u^2 = 16rg$$

$$\Rightarrow \quad u = \sqrt{8rg} = \sqrt{8(0.8)10} = 8\ \text{ms}^{-1}$$

(b) After breaking away from the string, the sphere moves along a parabolic path, and strikes the ground at G.

Vertical displacement of sphere

$$S_y = 5.8 - 0.8 = 5\text{m}$$

Let t = time after which the sphere hits the ground.

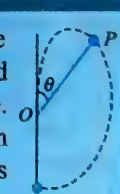
$$\Rightarrow \quad S_y = 0t + \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2S_y}{g}} = 1\ \text{s}$$

The horizontal displacement = $x = ut = 8 \times 1 = 8$ m.

Hence, the sphere hits the ground 1 s after breaking off the string and at the point G.

EXAMPLE 9.7

A point mass m connected to one end of inextensible string of length l and other end of string is fixed at peg. String is free to rotate in vertical plane. Find the minimum velocity given to the mass in horizontal direction so that it hits the peg in its subsequent motion.

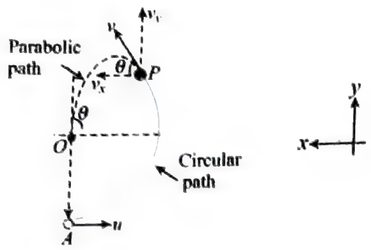


Sol. Tension in string is zero at point P in its subsequent motion, after this point its motion is projectile. Let the velocity at point P be v.

For tension at point P to be zero,

$$\Rightarrow \quad mg\cos\theta = \frac{mv^2}{l} \Rightarrow v = \sqrt{gl\cos\theta}$$

Assume its projectile motion start at point P and it passes through point C. So that equation of trajectory satisfy the co-ordinate of C ($l \sin\theta$, $-l\cos\theta$)



Equation of trajectory.

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$$

$$= (l\sin\theta)\tan\theta - \frac{g(l\sin\theta)^2}{2(gl\cos\theta)\cos^2\theta}$$

$$\Rightarrow \quad -\cos\theta = \frac{\sin^2\theta}{\cos\theta} - \frac{1}{2}\frac{\sin^2\theta}{\cos^3\theta}$$

$$\Rightarrow \quad -2\cos^4\theta = 2\sin^2\theta\cos^2\theta - \sin^2\theta$$

$$\Rightarrow \quad \sin^2\theta = 2\sin^2\theta\cos^2\theta + 2\cos^4\theta$$

$$= 2\cos^2\theta(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow \quad \tan^2\theta = 2$$

$$\Rightarrow \quad \tan\theta = \sqrt{2} \text{ hence, } \cos\theta = \frac{1}{\sqrt{3}}, \quad \sin\theta = \sqrt{\frac{2}{3}}$$

From conservation of mechanical energy between point P and A,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgl(1+\cos\theta)$$

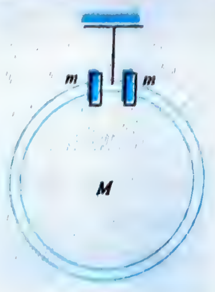
$$\Rightarrow \quad u^2 = v^2 + 2gl(1+\cos\theta)$$

$$= 2gl + 3gl\cos\theta = 2gl + 3gl\frac{1}{\sqrt{3}}$$

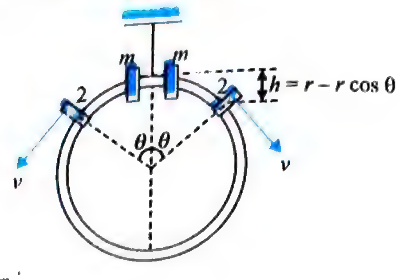
$$\Rightarrow \quad u = \left[(2+\sqrt{3})gl\right]^{1/2}$$

EXAMPLE 9.8

A hoop of mass M with two identical rings of mass m at its top hangs from a ceiling by an inextensible string. If the rings gently pushed horizontally in opposite directions, find the angular distance covered by each ring when the tension in the string vanishes for once during their motion.



Sol. Suppose the string slakens that means its tension T becomes zero when the rings slide through an angle θ with vertical as shown in figure. The speeds of the rings at the angular position θ can be given by conserving its energy between the position 1 and 2.



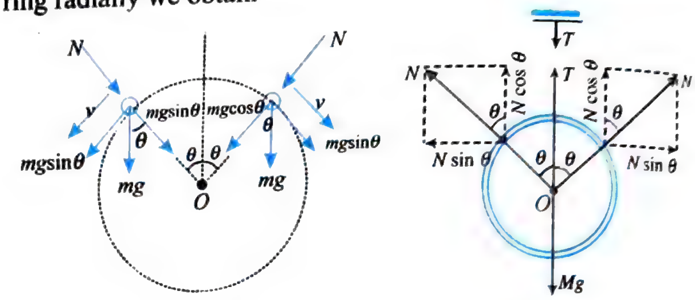
$$\Delta PE = \Delta KE$$

$$\Rightarrow \quad mgh = \frac{1}{2}mv^2$$

$$\Rightarrow \quad mg[r - r\cos\theta] = \frac{1}{2}mv^2$$

$$\Rightarrow \quad v = \sqrt{2gr(1-\cos\theta)} \qquad \text{...(i)}$$

Referring the following figure, resolving the force acting on the ring radially we obtain



F.B.D. of ring F.B.D. of hoop

$$F_{cp} = mg\cos\theta + N$$

where N is the contacting force between the hoop and the rings.

$$ma_r = mg\cos\theta + N$$

$$\Rightarrow \quad \frac{mv^2}{r} = mg\cos\theta + N$$

$$\Rightarrow \quad N = \frac{mv^2}{r} - mg\cos\theta \qquad \text{...(ii)}$$

Resolving the forces acting on the hoop for its equilibrium (refer FBD), we obtain

$$F_x = N\sin\theta - N\sin\theta = 0$$

and $\quad F_y = 2N\cos\theta + T - Mg = 0$

$$\Rightarrow \quad 2N\cos\theta + T - Mg = 0,$$

Since the string slackens,

$$T = 0 \Rightarrow N = \frac{Mg}{2\cos\theta} \qquad \text{...(iii)}$$

Using Eqs. (ii) and (iii),

$$\frac{Mg}{2\cos\theta} = \frac{mv^2}{r} - mg\cos\theta \qquad \text{...(iv)}$$

Using Eqs. (i) and (iv),

$$\frac{Mg}{2\cos\theta} = \frac{m}{r}[2gr(1-\cos\theta)] - mg\cos\theta$$

$$\Rightarrow \quad 3\cos^2\theta - 2\cos\theta + \frac{M}{2m} = 0 \qquad \text{...(v)}$$

$$\cos\theta = \frac{1}{3}\left[1 \pm \sqrt{1-\left(\frac{3}{2}\right)\left(\frac{M}{m}\right)}\right]$$

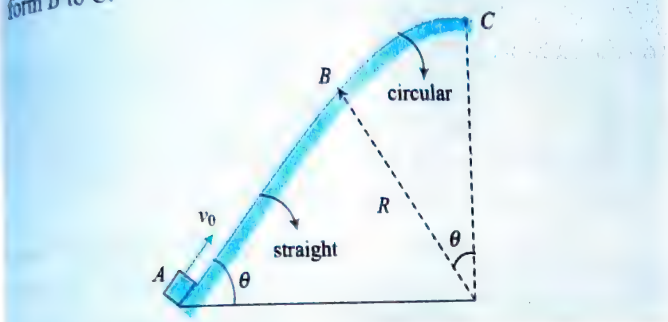
If $\frac{M}{m} \le \frac{2}{3}$ cos θ is real, which means the string is not slackened.

If $\frac{M}{m} > \frac{2}{3}$, cos θ is imaginary, which means the string cannot be slackened.

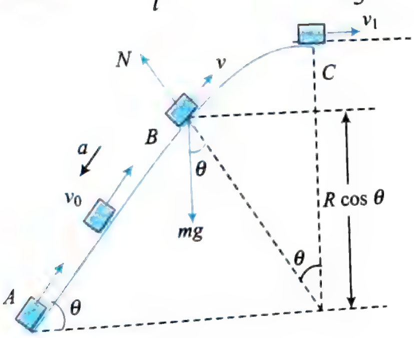
If $\frac{M}{m} = \frac{2}{3}$, the quadratic equation (v) has one root, which means the string is slacken at $\theta = \cos^{-1}(1/3)$.

EXAMPLE 9.9

A block of mass m is projected up with a velocity v_0 along an inclined plane of angle of inclination $\theta = 37°$. The coefficient of friction between the inclined plane AB and block is μ (= tan θ). Find the values of v_0 so that the block moves in a circular path form B to C.



Sol. In triangle ABO, $\frac{R}{l} = \tan\theta \Rightarrow l = \frac{4R}{3}$



For contact not to be lost at B:

$$mg\cos\theta - N = \frac{mv^2}{R}$$

$$\Rightarrow \quad N = mg\cos\theta - \frac{mv^2}{R} \ge 0$$

$$\Rightarrow \quad v \le \sqrt{gR\cos\theta} \qquad \text{...(i)}$$

From B to C: $\Delta K + \Delta U = 0$

$$\left(\frac{1}{2}mv_1^2 - \frac{1}{2}mv^2\right) + mg(R - R\cos\theta) = 0 \qquad \text{...(ii)}$$

$$\Rightarrow \quad v^2 = v_1^2 + 2gR\,(1 - \cos\theta)$$

From A to B:

$$a = g\sin\theta + \mu g\cos\theta = g\sin\theta + (\tan\theta)\,g\cos\theta$$

$$\Rightarrow \quad a = 2g\sin\theta$$

$$\Rightarrow \quad v^2 = v_0^2 - 2al$$

$$\Rightarrow \quad v^2 = v_0^2 - \frac{4g\sin\theta R}{\tan\theta}$$

$$\Rightarrow \quad v^2 = v_0^2 - 4gR\cos\theta \qquad \text{...(iii)}$$

From Eqs. (i) and (iii),

$$v_0^2 - 4gR\cos\theta \le gR\cos\theta \text{ or } v_0^2 \le 5gR\cos\theta$$

$$\Rightarrow \quad v_0^2 \le 5gR\left(\frac{4}{5}\right) = 4gR \Rightarrow v_0 \le \sqrt{4gR} \qquad \text{...(iv)}$$

From Eqs. (ii) and (iii),

$$v_1^2 + 2gR(1 - \cos\theta) = v_0^2 - 4gR\cos\theta$$

$$\Rightarrow \quad v_1^2 = v_0^2 - 2gR - 2gR\cos\theta$$

Now to reach at C: $v_1 \ge 0$

or $\quad v_0^2 - 2gR - 2gR\cos\theta \ge 0$

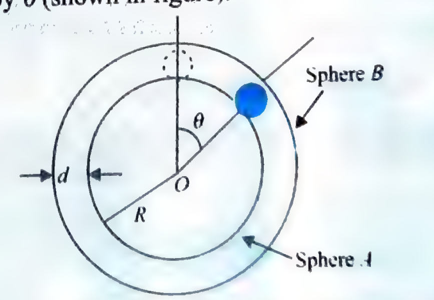
$$\Rightarrow \quad v_0 \ge \sqrt{2gR(1+\cos\theta)}$$

$$\Rightarrow \quad v_0 \ge \sqrt{2gR(1+4/5)} \Rightarrow v_0 \ge \sqrt{\frac{18gR}{5}} \qquad \text{...(v)}$$

From Eqs. (iv) and (v) $\sqrt{\frac{18gR}{5}} \le v_0 \le \sqrt{4gR}$

EXAMPLE 9.10

A spherical ball of mass m is kept at the highest point in the space between two fixed, concentric spheres A and B as shown in the figure. The small sphere A has a radius R, and the space between the two spheres has a width d. The ball has a diameter very slightly less than d. All surfaces are frictionless. The ball is given a gentle push (towards the right in figure). The angle made by the radius vector of the ball with the upward vertical is denoted by θ (shown in figure).



(a) Express the total normal reaction force exerted by the sphere on the ball as a function of angle θ.

(b) Let N_A and N_B denote the magnitudes of the normal reaction forces on the ball exerted by spheres A and B, respectively. Sketch the variations of N_A and N_B as functions of cos θ in the range $0 \le \theta \le \pi$ by drawing two separate graphs in your answer book, taking cos θ on the horizontal axes.

Sol. The ball is moving in a circular motion. The necessary centripetal force is provided by (mg cos $\theta - N$). Therefore,

$$mg\cos\theta - N_A = \frac{mv^2}{[R + (d/2)]} \qquad \text{...(i)}$$

According to energy conservation,

$$\frac{1}{2}mv^2 = mg\left(R + \frac{d}{2}\right)(1 - \cos\theta) \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$N_A = mg\,(3\cos\theta - 2) \qquad \text{...(iii)}$$

The above equation shows that as θ increases, N_A decreases. At a particular value of θ, N_A will become zero and the ball will lose contact with sphere A. This condition can be found by putting $N_A = 0$ in Eq. (iii).

$$0 = mg\,(3\cos\theta - 2)$$

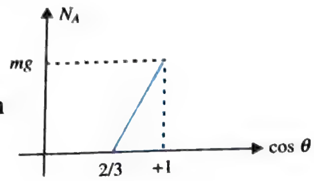
$$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

The graph between N_A and $\cos\theta$ is shown in figure.

From Eq. (iii), when $\theta = 0$, $N_A = mg$.

When $\theta = \cos^{-1}\left(\frac{2}{3}\right)$, $N_A = 0$.

The graph is a straight line as shown in figure.



When $\theta > \cos^{-1}\left(\frac{2}{3}\right)$

$$N_B + mg\cos\theta = \frac{mv^2}{\left(R + \frac{d}{2}\right)} \qquad \text{...(iv)}$$

Using energy conservation, we get

$$\frac{1}{2}mv^2 = mg\left[\left(R + \frac{d}{2}\right) - \left(R + \frac{d}{2}\right)\cos\theta\right]$$

$$\frac{mv^2}{\left(R + \frac{d}{2}\right)} = 2\,mg\,[1 - \cos\theta] \qquad \text{...(v)}$$

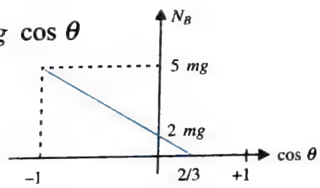
From Eqs. (iv) and (v), we get

$$N_B + mg\cos\theta = 2mg - 2mg\cos\theta$$

$$\Rightarrow \quad N_B = mg\,(2 - 3\cos\theta)$$

When $\cos\theta = \frac{2}{3}$, $N_B = 0$

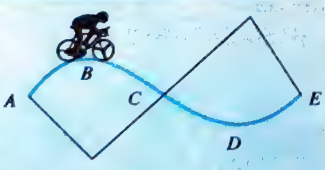
When $\cos\theta = -1$, $N_B = 5mg$



Therefore, the graph is as shown in figure.

EXAMPLE 9.11

A track consists of two circular parts ABC and CDE of equal radius 100 m and joined smoothly as shown in figure. Each part subtends a right angle at its center. A cycle weighing 100 kg together with the rider travels at a constant speed of 18 km h^{-1} on the track.



(a) Find the normal contact force by the road on the cycle when it is at B and at D.

(b) Find the force of friction exerted by the track on the tyres when the cycle is at B, C, and D.

(c) Find the normal force between the road and the cycle just before and just after the cycle crosses C.

(d) What should be the minimum friction coefficient between the road and the tyre, which will ensure that the cyclist can move with constant speed?

Sol. $v = 18\ \text{km h}^{-1} = \dfrac{1800}{60 \times 60} = 5\ \text{m s}^{-1}$

(a) At point B, various forces acting on the cyclist are

(i) Weight, mg, vertically downwards

(ii) Normal force, N_B, by the road upwards

Since under the action of these two forces, the cyclist moves in a circular path of radius r, from the dynamics of circular motion,

$$\frac{mv^2}{r} = mg - N_B$$

$$\Rightarrow \quad N_B = m\left(g - \frac{v^2}{r}\right) = 100\left(10 - \frac{5^2}{100}\right) = 975\ \text{N}$$

At point D, various forces acting on the cyclist are

(i) Weight, mg, vertically downwards

(ii) Normal force, N_D, by the road upwards

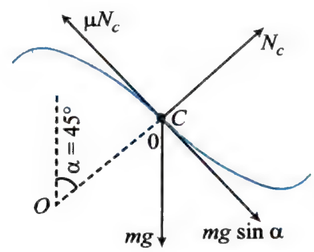
$$N_D - mg = \frac{mv^2}{r}$$

$$\Rightarrow \quad N_D = m\left(g + \frac{v^2}{r}\right) = 100\left(10 + \frac{5^2}{100}\right) = 1025\ \text{N}$$

(b) At points B and D, the tracks are almost horizontal; therefore, there is no component of g along the track at those points that will change the speed of the cyclist. Since the cyclist moves with a constant speed, the frictional force at these two points is necessarily zero.

At point C, however, the component of mg along the track that tends to accelerate the motion of the cyclist $= mg\sin\alpha$.

But since his speed remains constant, a force, equal and opposite to $mg\sin\alpha$, must be acting on it and this force is the force of friction μN_c, where μ is the friction coefficient between the road and the tyre and N_c is the normal reaction at point C.



Hence, frictional force at C is

$$\mu N_c = mg\sin\alpha = 100 \times 10 \times \sin 45^\circ = 707.12\ \text{N}$$

(c) Just before the cyclist crosses point C, various forces acting on him are

(i) His weight, mg, vertically downwards, and

(ii) Normal force, N_b, of the track.

Hence, from the dynamics of circular motion, we have

$$mg\cos\alpha - N_b = \frac{mv^2}{r}$$

$$N_b = m\left(g\cos\alpha - \frac{v^2}{r}\right) = 682.11\ \text{N}$$

Just after the cyclist crosses the point C, various forces acting on him are

(i) Weight, mg, vertically downwards, and

(ii) Normal force, N_a, of the track.

Hence, from the dynamics of circular motion

$$N_d - mg\cos\alpha = \frac{mv^2}{r}$$

$$\Rightarrow \quad N_d = m\left(g\cos\alpha + \frac{v^2}{r}\right) = 732.11 \text{ N}$$

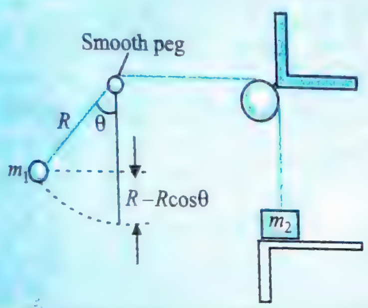
(d) The tendency of the cyclist to skid is maximum just before the point C, the point where the radius of curvature changes its direction. This is so because the curve is steepest at this point. Force along the track at this point = $mg \sin \alpha$. And frictional force = μN_b, where N_b is the normal force of the track just before the point C.

The cyclist can move with constant speed only if the two forces are equal and opposite.

$$\mu N_b = mg \sin\alpha \quad \Rightarrow \quad \mu = 1.037$$

Example 9.12

Two blocks are connected by a massless string that passes over a frictionless peg as shown in figure. One end of the string is attached to a mass m_1 = 3 kg, i.e., a distance R = 1.20 m from the peg. The other end of the string is connected to a block of mass m_2 = 6 kg restring on a table. From what angle θ, measured from the vertical, must the 3 kg block be released in order to just lift the 6 kg block off the table?



Sol. Here we can apply conservation of energy to find the speed of the block m_1 at the bottom of the circular path as a function of θ and the radius of the path, R. And from Newton's second law we will determine the tension at the bottom of its path as function of given parameters. Finally, the block m_2 will lift off the ground when the upward force (tension) exerted by the cord just exceeds the weight of the block. From conservation of energy, we have

$$\Delta K + \Delta U = 0 \text{ or } \Delta K = -\Delta U$$

$$\left(\frac{1}{2}m_1v^2 - 0\right) = -m_1 g(R - R\cos\theta)$$

$$v^2 = 2gR(1-\cos\theta) \qquad \text{...(i)}$$

Applying Newton's second law on block of mass m_2, we have

$$T - m_1 g = m_1\frac{v^2}{R}$$

$$T = m_1 g + \frac{m_1 v^2}{R} \qquad \text{...(ii)}$$

As the string is massless, tension T is constant throughout. When m_2 just lifts off the normal reaction becomes zero. For block m_2, we have

$$T = m_2 g \qquad \text{...(iii)}$$

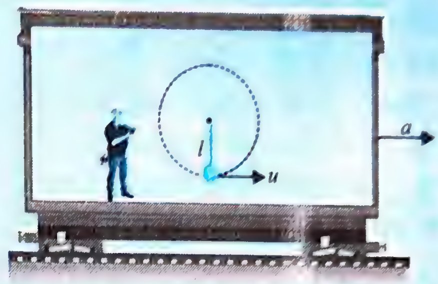
From Eqs. (i), (ii) and (iii), we get

$$m_2 g = m_1 g + m_1\frac{2gR(1-\cos\theta)}{R}$$

$$\cos\theta = \frac{3m_1 - m_2}{2m_1} = \frac{3\times3-6}{2\times3} = \frac{1}{2} \Rightarrow \theta = 60°$$

Example 9.13

A Physics professor perform an experiment inside a stationary train. He observes that minimum speed at lowest position needed by a pendulum bob connected with a light string to complete a vertical circle is 10 m/s. Find the minimum speed (u) needed at the lowest position so as to complete the vertical circle when the train is moving horizontally at an acceleration of a = 7.5 m/s². [g = 10 m/s²]



Sol. **First approach:** When train is at rest, the minimum velocity needed at lowest point to complete circle

$$v_{\min} = \sqrt{5gl} \quad [l = \text{length of the string of pendulum}]$$

$$\Rightarrow \qquad l = \frac{v_{\min}^2}{5g} = \frac{10^2}{5\times10} = 2 \text{ m}$$

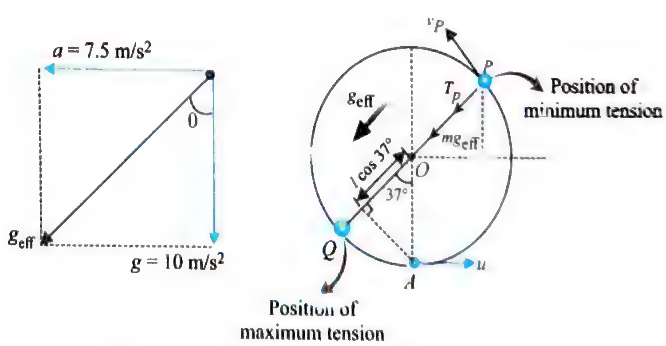
When the train is moving horizontally at an acceleration a, we can assume that inside the train effective acceleration due to gravity is

$$g_{\text{eff}} = \sqrt{g^2 + a^2} = \sqrt{10^2 + 7.5^2} = 12.5 \text{ m/s}^2$$

$$\tan\theta = \frac{a}{g} = \frac{7.5}{10} = \frac{3}{4} \Rightarrow \theta = 37°$$

In accelerated train, minimum tension will be at point P.

$$mg_{\text{eff}} + T_P = \frac{mv_P^2}{l}$$



(a) (b)

In limiting case $T_P = 0$

$$\frac{mv_P^2}{l} = mg_{\text{eff}} \qquad \text{...(i)}$$

Here the bob is rotating in non-inertial frame of reference, hence here we can not use conservation of mechanical energy principle. Let us apply here work energy theorem between points P and A.

$$W_{total} = \Delta K \Rightarrow W_{gravity} + W_{psudo} = \Delta K$$

$$-mgl(1+\cos 37°) - mal\sin 37° = \left(\frac{1}{2}mv_P^2 - \frac{1}{2}mu^2\right)$$

$$\frac{1}{2}mu^2 = \frac{1}{2}mv_P^2 + mgl\,(1+\cos 37°) + mal\sin 37°$$

$$\frac{1}{2}mu^2 = \frac{1}{2}mg_{eff}l + mgl(1+\cos 37°) + mal\,\sin 37°$$

$$u^2 = g_{eff}l + 2gl(1+\cos 37°) + 2al\sin 37°$$

On substituting the given values in above equation, we get $u = \sqrt{115}$ m/s

Second approach: Here we can apply the conservation of mechanical energy using the concept of g_{eff}. It make the calculations a lot easier. The conservation of mechanical energy principle can be applied in the non-inertial frame of reference by taking it in the line of right of g_{eff}. Now applying conservation of mechanical energy at points P and Q.

$$\frac{1}{2}mv_P^2 + mg_{eff}l(1+\cos 37°) = \frac{1}{2}mu^2$$

$$u^2 = v_P^2 + 2g_{eff}l(1+\cos 37°)$$

$$= lg_{eff} + \frac{18}{5}lg_{eff} = \frac{23}{5}l$$

$$g_{eff} = \frac{23}{5}\times 2\times 12.5$$

$\Rightarrow \quad u = \sqrt{115}$ m/s

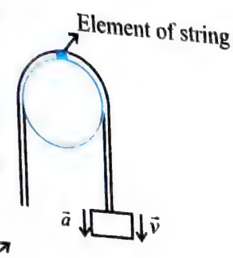
Exercises

...e Correct Answer Type

...particle moves in xy plane on a circular path of radius ...0 m (with centre at origin) with constant speed of 20 m/s. ...ts velocity vector at point (0, 10 m) if it's angular velocity ...s along z-axis is

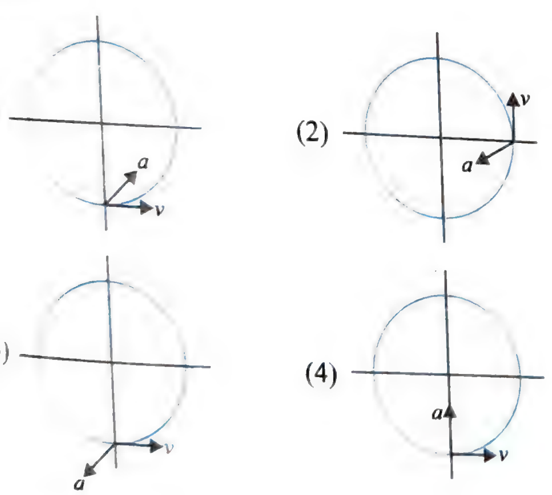
(1) $(20\hat{i})$m/s
(2) $(-20\hat{i})$m/s
(3) $(10\hat{i})$m/s
(4) $(-10\hat{i})$m/s

...Figure shows a real string on a real pulley. The string runs on the pulley without slipping as shown in figure. The direction of resultant force on an element shown in figure on the string may be



(1) $\longrightarrow$
(2) $\nearrow$
(3) $\searrow$
(4) $\downarrow$

...A particle is undergoing uniformly accelerated circular motion with angular retardation π rad/s^2. If the angular velocity of the particle at $t = 0$ is 2π rad/s, the velocity and acceleration vectors of the body at $t = 0$ s are best represented by



4. Which of the following is incorrect about non-uniform circular motion (a_c = centripetal acceleration, a_t = tangential acceleration)?

(1) $\vec{v}\cdot\vec{\omega}=0$
(2) $\vec{a}_c\cdot\vec{v}=0$
(3) $\vec{\omega}\cdot\vec{a}_c=0$
(4) $\vec{v}\cdot\vec{a}_t=0$

5. A body is moving in a circle with a speed of 1 m s^{-1}. This speed increases at a constant rate of 2 m s^{-1} every second. Assume that the radius of the circle described is 25 m. The total acceleration of the body after 2 s is

(1) 2 m s^{-2}
(2) 25 m s^{-2}
(3) $\sqrt{5}$ m s^{-2}
(4) $\sqrt{7}$ m s^{-2}

6. A body is moving in a circular path with a constant spe... It has

(1) A constant velocity
(2) A constant acceleration
(3) An acceleration of constant magnitude
(4) An acceleration which varies with time in magnitu...

7. A particle is moving along a circular path with unifor... speed. Through what angle does its angular velocity chan... when it completes half of the circular path?

(1) 0°
(2) 45°
(3) 180°
(4) 360°

8. A particle is moving along a circular path. The angul... velocity, linear velocity, angular acceleration, and centripet... acceleration of the particle at any instant, respectively, a... $\vec{\omega}, \vec{v}, \vec{\alpha}$, and $\vec{a}_c$. Which of the following relations is n... correct?

(1) $\vec{\omega}\perp\vec{v}$
(2) $\vec{\omega}\perp\vec{\alpha}$
(3) $\vec{\omega}\perp\vec{\alpha}_c$
(4) $\vec{v}\perp\vec{\alpha}_c$

9. The angular velocity of a particle moving in a circle ... radius 50 cm is increased in 5 min from 100 revolutions p... minute to 400 revolutions per minute. Find the tangent... acceleration of the particle.

(1) 60 m s^{-2}
(2) $\pi/30$ m s^{-2}
(3) $\pi/15$ m s^{-2}
(4) $\pi/60$ m s^{-2}

10. A vehicle is moving with a velocity v on a curved road of width b and radius of curvature R. For counteracting the centrifugal force on the vehicle, the difference in elevation require... between the outer and inner edges of the rod is

(1) v^2b/Rg
(2) vb/Rg
(3) vb^2/Rg
(4) vb/R^2g

11. A circular road of radius 1000 m has banking angle ... The maximum safe speed (in m s^{-1}) of a car having a m... 2000 kg will be (if the coefficient of friction between ... and road is 0.5)

(1) 172
(2) 124
(3) 99
(4) 86

12. A circular table of radius 0.5 m has a smooth diamet... groove. A ball of mass 90 g is placed inside the gro... along with a spring of spring constant 10^2 N cm^{-1}. One ... of the spring is tied to the edge of the table and the ot... end to the ball. The ball is at a distance of 0.1 m from ... center when the table is at rest. On rotating the table w... a constant angular frequency of 10^2 rad s^{-1}, the ball mo... away from the center by a distance nearly equal to

(1) 10^{-1} m
(2) 10^{-2} m
(3) 10^{-3} m
(4) 2×10^{-1} m

13. A coin is placed at the edge of a horizontal disc rotati... about a vertical axis through its axis with a uniform ang...

speed 2 rad s^{-1}. The radius of the disc is 50 cm. Find the minimum coefficient of friction between disc and coin so that the coin does not slip ($g = 10$ m s^{-2}).

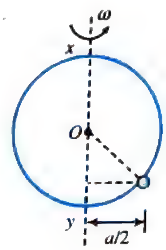
(1) 0.1 (2) 0.2
(3) 0.3 (4) 0.4

14. A man revolves a stone of mass m tied to the end of a string in a vertical circle of radius R. The net forces at the lowest and highest points of the circle directed vertical downwards are

	Lowest point	**Highest point**
(1)	$mg - T_1$	$mg + T_2$
(2)	$mg + T_1$	$mg - T_2$
(3)	$mg + T_1 - \frac{mv^2}{r}$	$mg - T_2 + \frac{mv^2}{r}$
(4)	$mg - T_1 - \frac{m_1 v_1^2}{R}$	$mg + T + \frac{m_1 v_1^2}{R}$

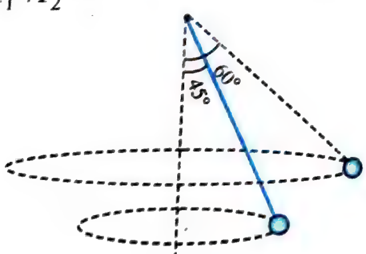
Here T_1, T_2 (and v_1, v_2) denote the tension in the string (and the speed of the stone) at the lowest and highest points, respectively.

15. A small ring P is threaded on a smooth wire bent in the form of a circle of radius a and center O. The wire is rotating with constant angular speed ω about a vertical diameter XY, while the ring remains at rest relative to the wire at a distance $a/2$ from XY. Then ω^2 is equal to



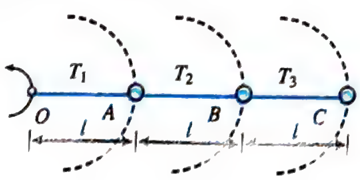
(1) $\frac{2g}{a}$ (2) $\frac{g}{2a}$
(3) $\frac{2g}{a\sqrt{3}}$ (4) $\frac{g\sqrt{3}}{2a}$

16. When the string of a conical pendulum makes an angle of 45° with the vertical, its time period is T_1. When the string makes an angle of 60° with the vertical, its time period is T_2. Then T_1^2/T_2^2 is



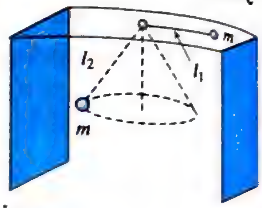
(1) 2 (2) $\sqrt{2}$
(3) 0.5 (4) none of these

17. Three identical particles are joined together by a thread as shown in figure. All the three particles are moving in a horizontal plane. If the velocity of the outermost particle is v_0, then the ratio of tensions in the three sections of the string ($T_1 : T_2 : T_3 = ?$) is



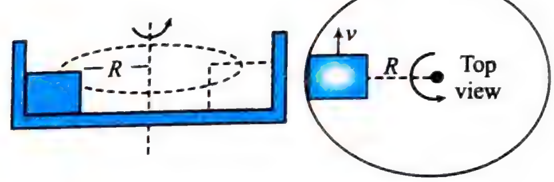
(1) 6 : 5 : 3 (2) 3 : 5 : 6
(3) 3 : 4 : 5 (4) none of these

18. Two identical particles are attached at the ends of a light string which passes through a hole at the center of a table. One of the particles is made to move in a circle on the table with angular velocity ω_1 and the other is made to move in a horizontal circle as a contact pendulum with angular velocity ω_2. If l_1 and l_2 are the length of the string over and under the table, then in order that particle under the table neither moves down nor moves up, the ratio l_1/l_2 is



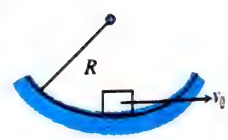
(1) $\frac{\omega_1}{\omega_2}$ (2) $\frac{\omega_2}{\omega_1}$
(3) $\frac{\omega_1^2}{\omega_2^2}$ (4) $\frac{\omega_2^2}{\omega_1^2}$

19. A block of mass m is revolving in a smooth horizontal plane with a constant speed v. If the radius of the circular path is R, find the total contact force received by the block



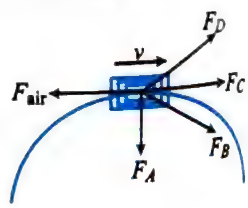
(1) $\frac{mv^2}{R}$ (2) mg
(3) $m\sqrt{\frac{v^4}{R^2} + 4g^2}$ (4) $m\sqrt{\frac{v^4}{R^2} + g^2}$

20. A particle is projected with a speed $v_0 = \sqrt{gR}$. The coefficient of friction between the particle and the hemi-spherical plane is $\mu = 0.5$. Then, the acceleration of the particle is



(1) g (2) $\frac{\sqrt{5}g}{2}$ ↖
(3) $\sqrt{2}g$ ↖ (4) none of these

21. A car travels with constant speed on a circular road on level ground. In figure, F_{air} is the force of air resistance on the car. Air resistance acts opposite to the motion of car. Which of the other forces shown best represents the horizontal force of the road on the car's tires?

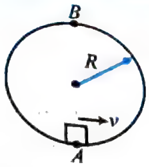


(1) F_C (2) F_D
(3) F_A (4) F_B

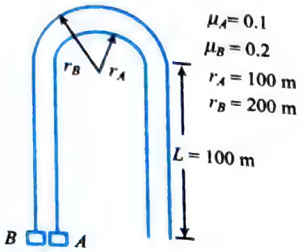
22. A block of mass m is projected on a smooth horizontal circular track with velocity v. What is the average normal force exerted by the circular walls on the block during its motion from A to B?

(1) $\dfrac{mv^2}{R}$ (2) $\dfrac{mv^2}{\pi R}$

(3) $\dfrac{2mv^2}{R}$ (4) $\dfrac{2mv^2}{\pi R}$

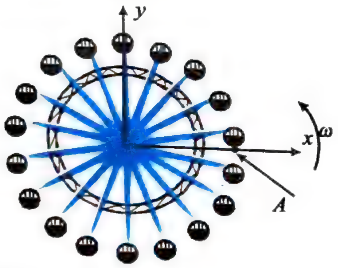


23. Two cars A and B start racing at the same time on a flat race track which consists of two straight sections each of length 100π and one circular section as shown in figure. The rule of the race is that each car must travel at constant speed at all times without even skidding.



(1) Car A completes its journey before car B
(2) Both cars complete their journey in same time
(3) Velocity of car A is greater than that of car B
(4) Car B completes its journey before car A.

24. Consider the setup of a Ferris wheel in an amusement park. The wheel is turning in a counterclockwise manner. Contrary to the illustration, not all seats are aligned horizontally, i.e. parallel to the x-axis. Determine the orientation of the normal to seat as it passes point A.



(1) parallel to the x-axis
(2) in the first/third quadrants
(3) parallel to the y-axis
(4) in the second/fourth quadrants

25. The kinetic energy K of a particle moving along a circle of radius R depends upon the distance s as $K = as^2$. The force acting on the particle is

(1) $2a\dfrac{s^2}{R}$ (2) $2as\left[1+\dfrac{s^2}{R^2}\right]^{1/2}$

(3) $2as$ (4) $2a$

26. A particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a_c is varying with time t as $a_c = k^2rt^2$, where k is a constant. The power delivered to the particle by the forces acting on it is

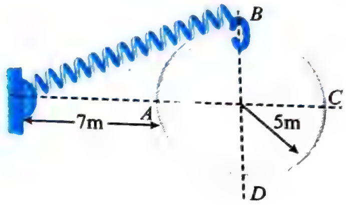
(1) Zero (2) $mk^2r^2t^2$
(3) mk^2r^2t (4) mk^2rt

27. A particle of mass m moves along a circular path of radius r with a centripetal acceleration a_n changing with time t as $a_n = kt^2$, where k is a positive constant. The average power developed by all the forces acting on the particle during the first t_0 seconds is

(1) $mkrt_0$ (2) $\dfrac{mkrt_0^2}{2}$

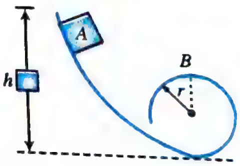
(3) $\dfrac{mkrt_0}{2}$ (4) $\dfrac{mkrt_0}{4}$

28. A collar B of mass 2 kg is constrained to move along a horizontal smooth and fixed circular track of radius 5 m. The spring lying in the plane of the circular track and having spring constant 200 N m^{-1} is undeformed when the collar is at A. If the collar starts from rest at B, the normal reaction exerted by the track on the collar when it passes through A is



(1) 360 N (2) 720 N
(3) 1440 N (4) 2880 N

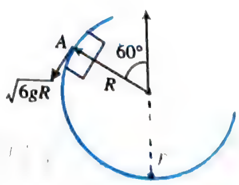
29. A mass m starting from A reaches B of a frictionless track. On reaching B, it pushes the track with a force equal to x times its weight, then the applicable relation is



(1) $h = \dfrac{(x+5)}{2}r$ (2) $h = \dfrac{x}{2}r$

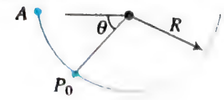
(3) $h = r$ (4) $h = \left(\dfrac{x+1}{2}\right)r$

30. Figure shows a smooth vertical circular track AB of radius R. A block slides along the surface AB when it is given a velocity equal to $\sqrt{6gR}$ at point A. The ratio of the force exerted by the track on the block at point A to that at point B is



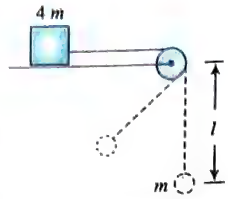
(1) 0.25 (2) 0.35
(3) 0.45 (4) 0.55

31. A bead of mass m is released from rest at A to move along the fixed smooth circular track as shown in figure. The ratio of magnitudes of centripetal force and normal reaction by the track on the bead at any point P_0 described by the angle $\theta\ (\neq 0)$ would



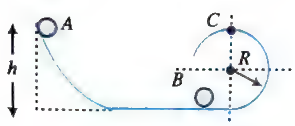
(1) Increase with θ
(2) Decrease with θ
(3) Remain constant
(4) First increase with θ and then decrease

32. Two bodies of masses m and $4m$ are attached to a light string as shown in figure. A body of mass m hanging from string is executing oscillations with angular amplitude 60°, while other body is at rest on a horizontal surface. The minimum coefficient of friction between mass $4m$ and the horizontal surface is (here pulley is light and smooth)



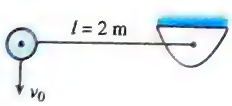
(1) $\frac{1}{4}$ (2) $\frac{3}{4}$
(3) $\frac{1}{2}$ (4) $\frac{1}{8}$

33. Ball A of mass m, after sliding from an inclined plane, strikes elastically another ball B of same mass at rest. Find the minimum height h so that ball B just completes the circular motion of the surface at C. (All surfaces are smooth.)



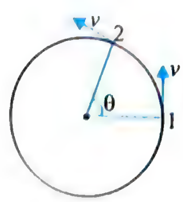
(1) $h=\frac{5}{2}R$ (2) $h=2R$
(3) $h=\frac{2}{5}R$ (4) $h=3R$

34. A small sphere is given vertical velocity of magnitude $v_0 = 5\ \text{ms}^{-1}$ and it swings in a vertical plane about the end of a massless string. The angle θ with the vertical at which string will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere, is



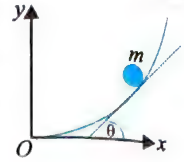
(1) $\cos^{-1}\left(\frac{2}{3}\right)$ (2) $\cos^{-1}\left(\frac{1}{4}\right)$
(3) 60° (4) 30°

35. Two particles describe the same circle of radius R in the same direction with the same speed v, then at the given instant relative angular velocity of 2 with respect to 1 will be



(1) $\frac{2v\sin\frac{\theta}{2}}{R}$ (2) $\frac{v}{2R\sin\frac{\theta}{2}}$
(3) $\frac{v}{R}$ (4) $\frac{v\cos\frac{\theta}{2}}{R}$

36. A block m slides down a curved frictionless track, starting from rest. The curve obeys the equation $y=\frac{x^2}{2}$. The tangential acceleration of the block is

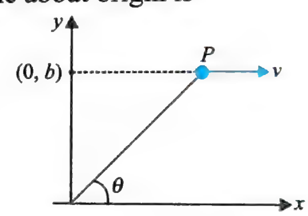


(1) g (2) $\frac{gx}{\sqrt{x^2+4}}$
(3) $\frac{g}{2}$ (4) $\frac{gx}{\sqrt{x^2+1}}$

37. A particle is moving with a velocity of $\vec{v}=\left(3\hat{i}+4t\hat{j}\right)$m/s Find the ratio of tangential acceleration to that of normal acceleration at $t=1$ sec.
(1) 4/3 (2) 3/4
(3) 5/3 (4) 3/5

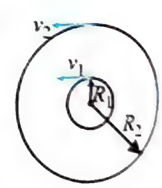
38. A car is moving on circular path of radius 100 m such that its speed is increasing at the rate of 5 m/s². At $t=0$ it starts from rest. What is the radial acceleration of car at the instant it makes one complete round trip?
(1) $20\pi\ \text{ms}^{-2}$ (2) $7\pi\ \text{ms}^{-2}$
(3) $5\ \text{ms}^{-2}$ (4) None of these

39. A particle is moving parallel to x-axis as shown in the figure such that at all instant the y-component of its position vector is constant and is equal to b. The angular velocities of the particle about origin is



(1) $\frac{v}{b}\sin^2\theta$ (2) $\frac{v}{b}$
(3) $\frac{v}{b}\sin\theta$ (4) vb

40. Two particles start moving on circles of radius $R_1 = 1$ m and $R_2 = 3$ m shown with velocity $v_1 = 2$ m/s, $v_2 = 15$ m/s, respectively. The minimum time after which the particles will again be collinear with centre is equal to



(1) $\frac{2\pi}{3}$sec (2) $\frac{2\pi}{5}$sec
(3) 2π sec (4) $\frac{\pi}{3}$sec

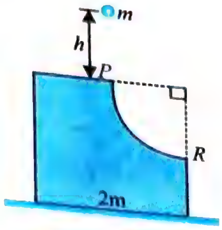
41. The track of motorcycle-race is circular and unbanked. There are two bikers on the road, one travels along a path of greater radius than the other. They both lean towards the centre at the same angle. Which one completes the circular path in less time?
(1) biker having path of smaller radius.
(2) biker having path of larger radius.
(3) will complete circle in same time.
(4) both can not lean at same angle if radius is different.

42. A stone is thrown from point O on ground with velocity $\vec{v}=5\hat{i}+10\hat{j}$ m/s, where x is horizontal and y is vertical. Its angular velocity about O when the stone is at maximum height of its trajectory, is

(1) $\frac{1}{2}$ rad/s (2) $\frac{1}{\sqrt{2}}$ rad/s

(3) 1 rad/s (4) $\sqrt{2}$ rad/s

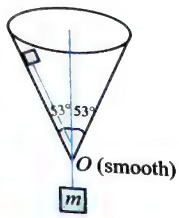
43. A small ball of mass m is released from height h on a wedge having circular track as shown in figure. Find the normal force on ball due to wedge just after it enters in circular track. (Assume it enters at point P tangentially and all surfaces are smooth)



(1) $\frac{2mgh}{R}$ (2) $\frac{4}{3}\frac{mgh}{R}$

(3) $\frac{mgh}{2R}$ (4) zero

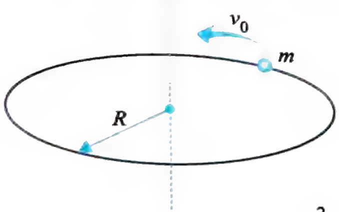
44. As shown in figure a mass m is rotating freely in a horizontal circle of radius 20 cm in a smooth fixed cone which supports a stationary mass m, attached to the other end of the string passing through smooth hole O in cone, hanging vertically. Find the angular velocity of rotation.



(1) 5 rad/s (2) 2 rad/s

(3) 10 rad/s (4) 15 rad/s

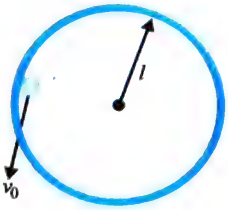
45. A bead of mass m is free to slide on a fixed horizontal circular wire of radius R. At time $t = 0$, it is given a velocity v_0 along the tangent to the circle. If the coefficient of kinetic friction between the bead and the wire is μ_k. Then magnitude of tangential acceleration at $t = 0$ will be



(1) $\mu_k g$ (2) $\mu_k \frac{v^2}{R}$

(3) $\frac{\mu_k}{R}\sqrt{v^2 + g^2R^2}$ (4) $\mu_k\left[g + \frac{v^2}{R}\right]$

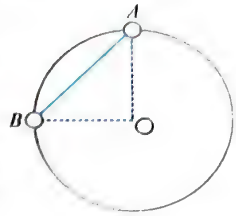
46. A small block of mass m slides on a frictionless horizontal table. It is constrained to move inside a ring of radius l which is fixed to the table. At $t = 0$, the block has a tangential velocity v_0. The coefficient of friction between the block and the ring is μ. The velocity of the block at time t is



(1) $\frac{v_0}{1 + \frac{\mu v_0 t}{l}}$ (2) $\frac{v_0}{1 + \frac{2\mu v_0 t}{l}}$

(3) $\frac{v_0}{1 - \left(\frac{\mu v_0 t}{l}\right)}$ (4) $\frac{v_0}{\left(\frac{\mu v_0 t}{l}\right)}$

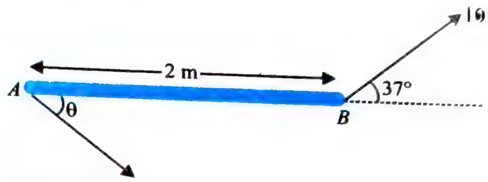
47. Beads A and B, each of mass m, are connected by a light inextensible cord. They are constrained (restricted) to move on a frictionless ring in a vertical plane as shown. The beads are released from rest at the positions shown. The tension in the cord just after the release is



(1) $\sqrt{2}\,mg$ (2) mg

(3) $\frac{1}{\sqrt{2}}mg$ (4) $2\,mg$

48. A uniform rod of length 2 m. Velocity of point 'B' is 10 m/s as shown in figure and angular velocity of rod is 7 rad/s. Then



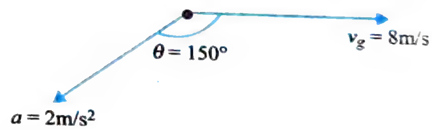
(1) Velocity of point A is 10 m/s at an angle of 53° with rod

(2) Velocity of point A is 10 m/s at an angle of 37° with rod

(3) Velocity of point A is 6 m/s perpendicular to the rod

(4) Velocity of point A is $8\sqrt{2}$ m/s at an angle of 45° with rod

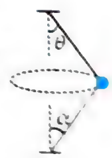
49. The figure shows the velocity and acceleration of a point like body at the initial moment of its motion. The acceleration vector of the body remains constant. The minimum radius of curvature of trajectory of the body is



(1) 2 m (2) 4 m

(3) 8 m (4) 16 m

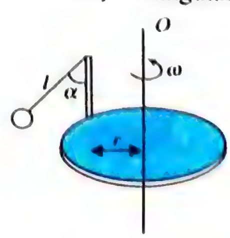
50. A massless thread passes through a ring (smooth). The ring is in a circular path with both part of the thread in tension. Which of the following values of θ and α are possible?



(1) $\theta > \alpha$ (2) $\alpha > \theta$

(3) $\theta = \alpha$ (4) Anything is possible

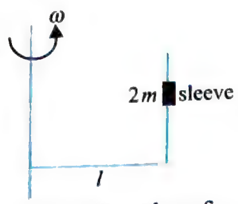
51. A ball of mass m is attached to the end of a thread fastened to the top of a vertical rod which is fitted to a horizontally revolving round table as shown. If the thread forms an angle α with the vertical, the angular velocity ω of table is



(1) $\sqrt{\dfrac{g}{l\cos\alpha}}$ (2) $\sqrt{\dfrac{r}{g\tan\alpha}}$

(3) $\sqrt{\dfrac{g\tan\alpha}{r+l\sin\alpha}}$ (4) $\sqrt{\dfrac{g\tan\alpha}{r}}$

52. A L shaped rod whose one rod is horizontal and other vertical is rotating about a vertical axis as shown with angular speed ω. The sleeve shown in figure has mass $2m$ and friction coefficient between rod and sleeve is μ. The minimum angular speed ω for which sleeve cannot slip on rod is



(1) $\omega=\sqrt{\dfrac{g}{\mu l}}$ (2) $\omega=\sqrt{\dfrac{\mu g}{l}}$

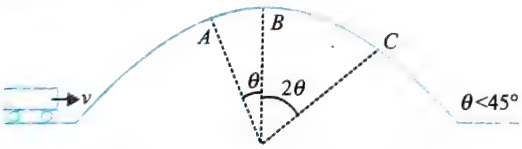
(3) $\omega=\sqrt{\dfrac{l}{\mu g}}$ (4) None of these

53. The angular acceleration of a particle moving along a circle is given by $\alpha = k\sin\theta$, where θ is the angle turned by particle and k is constant. The centripetal acceleration of the particle in terms of k, θ and R (radius of circle) is given by :

(1) $2kR(\sin^2\theta)$ (2) $2kR(\sin\theta)$

(3) $2kR(1+\cos\theta)$ (4) $2kR(1-\cos\theta)$

54. A self-propelled vehicle (assume it as a point mass) runs on a track with constant speed v. It passes through three positions A, B and C on the circular part of the track. Suppose N_A, N_B and N_C are the normal forces exerted by the track on the vehicle when it is passing through points A, B and C respectively then



(1) $N_A=N_B=N_C$ (2) $N_B>N_A>N_C$

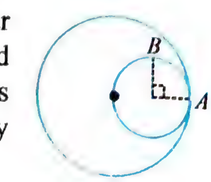
(3) $N_C>N_A>N_B$ (4) $N_B>N_C>N_A$

55. A body suspended on a spring balance in a ship weighs W, when ship is at rest. When the ship begins to move along the equator from west to east with the velocity v, the scale reading is approximately equal to (ω is angular velocity of earth)

(1) W (2) $W\left(1+\dfrac{v^2}{R}\right)$

(3) $W\left(1+\dfrac{2v\omega}{g}\right)$ (4) $W\left[1-\dfrac{2v\omega m}{W}\right]$

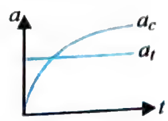
56. A and B are moving in 2 circular orbits with angular velocity 2ω and ω respectively. Their positions are as shown at $t=0$. Find the time when they will meet for first time.

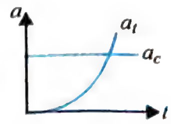


(1) $\dfrac{\pi}{2\omega}$ (2) $\dfrac{3\pi}{2\omega}$

(3) $\dfrac{\pi}{\omega}$ (4) they will never meet

57. A particle starts from rest and performing circular motion of constant radius with speed given by $v=\alpha\sqrt{x}$ where α is a constant and x is the distance covered. The correct graph of magnitude of its tangential acceleration (a_t) and centripetal acceleration (a_c) versus t will be

(1)


(2)


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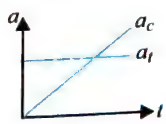

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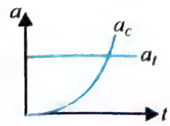


Multiple Correct Answers Type

1. A car moves on a circular road describing equal angles about the centre in equal intervals of time. Which of the following statements about the velocity of car is/are not true?

(1) Velocity is constant.

(2) Magnitude of velocity is constant but the direction changes.

(3) Both magnitude and direction of velocity change.

(4) Velocity is directed towards the center of circle.

2. A particle moves in a circle of radius 20 cm. Its linear speed is given by $v=2t$ where t is in seconds and v in $\mathrm{m\,s^{-1}}$. Then

(1) The radial acceleration at $t=2$ s is 80 $\mathrm{m\,s^{-2}}$.

(2) Tangential acceleration at $t=2$ s is 2 $\mathrm{m\,s^{-2}}$.

(3) Net acceleration at $t=2$ s is greater than 80 $\mathrm{m\,s^{-2}}$.

(4) Tangential acceleration remains constant in magnitude.

3. A car moving along a circular track of radius 50.0 m. accelerates from rest at 3.00 $\mathrm{m\,s^{-2}}$. Consider a situation when the car's centripetal acceleration equals its tangential acceleration. Then

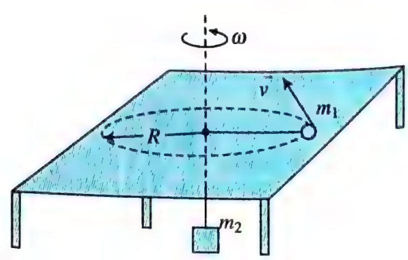
(1) The angle around the track does the car travel is 1 rad.

(2) The magnitude of the car's total acceleration at that instant is $3\sqrt{2}\ \mathrm{m\,s^{-1}}$.

(3) Time elapses before this situation is $\sqrt{\dfrac{50}{3}}$ s.

(4) The distance travelled by the car during this time is 25 m.

4. A particle of mass m_1 moves in a circular path of radius R on a rotating table. A string connecting the particles m_1 and m_2 passes over a smooth hole made on the table as shown in figure. If mass m_1 does not slide relative to the rotating table, mark the correct option(s) as applicable.



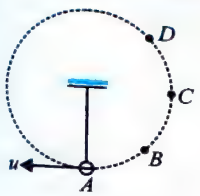
(1) The friction force acting on the block m_1 is $(m_1\omega^2R - m_2g)$ along radial rotation towards center of rotation.

(2) The friction force acting on the block m_1 is $(m_1\omega^2R - m_2g)$ along tangent direction in the direction opposite to $\vec{v}$.

(3) The maximum angular velocity of the particle is $\sqrt{\left(\frac{m_2+\mu m_1}{m_1}\right)\frac{g}{R}}$.

(4) The minimum angular velocity of the particle is $\sqrt{\left(\frac{m_2-\mu m_1}{m_1}\right)\frac{g}{R}}$.

5. A bob is attached to a string as shown in the figure below. At the lowest point, it is given a horizontal velocity u. The bob rotates complete circle in vertical plane. For different values of u, it can have different total acceleration at different points. For each of the points marked in column I, match the possible direction(s) of the total acceleration given by statements (i), (ii), (iii), (iv), and (v).

Column I		Column II
	(I)	←
	(II)	↑
	(III)	↓
	(IV)	↙
	(V)	↖

Match the situation of the particle with the statement given in Column II.

(1) A-II
(2) B-I, IV, V
(3) C-I, II
(4) D-III, IV

6. A particle is projected from ground at an angle θ. At a certain instant the velocity vector $\vec{v}$ of particle makes an angle α with horizontal. Choose the correct option(s).

(1) $\frac{d\vec{v}}{dt} = \vec{g}$

(2) Modulus of $\frac{d|\vec{v}|}{dt} = g\sin\alpha$

(3) Tangential acceleration has magnitude of g sinα

(4) Normal acceleration = g cos α

7. A particle moves in a circle uniformly such that it completes a circle in 2 sec. During half a turn its centripetal acceleration change by 6 m/s². Choose the correct option(s).

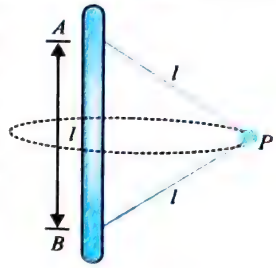
(1) the radius of the circle is $(3/\pi^2)$m.

(2) the magnitude of change in velocity during 1 s is $3/\pi$ m/s

(3) the tangential velocity of the particle is $3/\pi$ m/s.

(4) the particle is moving in anticlockwise sense as seen from above.

8. A particle P of mass m is attached to a vertical axis by two strings AP and BP of length l each. The separation $AB = l$. P rotates around the axis with an angular velocity ω. The tensions in the two strings are T_1 and T_2. Choose the correct option(s).



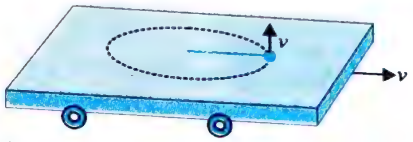
(1) $T_1 = T_2$

(2) $T_1 = T_2 = m\omega^2 l$

(3) $T_1 + T_2 = 2mg$

(4) BP will remain taut only if $\omega \geq \sqrt{2g/l}$

9. On a train moving along east with a constant speed v, a boy revolves a bob with string of length l on smooth surface of a train, with equal constant speed v relative to train. Mark the correct option(s).



(1) Maximum speed of bob is $2v$ in ground frame.

(2) Tension in string connecting bob is $4mv^2/l$ at any instant.

(3) Tension in string is mv^2/l at all the moments.

(4) Minimum speed of bob is zero in ground frame.

10. A block is resting on a rotating rough table as shown in the figure below. Choose the correct option(s).

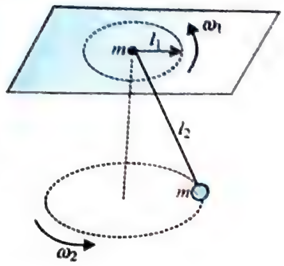
(1) According to A, centripetal acceleration is provided by friction force.

(2) According to A, friction is kinetic and according to B friction is static.

(3) According to B, centrifugal force is balanced by friction force.

(4) According to A, centrifugal force is balanced by friction force.

11. One end of a light inextensible string of length l is attached to a particle of mass m resting on a smooth horizontal table. The string passes through a smooth hole in the table and to its other end is attached a small particle of same mass. The system is set in rotation as shown in figure with constant angular speed. Choose the correct option(s).



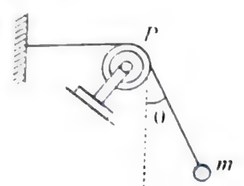
(1) $\frac{l_1}{l_2} = \frac{\omega_2^2}{\omega_1^2}$

(2) $\frac{l_1}{l_2} = \frac{\omega_1^2}{\omega_2^2}$

(3) The motion is possible if, $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} < \frac{l}{g}$

(4) The motion is possible if, $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} > \frac{l}{g}$

12. A bob of mass m hangs by a light inextensible string which is connected with a rigid support after passing over a smooth pulley P. If the bob is released from an angular position $\theta = 30°$, then



(1) The force exerted by the string on the pulley is $\sqrt{3}mg/2$

(2) The net horizontal force exerted by the string on the pulley is $\sqrt{3}mg/2$

(3) Acceleration of the bob is $g/2$

(4) The force exerted by the string on the wall is $2mg/\sqrt{3}$

13. A heavy particle is tied to the end A of a string of length 1.6 m. Its other end O is fixed. It revolves as a conical pendulum with the string making 60° with the vertical. Then

(1) its period of revolution is $4\pi/7$ s.

(2) the tension in the string is doubled the weight of the particle

(3) the velocity of the particle = $2.8\sqrt{3}$ m/s

(4) the centripetal acceleration of the particle is $9.8\sqrt{3}$ m/s².

14. A force of constant magnitude F acts on a particle (mass m) moving in a plane such that it is always perpendicular to the velocity $\vec{v}\,(|\vec{v}| = v)$ of the body. Select the correct option(s).

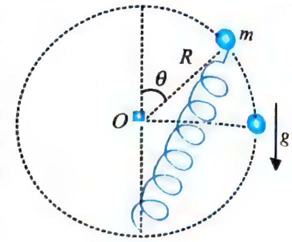
(1) The time taken by the particle to cover a distance S is $(t = S/v)$

(2) The angle turned by the velocity vector of the particle during a distance S is $\left(\frac{FS}{mv^2}\right)$.

(3) The magnitude of change in velocity vector during the time t is $2v\sin\left(\frac{Ft}{mv}\right)$.

(4) The magnitude of change in velocity vector during the time t is $2v\sin\left(\frac{Ft}{2mv}\right)$.

15. A bead of mass m can slide without friction on a fixed vertical loop of radius R. The bead moves under the combined effect of gravity and a spring with spring constant k only. The spring is rigidly attached to the bottom of the loop. Assume that the natural length of spring is zero. The bead is released from rest at $\theta = 0°$ with non-zero but negligible speed to the bead. The gravitational acceleration is directed downward as shown in the figure. Now choose the correct option(s). (Neglect any type of friction between any contact) (given $kR = mg/2$)



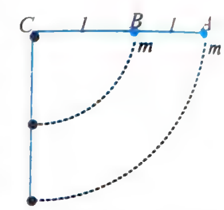
(1) The speed v of the bead when $\theta = 90°$ is $\sqrt{3gR}$.

(2) The speed v of the bead when $\theta = 90°$ is $\sqrt{5gR}$.

(3) The magnitude of the force that ring exerts on the bead when $\theta = 90°$ is $3mg$.

(4) The magnitude of the force that ring exerts on the bead when $\theta = 90°$ is $5mg$.

16. A weightless rod of length $2l$ carries two equal masses m. One is tied at end A and other at middle of rod B. The rod can rotate in vertical plane about a fixed horizontal axis passing through C.



The rod is released from rest in horizontal position. Then which of the following option(s) is/are correct?

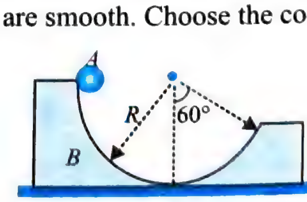
(1) The speed of mass B at the instant rod becomes vertical is $\sqrt{\frac{6}{5}gl}$.

(2) The tension in thread AB is $\frac{18}{5}mg$, when the rod becomes vertical.

(3) The tension in thread BC is $\frac{28}{5}mg$ when the rod becomes vertical.

(4) The hinge reaction at the hinge point C is $\frac{28}{5}mg$ when the rod becomes vertical.

17. In the adjacent figure, a wedge of mass $2m$ is lying at rest on a horizontal surface. The wedge has a cavity which is the portion of a sphere of radius R. A small sphere of mass m is released from the top edge of the cavity to slide down. All surface are smooth. Choose the correct option(s).



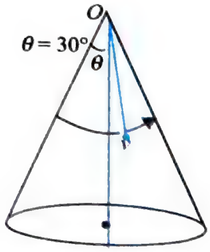
(1) The maximum speed acquired by the sphere is $\sqrt{\frac{4gR}{3}}$.

(2) The maximum height (from ground) to which the sphere will rise after breaking off the wedge is $\frac{10R}{11}$.

(3) The wedge will acquire its maximum speed when the sphere is about to leave it.

(4) The maximum speed acquired by the wedge is $\sqrt{\frac{gR}{3}}$.

18. A bob of mass 2 kg is suspended from point O of a cone with an inextensible string of length $\sqrt{3}$ m. It is moving in horizontal circle over the surface of cone as shown in the figure. Then $\left(g = 10 \text{ m/s}^2\right)$



(1) normal force on bob is 19 N when $v = 2$ m/s

(2) tension in string is $\frac{38}{\sqrt{3}}$ N when $v = 2$ m/s

(3) normal force on bob is $\frac{17}{\sqrt{3}}$ N when $v = 2$m/s

(4) bob looses contact with cone if $v > \sqrt{5}$ m/s

19. A ball tied to the end of a string performs vertical circle (complete circle) under the influence of gravity. Choose the correct option(s).

(1) When the string makes an angle 90° with the vertical, the tangential acceleration is zero & radial acceleration is somewhere between maximum and minimum.

(2) When the string makes an angle 90° with the vertical, the tangential acceleration is maximum & radial acceleration is somewhere between maximum and minimum.

(3) At no place in the circular motion, tangential acceleration is equal to radial acceleration.

(4) Throughout the path whenever radial acceleration has its extreme value, the tangential acceleration is zero.

20. A particle tied at one end of an inextensible string of length R, whose another end is fixed. Particle is performing circular motion in vertical plane. Ratio of maximum to minimum speed is λ. Choose the correct option(s).

(1) Speed at the top most point is $\sqrt{\left(\frac{4\lambda^2}{\lambda^2 - 1}\right)gR}$

(2) Speed at the top most point is $\sqrt{\left(\frac{4}{\lambda^2 - 1}\right)gR}$

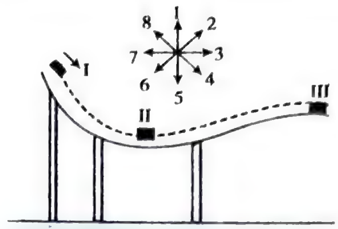
(3) Ratio of maximum to minimum tension is $\left(\frac{5\lambda^2 - 1}{5 - \lambda^2}\right)$

(4) Ratio of tension in string to weight when string is horizontal is $2\left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right)$

Linked Comprehension Type

For Problems 1–3

The figure below depicts a block sliding along a frictionless ramp in vertical plane. Eight numbered arrows in the diagram represent directions to be referred to when answering the questions.



Position-I: Starts sliding on curve path
Position-II: Lowest position of curve part
Position-III: Just outside of the curve part

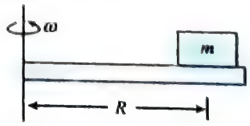
1. The direction of the acceleration of the block, when in position I, is best represented by which of the arrows in the diagram.
   (1) 4
   (2) 5
   (3) 2
   (4) None of the arrows, the acceleration is zero.

2. The direction of the acceleration of the block when in position II is best represented by which of the arrows in the diagram?
   (1) 5 (2) 8
   (3) 1 (4) 3

3. The direction of the acceleration of the block (after leaving the ramp) at position III is best represented by which of the arrows in the diagram?
   (1) 5
   (2) 6
   (3) 2
   (4) None of the arrows, the acceleration is zero.

For Problems 4 and 5

The coin is rotating with a plate without sliding. If the coefficient of friction between the coin and plate is $\mu = 0.75$, the friction between the coin and plate is



4. If the angular frequency of the rotation of the plate is $\omega = \sqrt{\frac{g}{2R}}$. The friction force acting on coin is
   (1) $\frac{3}{4}mg \rightarrow$ (2) $\frac{mg}{4} \leftarrow$
   (3) $\frac{mg}{2} \leftarrow$ (4) $\frac{mg}{2} \rightarrow$

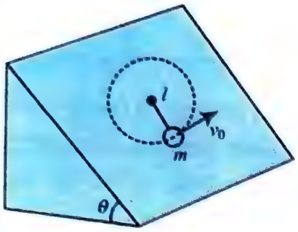
5. If the plate is rotating alone and the coin is gently placed on the rotating plate, the frictional force on the coin is

(1) mg (2) $\frac{3}{2}mg$

(3) $\frac{mg}{2}$ (4) $\frac{3}{4}mg$

For Problems 6–8

A pendulum bob swings along a circular path on a smooth inclined plane as shown in figure, where $m = 3$ kg, $l = 0.75$ m, $\theta = 37°$. At the lowest point of the circle the tension in the string is $T = 274$ N. Take $g = 10$ m s^{-2}.



6. The speed of the bob at the lowest point is

(1) 9.2 m s^{-1} (2) 9 m s^{-1}

(3) 6.5 m s^{-1} (4) 8 m s^{-1}

7. The speed of the bob at the highest point on the circle is

(1) $\sqrt{46}$ m s^{-1} (2) $\sqrt{26}$ m s^{-1}

(3) $\sqrt{52}$ m s^{-1} (4) $\sqrt{35}$ m s^{-1}

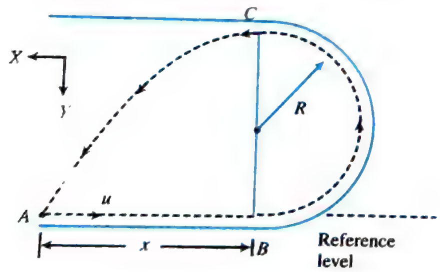
8. The tension in the string at the highest position is

(1) 166 N (2) 67.56 N

(3) 68 N (4) 152 N

For Problems 9 and 10

A small ball is rolled with speed u from point A along a smooth circular track as shown in figure. If $x = 3R$, then



9. Determine the required speed u so that the ball returns to A, the point of projection after passing through C, the highest point.

(1) $\frac{3}{2}\sqrt{gR}$ (2) $\frac{1}{2}\sqrt{gR}$

(3) $\frac{5}{3}\sqrt{gR}$ (4) $\frac{5}{2}\sqrt{gR}$

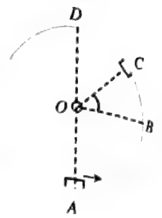
10. What is the minimum value of x for which the ball can reach the point of projection after reaching C?

(1) $2R$ (2) $5R$

(3) $3R$ (4) $\frac{5}{2}R$

For Problems 11–13

A small block of mass m is pushed on a smooth track from position A with a velocity $2/\sqrt{5}$ times the minimum velocity required to reach point D. The block will leave the contact with track at the point where normal force between them becomes zero.



11. At what angle θ with horizontal does the block gets separated from the track?

(1) $\sin^{-1}\left(\frac{1}{3}\right)$ (2) $\sin^{-1}\left(\frac{3}{4}\right)$

(3) $\sin^{-1}\left(\frac{2}{3}\right)$

(4) Never leaves contact with the track

12. When the block reaches point B, what is the direction (in terms of angle with horizontal) of acceleration of the block?

(1) $\tan^{-1}\left(\frac{1}{2}\right)$ (2) $\tan^{-1}(2)$

(3) $\sin^{-1}\left(\frac{2}{3}\right)$

(4) The block never reaches point B.

13. Find where the maximum contact force occurs between the block and the track.

(1) At B (2) At C

(3) Somewhere between A and B

(4) At A

For Problems 14 and 15

A motorcycle moves around a vertical circle with a constant speed under the influence of the force of gravity $\vec{W}$, the force of friction between the wheels and the track $\vec{f}$, and the normal force between the wheels and the track $\vec{N}$.

14. Which of the following vectors has a constant magnitude?

(1) $\vec{N}$ (2) $\vec{N}+\vec{f}$

(3) $\vec{f}+\vec{W}$ (4) $\vec{N}+\vec{W}+\vec{f}$

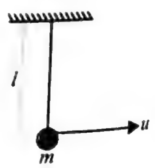
15. Which of the following vectors, when nonzero, is always directed toward the center of the circle or away from the centre of the circle?

(1) $\vec{f}$ (2) $\vec{W}$

(3) $\vec{f}+\vec{W}$ (4) $\vec{N}+\vec{f}$

For Problems 16 and 17

A particle of mass m is attached to one end of the light inextensible string and other end of the string is fixed in vertical plane as shown. Particle is given the horizontal velocity $u = \sqrt{\frac{5}{2}gl}$.



16. The maximum angle made by the particle with downward vertical is

(1) $\cos^{-1}\left(\frac{1}{4}\right)$ (2) $\sin^{-1}\left(\frac{1}{4}\right)$

(3) $\frac{\pi}{2}+\cos^{-1}\left(\frac{1}{4}\right)$ (4) $\pi-\cos^{-1}\left(\frac{1}{4}\right)$

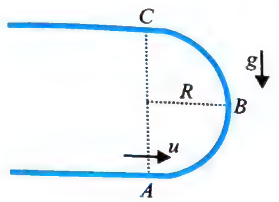
17. The tension in string at an instant when acceleration of the particle is horizontal is

(1) mg (2) $2mg$

(3) $4mg$ (4) $6mg$

For Problems 18 and 19

Figure shows a smooth fixed track with two straight parts connected by a semicircular arc ABC. The shown track is placed in a vertical plane. A small ball is given an initial speed u along the track as shown. The ball loses contact with the track somewhere in the part BC and collides with the track at A during subsequent motion. Now answer the following questions.



18. Find the value of 'u'.

(1) $\sqrt{\frac{9}{2}Rg}$ (2) $\sqrt{3Rg}$

(3) $\sqrt{\frac{7}{2}Rg}$ (4) $\sqrt{7Rg}$

19. The time after which it collides at 'A' after losing the contact with the shown track, is

(1) $\sqrt{\frac{3}{4}\frac{R}{g}}$ (2) $\sqrt{\frac{6R}{g}}$

(3) $\sqrt{\frac{3}{2}\frac{R}{g}}$ (4) $\sqrt{\frac{3R}{g}}$

For Problems 20 and 21

At a turn a track is banked for optimum speed of 40 km/h. At the instant shown in the figure, a car is traveling out of the plane of the figure.

20. If the car travels at 60 km/h, the net frictional force acting on the wheels must be

(1) static in nature and point downward along the bank of the track for safe driving.

(2) static in nature and points upward along the bank of the track for safe driving.

(3) kinetic in nature and points upward along the bank of the track for safe driving.

(4) kinetic in nature and points downwards along the bank of the track for safe driving.

21. When the car is on the turn the driver realizes that his speed is reaching the maximum safe limit so he applies brakes to reduce the speed till the car reaches the optimum speed. While he is applying the brake, the frictional force acting on the wheels of the car must be

(1) Static in nature and directed somewhere in between downward along bank of the track and into the plane of the figure.

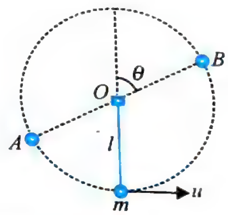
(2) static in nature and directed somewhere in between upward along bank of the track and into the plane of the figure.

(3) Static in nature and directed somewhere in between downward along bank of the track and out of the plane of the figure.

(4) Static in nature and directed somewhere in between upward along bank of the track and out of the plane of the figure.

For Problems 22 and 23

A ball is hanging vertically by a light extensible string of length l from fixed point O. The ball of mass m is given a speed u at the lowest position such that it completes a vertical circle with centre at O as shown. Let AB be a diameter of circular path of ball making an angle θ with vertical as shown, (g is acceleration due to gravity)



22. Choose the correct statements

(1) Let T_A and T_B be tension in string when ball is at A and B respectively, then $T_A - T_B$ is equal to $12mg\cos\theta$

(2) Let $\vec{a}_A$ and $\vec{a}_B$ be acceleration of ball when it is at A and B respectively, then $|\vec{a}_A+\vec{a}_B|$ is equal to $2g\sin\theta$

(3) Let a_A and a_B be acceleration of ball when it is at A and B respectively, then $|\vec{a}_A+\vec{a}_B|$ is equal to $g\sqrt{12\cos^2\theta+4}$

(4) Let T_A and T_B be tension in string when ball is at A and B respectively, then $T_A - T$ is equal to $6mg\cos\theta$.

23. Let K_A and K_B be kinetic energy of ball when it is at A and B respectively, then $K_A - K_B$ is equal to.

(1) $2mgl\cos\theta$ (2) $4mgl\cos\theta$

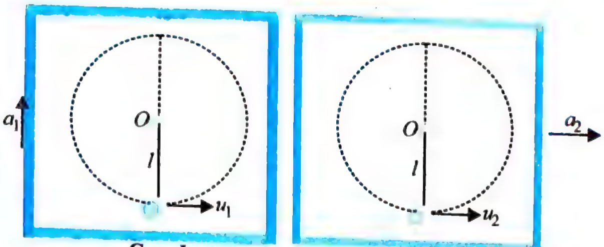
(3) $mgl\cos\theta$ (4) None of these

For Problems 24 and 25

A man inside the lift swings a stone tied to a string of length l in a vertical plane. The string remains taut throughout the time of rotation of the stone.

Case I: The lift accelerates vertically upwards with acceleration $a_1 = \frac{g}{2}$

Case II: The lift accelerates horizontally towards right with acceleration $a_2 = g$



Case I   Case II

24. For case-I choose the correct option(s).

(1) The minimum speed u_1 of the stone (relative to the lift) required at the lowest point of its path for successful completion of the loop is $\sqrt{5gl}$.

(2) The tension at the lowest point of its path for just completing the circular motion is $6mg$.

(3) The tension at the lowest point of its path for just completing the circular motion is $9mg$.

(4) The minimum speed u_1 of the stone (relative to the lift) required at the lowest point of its path for successful completion of the loop is $\sqrt{15gl}$.

25. For case-II choose the correct option(s).

(1) The minimum speed u_2 of the stone (relative to the lift) required at the lowest point of its path for successful completion of loop is $\sqrt{(2+3\sqrt{2})gl}$.

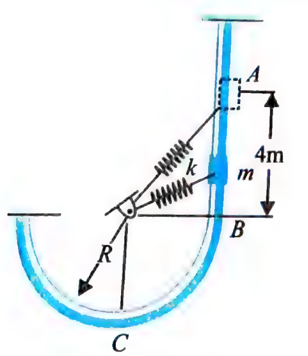
(2) The minimum speed u_2 of the stone (relative to the lift required at the lowest point of its path for successful completion of loop is $\sqrt{(2+4\sqrt{2})gl}$.

(3) The maximum possible tension in the string, if the particle just completes the circular motion, is $6mg$.

(4) The maximum possible tension in the string, if the particle just completes the circular motion, is $7\sqrt{2}\,mg$.

For Problems 26 and 27

The spring of constant $k = 50$ N/m is unstretched when the slider of mass $m = 2$ kg passes through position B. The slider is released from rest in position A. There is friction between slider and guide whose work done from A to B is –11 J and from B to C is –4 J. (Radius of guide $R = 3$ m). The whole system is in vertical plane.



26. If speed of slider at position B and C are v_B and v_C, respectively, then

(1) $v_B = 6.5$ m/s   (2) $v_B = 13$ m/s

(3) $v_C = 16$ m/s   (4) $v_C = 15$ m/s

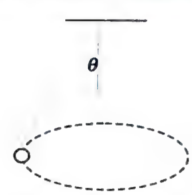
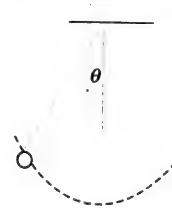
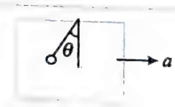
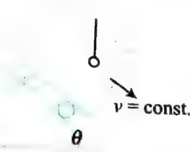
27. If the normal force at position B and C are N_B and N_C, respectively, then

(1) $N_B = 112.67$ N   (2) $N_B = 216.33$ N

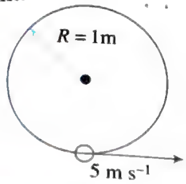
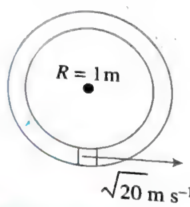
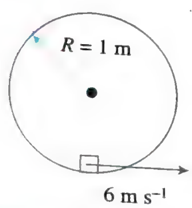
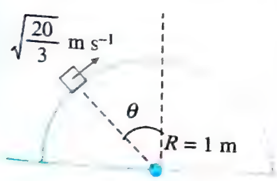
(3) $N_C = 150$ N   (4) $N_C = 170$ N

Matrix Match Type

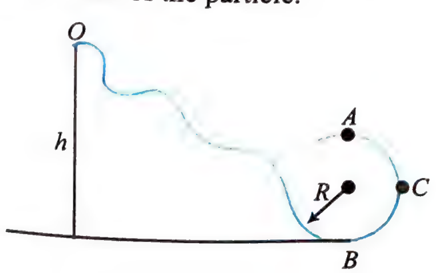
1. In all the four situations depicted in column I, a ball of mass m is connected to a string. In each case, find the tension in the string and match the appropriate entries in column II. θ is the extreme position; T is tension in extreme position.

Column I		Column II	
i.	 Conical pendulum	a.	$T = mg\cos\theta$
ii.	 Pendulum is swinging in angular position	b.	$T\cos\theta = mg$
iii.	 The car is moving with constant acceleration. The ball is at rest with respect to car.	c.	Speed of ball with respect to ground is constant
iv.	 The car is moving with constant velocity. The ball is at rest with respect to car.	d.	Velocity of ball with respect to ground is changing continuously

2. In column I, a situation is depicted each of which is in vertical plane. The surfaces are frictionless. Match with appropriate entries in column II.

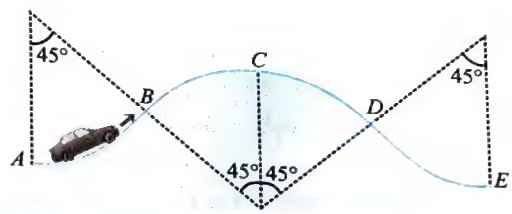
Column I		Column II
i. Bead is threaded on a circular fixed wire and is projected from the lowest point. 	**a.**	Normal force is zero at the top-most point of its trajectory.
ii. Block loosely fits inside the fixed small tube and is projected from lowest point 	**b.**	Velocity of the body is zero at the top-most point of its trajectory
iii. Block is projected horizontally from lowest point of a smooth fixed cylinder. 	**c.**	Acceleration of the body is zero at the top-most point of its trajectory
iv. Block is projected on a fixed hemisphere from angular position θ.  $\cos\theta = 2/3$	**d.**	Normal force is radially outward at the top-most point of trajectory.

3. A particle is released from height h on a smooth track terminating in a circular path of radius R. A and C are points at top and at horizontal level of centre, respectively, of the circular path. Column I represents the different values of height of inclined plane and Column II gives the conditions during the motion of the particle.



Column I	Column II
i. If $h = 3.2R$	**a.** the particle is able to complete vertical circular motion
ii. If $h = 2.7R$	**b.** the force exerted by the particle on the track at point A is zero
iii. If $h = 2.5R$	**c.** the force exerted by the particle on the track at point A is more than its weight
iv. If $h = 4R$	**d.** the force exerted by the particle on the track at point C is more than its weight

4. A car moves with constant speed on a track consisting of four circular segments AB, BC, CD and DE, all in vertical plane. Size of car is negligible as compared to the radius of the tracks.



Column I	Column II
i. Point A	**a.** Friction force is zero
ii. Point B	**b.** Normal reaction may be zero
iii. Point C	**c.** Friction is up the slope
iv. Point D	**d.** Friction is down the slope
	e. Friction is maximum

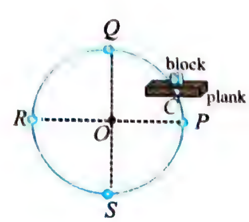
(1) i $\to$ a, d, e; ii $\to$ a,c; iii $\to$ a,c,d,e; iv $\to$ a

(2) i $\to$ a,d; ii $\to$ b,c; iii $\to$ a,c,e; iv $\to$ a,b

(3) i $\to$ a,e; ii $\to$ c,d; iii $\to$ a,b,d,e; iv $\to$ b,c

(4) i $\to$ a,b; ii $\to$ a,c; iii $\to$ c,d,e; iv $\to$ d

5. A small block lies on a rough horizontal platform above its centre C as shown in figure. The plank is moved in vertical plane such that it always remains horizontal and its centre C moves in a vertical circle of centre O with constant angular velocity ω. There is no relative motion between block and the plank and the block does not loose contact with the plank anywhere. P, Q, R and S are four points on circular trajectory of centre C of platform. P and R lie on same horizontal level as O. Q is the highest point on the circle and S is the lowest point on the shown circle. Match the statements in column-I with points in column-II.



Column I	Column II
i. Magnitude of frictional force on block is maximum	**a.** when block is at position P
ii. Magnitude of normal reaction on block is equal to mg	**b.** when block is at position Q
iii. Magnitude of frictional force is zero	**c.** when block is at position R
iv. Net contact force on the block is directed toward centre	**d.** when block is at position S

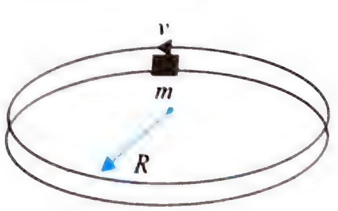
(1) i → a, b; ii → a,c,d; iii → b,c; iv → b,d
(2) i → a,c,d.; ii → a,b; iii → b,d; iv → a,b,d
(3) i → a,c; ii → a,c; iii → b, d; iv → b,d
(4) i → a,b,c; ii → a,b; iii → a,b,d; iv → b,d

6. A block is placed on a horizontal table which can rotate about its axis. A block is placed at a certain distance from centre as shown in figure. Table rotates such that particle does not slide. Select possible direction of net acceleration of block at the instant shown in figure.

Column I	Column II
i. When rotation is clockwise with constant ω	**a.** 1
ii. When rotation is clockwise with decreasing ω	**b.** 2
iii. When rotation is clockwise with increasing ω	**c.** 3
iv. Just after clockwise rotation begins from rest	**d.** 4

(1) i → b; ii → c, d; iii → a; iv → b
(2) i → c; ii → d; iii → c; iv → a,b
(3) i → a; ii → c, iii → c, iv → a
(4) i → c, ii → d, iii → b, iv → a

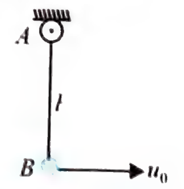
7. A block of mass 1 kg moves in a horizontal circle against the inner wall of a fixed circular hoop of radius 1 m. The block is given a speed 5 m/s tangentially, match the friction force on the block for the various conditions given in column I with the values given in column II. [Take coefficient friction $\mu = 0.2$, wherever required.]



Column I	Column II
i. hoop smooth, ground rough	**a.** 5 N
ii. hoop smooth ground smooth	**b.** 2 N
iii. hoop rough, ground rough	**c.** 7 N
iv. hoop rough, ground smooth	**d.** 0

(1) i → a; ii → b; iii → d; iv → c
(2) i → b; ii → d; iii → c; iv → a
(3) i → b; ii → c; iii → d; iv → a
(4) i → a; ii → d; iii → c; iv → b

8. In the given figure, the bob of the pendulum is given an initial velocity u_0 in horizontal direction.



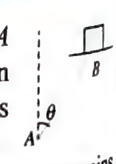
Column I (Nature of AB and u_0)	Column II (Maximum height attained by bob)
i. $AB \to$ massless rod; $u_0 = \sqrt{3gl}$	**a.** $\frac{40}{27}l$
ii. $AB \to$ massless rod; $u_0 = \sqrt{4gl}$	**b.** $\frac{3l}{2}$
iii. $AB \to$ massless string; $u_0 = \sqrt{5gl}$	**c.** $2l$
iv. $AB \to$ massless string; $u_0 = \sqrt{3gl}$	**d.** $\frac{4l}{3}$

Codes:

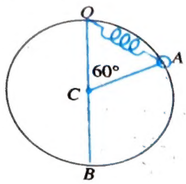
	i	ii	iii	iv
(1)	b	d	c	a
(2)	d	c	b	a
(3)	b	c	c	a
(4)	d	c	a	b

Numerical Value Type

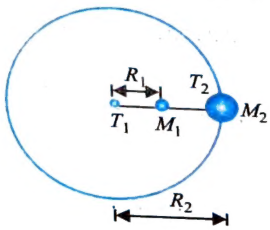
1. A rod AB of length 2 m is hinged at point A and its other end B is attached to a platform on which a block of mass m is kept. Rod rotates about point A maintaining angle $\theta = 30°$ with the vertical in such a way that platform remains horizontal and revolves on the horizontal circular path. If the coefficient of static friction between the block and platform is $\mu = 0.1$, then find the maximum angular velocity in rad s^{-1} of rod so that the block does not slip on the platform. ($g = 10$ m s^{-2})



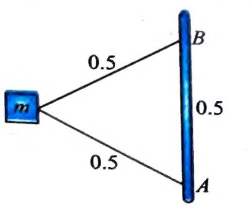
2. A particle of mass 5 kg is free to slide on a smooth ring of radius $r = 20$ cm fixed in a vertical plane. The particle is attached to one end of a spring whose other end is fixed to the top point O of the ring. Initially, the particle is at rest at a point A of the ring such that $\angle OCA = 60°$, C being the center of the ring. The natural length of the spring is also equal to $r = 20$ cm. After the particle is released and slides down the ring, the contact force between the particle and the ring becomes zero when it reaches the lowest position B. Determine the force constant (in $\times 10^2$ Nm^{-1}) of the spring.



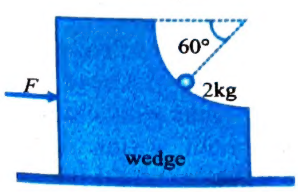
3. Two different masses are connected to two light and inextensible strings as shown in the figure. Both masses rotate about a central fixed point with constant angular speed of 10 rad s^{-1} on a smooth horizontal plane. Find the ratio of tensions T_1/T_2 in the strings.
(Given: $M_1 = 0.25$ kg, $M_2 = 1.0$ kg, $R_1 = 5$ cm, $R_2 = 10$ cm)



4. Two strings of length $l = 0.5$ m each are connected to a block of mass $m = 2$ kg at one end and their ends are attached to the point A and B, 0.5 m apart on a vertical pole which rotates with a constant angular velocity $\omega = 7$ rad/s. Find the ratio T_1/T_2 of tension in the upper string (T_1) and the lower string (T_2). [Use $g = 9.8$ m/s^2]

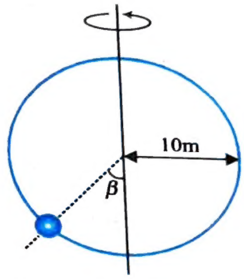


5. A wedge is placed on a smooth horizontal surface. Wedge contains a circular quadrant of radius 25 cm as shown. An insect crawls on the circular part with a constant speed 0.5 m/s. A force F is applied on the wedge, so that it does not move. Find the value of F in newton when radial line of position of insect makes an angle 60° with horizontal.

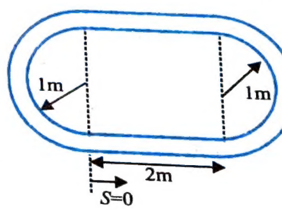


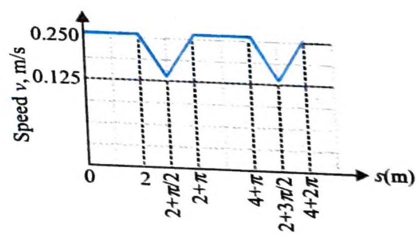
6. A force of constant magnitude $F = 10$ N acts on a particle moving in a plane such that it is perpendicular to the velocity $\vec{v}\left(|\vec{v}| = v = 5\,\text{m/s}\right)$ of the body, and the force is always directed towards a fixed point. Then the angle (in radian) turned by the velocity vector of the particle as it covers a distance $S = 10$ m is (take: mass of the particle as $m = 2$ kg)

7. A small bead can slide without friction on a circular hoop that is in a vertical plane and has a radius of 10 m. The hoop rotates at a constant rate about a vertical diameter. Find the height of the bead (in m) above the bottommost point if the angular velocity of rotation is 0.5 rad/s.

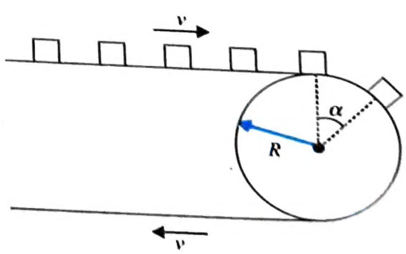


8. A toy car is moving anticlockwise on a closed track whose curved portions are semicircles of radius 1 m. The adjacent graph describes the variation of speed of the car with distance moved by it (starting from point P). Find the time (t) required for the car to complete one lap in second. (take $\pi \ln 2 \approx 2$)





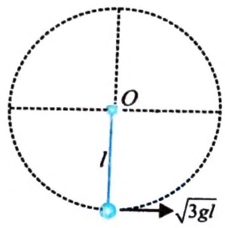
9. Small parts each of mass m on a conveyer belt moving with constant velocity are allowed to drop into a bin. Radius of circular portion is 2 m. The static coefficient of friction between the parts and belt μ_s is 1. If the parts start sliding on the belt at the angle $\alpha = 37°$. Find the velocity (in m/s) of conveyer belt.



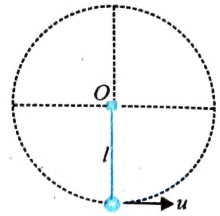
10. One end of a massless spring of spring constant 100 N/m and natural length 0.5 m is fixed and the other end is connected to a particle of mass 0.5 kg lying on a frictionless horizontal table. The spring remains horizontal. If the mass is made to rotate at an angular velocity of 2 rad/s find the elongation of the spring (in cm).

11. A particle is moving in xy plane. Its velocity and acceleration at a certain instant is given by $\vec{v} = 3\hat{i} + 4\hat{j}$ m/s and $\vec{a} = 5\hat{i} + \frac{15}{4}\hat{j}$. Find the rate of change of speed (in m/s²) of particle at that instant.

12. A ball is given a velocity of $\sqrt{3gl}$ as shown. If the ratio of centripetal acceleration to tangential acceleration is $1 : y\sqrt{2}$ at the point where the ball leaves circular path then write the value of y. [Neglect the size of ball]



13. A bob of mass 100 g is connected to a thread of length l and projected horizontally with speed of $\sqrt{6gl}$ so that it completes vertical circle. What is the difference between maximum tension and minimum tension (in N)?

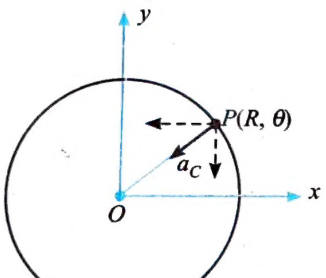


Archives

JEE MAIN

Single Correct Answer Type

1. For a particle in uniform circular motion the acceleration $\vec{a}$ at a point $P(R, \theta)$ on the circle of radius R is (here θ is measured from the x–axis).



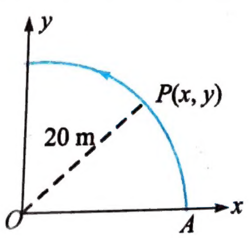
(1) $-\frac{v^2}{R}\cos\theta\,\hat{i} + \frac{v^2}{R}\sin\theta\,\hat{j}$

(2) $-\frac{v^2}{R}\sin\theta\,\hat{i} + \frac{v^2}{R}\cos\theta\,\hat{j}$

(3) $-\frac{v^2}{R}\cos\theta\,\hat{i} - \frac{v^2}{R}\sin\theta\,\hat{j}$

(4) $\frac{v^2}{R}\hat{i} + \frac{v^2}{R}\hat{j}$ **(AIEEE 2010)**

2. A point P moves in counter-clockwise direction on a circular path as shown in the figure. The movement of 'P' is such that it sweeps out a length $s = t^3 + 5$, where s is in metres and t is in seconds. The radius of the path is 20 m. The acceleration of 'P' when $t = 2$ s is nearly



(1) 13 m/s² (2) 12 m/s²
(3) 7.2 m/s² (4) 14 m/s² **(AIEEE 2010)**

3. Two cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2, respectively. Their speeds are such that they make complete circles in the same time t. The ratio of their centripetal acceleration is

(1) $m_1 r_1 : m_2 r_2$ (2) $m_1 : m_2$
(3) $r_1 : r_2$ (4) $1 : 1$ **(AIEEE 2012)**

4. A particle is moving with a uniform speed in a circular orbit of radius R in a central force inversely proportional to the n^{th} power of R. If the period of rotation of the particle is T, then

(1) $T \propto R^{n/2}$ (2) $T \propto R^{3/2}$ for any n
(3) $T \propto R^{\frac{n}{2}+1}$ (4) $T \propto R^{(n+1)/2}$
(JEE Main 2018)

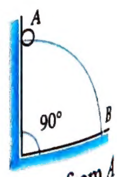
5. A particle is moving in a circular path of radius a under the action of an attractive potential $U = -k/2r^2$. Its total energy is

(1) $-\frac{3}{2}\frac{k}{a^2}$ (2) $-\frac{k}{4a^2}$
(3) $\frac{k}{2a^2}$ (4) Zero
(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. A wire, which passes through the hole in a small bead, is bent in the form of quarter of a circle. The wire is fixed vertically on ground as shown in the figure. The bead is released from near the top of the wire and it slides along the wire without friction. As the bead moves from A to B, the force it applies on the wire is



(1) always radially outwards.

(2) always radially inwards.

(3) radially outwards initially and radially inwards later.

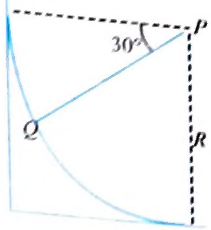
(4) radially inwards initially and radially outwards later.

(JEE Advanced 2014)

Linked Comprehension Type

For Problems 1 and 2

A small block of mass 1 kg is released from rest at the top of a rough track. The track is circular arc of radius 40 m. The block slides along the track without toppling and a frictional force acts on it in the direction opposite to the instantaneous velocity. The work done in overcoming the friction up to the point Q, as shown in the figure, is 150 J. (Take the acceleration due to gravity, $g = 10\ \text{m s}^{-2}$)



(JEE Advanced 2013)

1. The speed of the block when it reaches the point Q is

(1) $5\ \text{m s}^{-1}$ (2) $10\ \text{m s}^{-1}$

(3) $10\sqrt{3}\ \text{m s}^{-1}$ (4) $20\ \text{m s}^{-1}$

2. The magnitude of the normal reaction that acts on the block at the point Q is

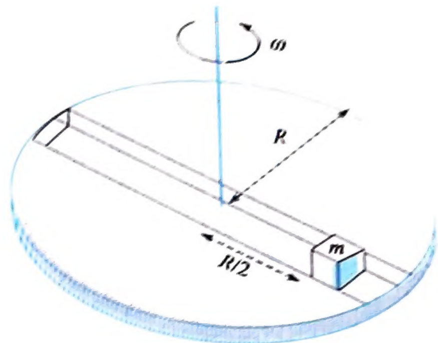
(1) 7.5 N (2) 8.6 N

(3) 11.5 N (4) 22.5 N

For Problems 3 and 4

A frame of reference that is accelerated with respect to an inertial frame of reference is called a noninertial frame of reference. A coordinate system fixed on a circular disc rotating about a fixed axis with a constant angular velocity ω is an example of a non-inertial frame of reference. The relationship between the force $\vec{F}_{rot}$ experienced by a particle of mass m moving on the rotating disc and the force in $\vec{F}_{in}$ experienced by the particle in an inertial frame of reference is, $\vec{F}_{rot} = \vec{F}_{in} + 2m(\vec{v}_{rot} \times \vec{\omega}) + m(\vec{\omega} \times \vec{r}) \times \vec{\omega}$, where $\vec{v}_{rot}$ the velocity of the particle in the rotating frame of reference and $\vec{r}$ is the position vector of the particle with respect to the centre of the disc.

Now consider a smooth slot along a diameter of a disc of radius R rotating counter-clockwise with a constant angular speed ω about its vertical axis through its center. We assign a coordinate system with the origin at the center of the disc, the x-axis along the slot, the y-axis perpendicular to the slot and the z-axis along the rotation axis ($\vec{\omega} = \omega\hat{k}$). A small block of mass m is gently placed in the slot at $\vec{r} = (R/2)\hat{i}$ at $t = 0$ and is constrained to move only along the slot.



(JEE Advanced 2016)

3. The distance r of the block at time t is:

(1) $\frac{R}{2}\cos 2\omega t$ (2) $\frac{R}{4}(e^{2\omega t} + e^{-2\omega t})$

(3) $\frac{R}{2}\cos \omega t$ (4) $\frac{R}{4}(e^{\omega t} + e^{-\omega t})$

4. The net reaction of the disc on the block is:

(1) $m\omega^2 R \sin \omega t\, \hat{j} - mg\hat{k}$

(2) $-m\omega^2 R \cos \omega t\, \hat{j} - mg\hat{k}$

(3) $\frac{1}{2}m\omega^2 R(e^{\omega t} - e^{-\omega t})\hat{j} + mg\hat{k}$

(4) $\frac{1}{2}m\omega^2 R(e^{2\omega t} - e^{-2\omega t})\hat{j} + mg\hat{k}$

Numerical Value Type

1. A bob of mass m, suspended by a string of length l_1, is given a minimum velocity required to complete a full circle in the vertical plane. At the highest point, it collides elastically with another bob of mass m suspended by a string of length l_2, which is initially at rest. Both the strings are massless and inextensible. If the second bob, after collision, acquires the minimum speed required to complete a full circle in the vertical plane, the ratio l_1/l_2 is

(JEE Advanced 2013)

Answers Key

EXERCISES

Single Correct Answer Type

1. (2) **2.** (3) **3.** (2) **4.** (4) **5.** (3)
6. (3) **7.** (1) **8.** (2) **9.** (4) **10.** (1)
11. (1) **12.** (2) **13.** (2) **14.** (1) **15.** (3)
16. (3) **17.** (1) **18.** (4) **19.** (4) **20.** (2)
21. (4) **22.** (4) **23.** (4) **24.** (4) **25.** (2)
26. (3) **27.** (3) **28.** (3) **29.** (1) **30.** (4)
31. (3) **32.** (3) **33.** (1) **34.** (2) **35.** (3)
36. (4) **37.** (1) **38.** (1) **39.** (1) **40.** (4)
41. (1) **42.** (1) **43.** (1) **44.** (3) **45.** (3)
46. (1) **47.** (3) **48.** (4) **49.** (3) **50.** (2)
51. (3) **52.** (1) **53.** (4) **54.** (2) **55.** (4)
56. (4) **57.** (4)

Multiple Correct Answers Type

1. (1),(3),(4) **2.** (1),(2),(3),(4)
3. (2),(3),(4) **4.** (1),(3),(4)
5. (1),(2),(4) **6.** (1),(2),(3),(4)
7. (1),(3) **8.** (2),(3),(4)
9. (1),(3),(4) **10.** (1),(3)
11. (1),(3) **12.** (1),(2),(3)
13. (1),(2),(3),(4) **14.** (1),(2),(4)
15. (1),(3) **16.** (1),(3),(4)
17. (1),(2),(4) **18.** (1),(3)
19. (2),(3),(4) **20.** (2),(3)

Linked Comprehension Type

1. (1) **2.** (3) **3.** (1) **4.** (3) **5.** (4)
6. (4) **7.** (1) **8.** (1) **9.** (4) **10.** (1)
11. (3) **12.** (1) **13.** (4) **14.** (4) **15.** (3)
16. (4) **17.** (2) **18.** (3) **19.** (2) **20.** (1)
21. (1) **22.** (3, 4) **23.** (1) **24.** (1) **25.** (1)
26. (2, 4) **27.** (1, 4)

Matrix Match Type

1. i $\rightarrow$ b, c, d; ii $\rightarrow$ a, d; iii $\rightarrow$ b, d; iv $\rightarrow$ c
2. i $\rightarrow$ b, d; ii $\rightarrow$ a, b; iii $\rightarrow$ a; iv $\rightarrow$ b, c, d
3. i $\rightarrow$ a, c, d; ii $\rightarrow$ a, d; iii $\rightarrow$ a, b, d; iv $\rightarrow$ a, c, d
4. (1) **5.** (3) **6.** (4) **7.** (2) **8.** (3)

Numerical Value Type

1. (1) **2.** (5) **3.** (1.125) **4.** (9) **5.** (1)
6. (2) **7.** (0) **8.** (48) **9.** (2) **10.** (1)
11. (6) **12.** (2) **13.** (6)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3) **2.** (4) **3.** (3) **4.** (4) **5.** (4)

JEE Advanced

Single Correct Answer Type

1. (4)

Linked Comprehension Type

1. (2) **2.** (1) **3.** (4) **4.** (3)

Numerical Value Type

1. (5)